BREAKING SYMMETRY ON COMPLETE BIPARTITE GRAPHS OF ODD SIZE

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Abstract

Players $\mathcal{A}$ and $\mathcal{B}$ alternatively colour edges of a graph G, red and blue respectively. Let $L_{\rm sym}(G)$ be the largest number of moves during which $\mathcal{B}$ can keep the red and blue subgraphs isomorphic, no matter how $\mathcal{A}$ plays.

This function was introduced by Harary, Slany and Verbitsky who in particular showed that for complete bipartite graphs we have $L_{\rm sym}(K_{m,n}) = \frac{mn}{2}$ if mn is even and that $L_{\rm sym}(K_{2m+1,2n+1}) \ge \max(m,n)$. Here we prove that

$$L_{\rm sym}(K_{2m+1,2n+1}) = O(n), \quad \text{if } m \le n \le m^{O(1)},$$

answering a question posed by Harary, Slany and Verbitsky.

1. Introduction

Let G be a graph. The following *symmetry breaking-preserving game* on G was introduced by Harary, Slany and Verbitsky [1, 2]. We have two players, $\mathcal{A}$ and $\mathcal{B}$, who alternatively select a previously uncoloured edge of G and colour it red and blue respectively. Player $\mathcal{A}$ starts the game. A move of $\mathcal{A}$ followed by a move of $\mathcal{B}$ is called a *round*. Clearly, we have the same number of red and blue edges after every round. The aim of Player $\mathcal{B}$ is to keep the red and blue subgraphs isomorphic after every round of the game; as soon as $\mathcal{B}$ fails to do so, he loses.

Let $L_{\rm sym}(G)$ be the maximum number of moves during which $\mathcal{B}$ can keep the red and blue subgraphs isomorphic, no matter how $\mathcal{A}$ plays. Equivalently (see [2, Proposition 2.1])

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$L_{\rm sym}(G)$ is the smallest k such that $\mathcal{A}$ can guarantee his win in at most $k+1$ moves. It is not quite clear how to define $L_{\rm sym}(G)$ in the cases when $\mathcal{B}$ can preserve the symmetry until the players run out of edges; following [2] we define $L_{\rm sym}(G) = \lfloor e(G)/2 \rfloor$ then.

One of the motivations of Harary, Slany and Verbitsky for introducing these notions was that $L_{\rm sym}(G)$ is clearly a lower bound on how long the second player can survive in any *graph avoidance game* on G. (The rules of the avoidance game are the same except that the player who first constructs a monochromatic copy of a certain forbidden subgraph loses.)

As it is observed in [1], if G has an involutary automorphism ψ without fixed edges, then $L_{\rm sym}(G) = e(G)/2$. Indeed, every orbit of the induced action ψ^* on $E(G)$ consists of two edges, so $\mathcal{B}$ can use the *copycat strategy* of choosing $\psi^*(e) \in E(G)$ where e is the edge previously coloured by $\mathcal{A}$.

The determination of $L_{\rm sym}(G)$ is suddenly getting rather complicated and deep when we consider graphs which do not admit a copycat strategy but for which this cannot be derived by looking at a part of the graph. For example, when one considers the path P_n with n vertices, one has to know (the parity of) its order n in order to ascertain the existence of an appropriate automorphism ψ. So, if $\mathcal{A}$ follows some 'local' strategy, $\mathcal{B}$ might put up a strong resistance by playing copycat on a few separate parts of the graph. The surprising (at least to me) result of Harary, Slany and Verbitsky [2, Corollary 3.7 & Proposition 3.8] states that

$$(0.5 + o(1))\, \log_2 n \le L_{\rm sym}(P_{2n}) \le (3.5 + o(1))\, (\log_2 n)^2.$$

Its proof exploits some beautiful connections to the so-called *Ehrenfeucht-Fraïssé game.*

Complete bipartite graphs of even size clearly admit a copycat strategy. The following argument from [2] shows that

$$L_{\rm sym}(K_{m,n}) \ge \max(\tfrac{m-1}{2}, \tfrac{n-1}{2}), \quad \text{for odd } mn. \tag{1}$$

Indeed, let $X \subset V(K_{m,n})$ be the bigger part of $K_{m,n}$. Starting with ψ being the identity automorphism, $\mathcal{B}$ uses the ψ-copycat strategy, that is, responds with $\psi^*(e)$ to the previous move e. This works unless $\psi^*(e) = e$ in which case $\mathcal{B}$ locally modifies ψ so that it exchanges x and y now, where $\{x\} = X \cap e$ and $y \in X$ is a vertex not incident to any coloured edge. Such y always exists during the first $(|X|-1)/2$ rounds during which ψ remains an involutary automorphism of $K_{m,n}$ swapping the blue and red subgraphs.

Harary, Slany and Verbitsky [2, Question 4.3] asked for the rate of growth of the function $L_{\rm sym}(K_{n,n})$ for odd n. The following more general result implies that $L_{\rm sym}(K_{m,n})$ grows linearly in n if $m \le n \le m^{O(1)}$ and mn is odd.

Theorem 1 *Let odd integers m and n and an integer $k \ge 8$ satisfy*

$$51 \le m \le n \le \frac{(m-2k-3)^k}{(k+1)!\, m}. \tag{2}$$

Then we have

$$L_{\rm sym}(K_{m,n}) \le k(n-m+3)+2m+14. \tag{3}$$

In particular, if $m=n$, then letting $k=8$ we obtain

$$L_{\rm sym}(K_{n,n}) \le 2n+38, \quad \textit{for odd } n \ge 51. \tag{4}$$

In Section 3 we present the bound (3) as a more digestible, explicit function of m and n.

Let us describe the main idea behind the proof of Theorem 1. In outline, $\mathcal{A}$ builds a red graph which has no non-trivial automorphism. If $\mathcal{B}$ has not lost yet, the isomorphism ψ between the red and blue graphs is unique and $\mathcal{B}$ is *forced* to play the ψ^*-copycat strategy now. As the total number of edges is odd, ψ^* has at least one odd orbit $D \subset E(G)$. No matter how D has been coloured, Player $\mathcal{A}$ can beat the ψ^*-copycat strategy on D with at most two moves. So, if ψ remains the unique isomorphism during these two moves of $\mathcal{A}$, then $\mathcal{B}$ loses.

This method might be applicable to many graphs with odd size. Here is its concrete realisation for complete bipartite graphs.

2. Proof of Theorem 1

We will describe the appropriate strategy of $\mathcal{A}$ which consists of a few phases. Assume that $\mathcal{B}$ keeps the red and blue subgraphs isomorphic throughout our strategy.

Let $V \cup V' = V(K_{m,n})$ be the vertex classes, where $|V| = m$.

PHASE 1. $\mathcal{A}$ builds a red cycle of length $2l \in [2m-6, 2m-2]$, say visiting vertices $x_1, x_1', \dots, x_l, x_l'$ in this order, plus one edge from some vertex $y_0 \in Y$ to X', where we denote $X = \{x_1, \dots, x_l\} \subset V$, $X' = \{x_1', \dots, x_l'\} \subset V'$, $Y = V \setminus X$, and $Y' = V' \setminus X'$.

The red/blue subgraph will have maximum degree at most 2 at any moment before the end of Phase 1, so in particular $\mathcal{A}$ can easily create a red path of length $2m-7$ by choosing vertices $x_1, x_1', \dots, x_{m-3}, x_{m-3}'$ one by one in this order. If the edge $\{x_1, x_{m-3}'\}$ is available, then $\mathcal{A}$ colours it and we are done because the addition of an edge between X' and Y is always possible as $m-3>2$. Otherwise, $\mathcal{A}$ extends the path by two more edges. Now, if $\{x_1, x_{m-2}'\}$ is available, then $\mathcal{A}$ selects it and we are done again. Otherwise, $\mathcal{A}$ can add x_{m-1} to the path: indeed, the vertex x_{m-2}' sends a blue edge to x_1, so it sends at most one blue edge to $V \setminus X$. If $\{x_{m-1}, x_1'\}$ is available, then $\mathcal{A}$ selects it, obtaining the desired configuration (up to relabelling). Otherwise, the edge $\{x_{m-1}, x_1'\}$ is blue and $\mathcal{A}$ can extend the path to x_{m-1}'. In the next move $\mathcal{A}$ colours the edge $\{x_1, x_{m-1}'\}$ which cannot be blue because we have already encountered two blue edges incident to

x_1. Finally, $\mathcal{A}$ connects some $y_0 \in Y$ to X', obtaining the desired configuration in all cases.

PHASE 2. $\mathcal{A}$ connects Y to X'.

The vertex $y_0 \in Y$ will play a special role. Assume that x'_1 is the vertex in X' connected to y_0 by a red edge.

Consider first the case when y_0 is incident to at least one blue edge. By reversing, if necessary, the direction of the red $2l$-cycle, we can assume that y_0 has at least one blue neighbour *outside* $\{x'_1, \ldots, x'_{10}\}$. In the next five moves $\mathcal{A}$ colours $\{y_0, x'_i\}$ for the smallest possible index $i \geq 2$ each time. Suppose that the last such edge was $\{y_0, x'_s\}$. When $\mathcal{A}$ was colouring it, at most 5 edges incident to y_0 were blue of which at most 4 lie in $X'_0 = \{x'_1, \ldots, x'_s\}$. Hence $s \leq 10$.

In the next three moves $\mathcal{A}$ colours $\{y_0, x'_i\}$, where i is the smallest available index with $i \geq 2s$ except, when colouring the third edge in the case $s = 6$, the additional condition on i is that the indexes of the last three red neighbours of y_0 do not form an arithmetic progression. (Note that for $s \geq 7$ the indexes of the *first* six neighbours of y_0 cannot form a six-term arithmetic progression as $s \leq 10$.)

Let $t > 2s$ be the largest index of a red neighbour of y_0 and let $X'_1 = \{x'_{2s}, \ldots, x'_t\}$. We claim that $t \leq 26$. If $s \geq 7$, then y_0 sends at most $8 - (s - 6) = 14 - s$ blue edges to X'_1 because y_0 sends $s - 6$ blue edges to X'_0; thus

$$t - 2s + 1 = |X'_1| \leq 3 + (14 - s), \tag{5}$$

which gives the required in view of $s \leq 10$. If $s = 6$, then we have to add 1 to the bound (5), still obtaining the claimed inequality $t \leq 26$.

If y_0 is not incident to a blue edge at the end of Phase 1, then this stays so, in whichever manner we add red edges incident to y_0. In particular, $\mathcal{A}$ can connect y_0 to

$$\{x'_1, x'_2, \ldots, x'_6, x'_{12}, x'_{13}, x'_{15}\}$$

and we have $s = 6$ and $t = 15$.

Next, $\mathcal{A}$ connects the vertices of $Y \setminus \{y_0\}$ to X', by five edges each, so that distinct vertices of Y have disjoint neighbourhoods in X', which is possible as $l > 2\cdot 9 + 5\cdot(|Y| - 1)$.

The first two phases last for $r_2 = 2l + 9 + 5(m - l - 1)$ rounds.

PHASE 3. If $m = n = l + 1$, then $\mathcal{A}$ connects the (unique) vertex of Y' to arbitrary 12 vertices of X. Otherwise, $\mathcal{A}$ picks vertices of Y' one by one and connects each selected vertex by k edges to X. Suppose that $\mathcal{A}$ has already dealt with $y'_1, \ldots, y'_i \in Y'$ and connected the next vertex $y' \in Y'$ to a set $H \subset X$ of size $h \in [0, k - 1]$. Let $\Gamma_{\mathrm{red}}(y)$ (resp. $\Gamma_{\mathrm{blue}}(y)$) denotes the red (resp. blue) neighbourhood of a vertex y.

We claim that f, the number of vertices in $F = \Gamma_{\text{blue}}(y') \cap X$, will be at most $k+1$ when we deal with the vertex y'. This is true for $i \leq 3$ when the maximum degree of the blue graph is at most $\max(k, 9)$. If $i \geq 4$, then the isomorphism between the red and blue graphs must respect the classes V and V' because the non-trivial red component has at least $l+i > |V|$ vertices in V'. So the blue degree of y' is at most the maximum red degree of a vertex in V' which is k, giving the desired bound on f.

Here is the strategy: $\mathcal{A}$ connects y' to a vertex $x \in X \setminus (F \cup H)$ such that $\mu(H \cup \{x\})$ is minimum, where

$$\mu(Z) = \sum_{j=1}^{i} c_{|Z \setminus \Gamma_{\text{red}}(y'_j)|}, \quad Z \subset X,$$

where $c_0 = l$, $c_1 = 1$, and all other c's are zero. In other words,

$$\mu(Z) = l \left|\left\{j \in [i] \mid Z \subset \Gamma_{\text{red}}(y'_j)\right\}\right| + \left|\left\{j \in [i] \mid |Z \setminus \Gamma_{\text{red}}(y'_j)| = 1\right\}\right|.$$

It is easy to see that

$$\begin{aligned}\sum_{x \in X \setminus (F \cup H)} \mu(H \cup \{x\}) \;&\leq\; \sum_{\substack{j \in [i] \\ H \subset \Gamma_{\text{red}}(y'_j)}} (c_0(k-h) + c_1(l-k)) \\ &+ \sum_{\substack{j \in [i] \\ |H \setminus \Gamma_{\text{red}}(y'_j)|=1}} c_1(k-h+1) \;\leq\; (k-h+1)\mu(H),\end{aligned}$$

by straightforwardly comparing the corresponding terms. Hence, $\mathcal{A}$ can choose an $x \in X \setminus (F \cup H)$ such that

$$\mu(H \cup \{x\}) \leq \frac{k-h+1}{l-f-h}\,\mu(H) \leq \frac{k-h+1}{m-2k-3}\,\mu(H).$$

As $\mu(\emptyset) = li < mn$, the inequality

$$\mu(H) < \frac{(k+1)!}{(m-2k-3)^k}\,mn \leq 1$$

holds when we reach the case $|H| = k$. As μ is integer-valued, it must be the case that $\mu(H) = 0$, which means by the definition that $|H \setminus \Gamma_{\text{red}}(y'_j)| \geq 2$ for any $j \in [i]$. Thus $\mathcal{A}$ can ensure that the hamming distance between any two sets in $\{\Gamma_{\text{red}}(y') \mid y' \in Y'\}$ is at least 4 at the end of Phase 3.

The first three phases took $r_3 = r_2 + 12$ rounds if $n = m = l+1$ and $r_3 = r_2 + k(n-l)$ rounds otherwise.

Phase 4. $\mathcal{A}$ adds at most two more edges and wins.

This phase needs some analysis before we can specify the moves of $\mathcal{A}$. Let A_i (resp. B_i) consist of red (resp. blue) edges after i rounds, viewed as a subset of $E(K_{m,n})$. Thus

$A = A_{r_3}$ is the red graph at the end of Phase 3. Note that $X \cup X'$ spans in A an induced $2l$-cycle which we denote by $C \subset A$.

Claim 1. Let A' be obtained by adding to A at most two edges of the encompassing graph $K_{m,n}$. Let $\phi : V(K_{m,n}) \to V(K_{m,n})$ be a bijection such that $\phi^*(A) \subset A'$, where ϕ^* denotes the induced action on 2-point sets. Then $\phi(y_0) = y_0$, $\phi(Y) = Y$, $\phi(X) = X$, $\phi(Y') = Y'$ and $\phi(X') = X'$.

If, furthermore, $\phi^*(C) = C$, then ϕ is the identity map.

Proof. As A, A' are connected bipartite graphs, we have $\phi(V) = V$ or $\phi(V) = V'$.

Let us show first that $\phi(V) = V$, which needs justification when $|V| = |V'|$. If $|Y'| = 1$, then the (unique) vertex of Y' of degree at least 12 must be preserved by ϕ as any other vertex has A'-degree at most 11; thus $\phi(V) = V$, as required. Suppose that $|Y'| = |Y| \geq 2$. The vertices in X are incident to at most $4 + |Y'| \leq 7 < k$ A'-edges each. Thus V contains at most one vertex of A'-degree at least k and ϕ must map Y' into V'. Now it follows that $\phi(V) = V$ and $\phi(V') = V'$.

As each vertex of Y' has degree at least 8 while the A'-degrees in X' are all at most 5, we conclude that $\phi(Y') = Y'$. As each vertex of $\phi(Y)$ sends at least five edges to $\phi(X') = X'$, we have $\phi(Y) = Y$. Similarly, $\phi(y_0) = y_0$, which proves the first part of the claim.

Suppose furthermore that $\phi^*(C) = C$. This means that the restriction of ϕ to $X \cup X'$ is a cyclic rotation, possibly composed with the reflection $x_i \mapsto x'_{l-i+1}$, $x'_i \mapsto x_{l-i+1}$. We are going to show that $\phi|_{X \cup X'}$ is the identity by considering the neighbourhood of y_0.

Recall that the set $X'_0 = \{x'_1, \ldots, x'_s\}$ contains six A-neighbours of y_0 and $X'_1 = \{x'_{2s}, \ldots, x'_t\}$ the remaining three. The sets X'_0 and X'_1 cannot both intersect $\phi(X'_0)$ as they are separated by $s - 1$ other vertices of X'. Hence, $\phi(X'_0) \cap X'_1 = \emptyset$ and $\phi(X'_0) \cap X'_0$ contains at least four A-neighbours of y_0. If $\phi(X'_1)$ is situated at the 'wrong' side of $\phi(X'_0)$, then (as $l \geq 2t - 4$) we have $\phi(X'_1) \cap X'_1 = \emptyset$ and at least three A'-neighbours of y_0 fall outside $\phi(X'_0 \cup X'_1)$, a contradiction. Thus $\phi|_{X \cup X'}$ is a cycle rotation (without any reflection). Moreover, it is the identity rotation for otherwise we obtain the contradiction $|\phi(\Gamma_{\mathrm{red}}(y_0)) \setminus \Gamma_{\mathrm{red}}(y_0)| \geq 3$. (The latter inequality holds because in Phase 2 we excluded the possibility that the indexes of the red neighbours of y_0 form two arithmetic progressions.)

Now, for any two vertices of $Y \cup Y'$ the hamming distance between their A-neighbourhoods is at least 4. This clearly implies that ϕ is the identity bijection. ∎

Claim 1 implies in particular that A has no non-trivial automorphism. Thus $\psi = \psi_{r_3}$ is uniquely determined, where ψ_r denotes the red-blue isomorphism after r rounds. The bipartition $V \cup V'$ is clearly preserved (or reversed) by ψ so we have the induced action ψ^* on $E(K_{m,n})$.

Claim 2. For any (r_3+1)st move e of $\mathcal{A}$, the player $\mathcal{B}$ is forced to reply with $\psi^*(e)$.

Proof. By Claim 1 any bijection ϕ with $\phi^*(A) \subset A_{r_3+1}$ preserves $X \cup X'$. As $X \cup X'$ induces in A_{r_3+1} a cycle with at most one chord added, we have $\phi^*(C) = C$ and, again by Claim 1, ϕ is the identity map. This means that A is the only subgraph of A_{r_3+1} isomorphic to A and the analogous claim holds for the blue edges. Thus $\psi^*_{r_3+1}(A) = B_{r_3}$, which implies that $\psi_{r_3+1} = \psi$. Now, ψ^* must map $\{e\} = A_{r_3+1} \setminus A$ into $B_{r_3+1} \setminus B_{r_3}$, as required. ∎

As the total number of edges mn is odd, some ψ^*-orbit $D \subset E(K_{m,n})$ has the odd number of elements.

If at least one element of D is already coloured, then we can find via a parity argument an uncoloured $e \in D$ such that $\psi^*(e)$ is red. Now, $\mathcal{A}$ colours e red and $\mathcal{B}$ loses by Claim 2.

Suppose that no edge of D has been coloured. If $D = \{e\}$, then $\psi^*(e) = e$ and $\mathcal{A}$ wins by choosing e by Claim 2. Suppose that $|D| > 1$ and let $e \in D$. Now $e' = \psi^*(e)$ and $e'' = (\psi^*)^{-1}(e)$ are two distinct edges belonging to D.

Suppose first that $e = \{x_i, x'_j\}$ and that both $C \cup \{e, e'\}$ and $C \cup \{e, e''\}$ have a $2l$-cycle passing through the edge e. Trivial considerations show that

$$\{e', e''\} = \Big\{ \{x_j, x'_{i-1}\}, \{x_{j+1}, x'_i\} \Big\}. \tag{6}$$

(Of course, we do all index calculations modulo l.) If $\psi(V) = V$, then (6) means that for some $\delta = \pm 1$ we have $\psi^\delta(x_i) = x_{j+1}$, $\psi^\delta(x'_j) = x'_i$, $\psi^{-\delta}(x_i) = x_j$ and $\psi^{-\delta}(x'_j) = x'_{i-1}$. But then ψ^δ maps the red edge $\{x_j, x'_j\}$ into the red edge $\{x_i, x'_i\}$, a contradiction. In the same way we obtain a contradiction in the case $\psi(V) = V'$.

Thus, by the first part of Claim 1, we can assume that we can recover C from knowing, for example, $A \cup \{e, e'\}$. Now, $\mathcal{A}$ selects e' to which, by Claim 2, $\mathcal{B}$ is forced to reply by colouring $\psi^*(e')$. Then $\mathcal{A}$ selects $e \neq \psi^*(e')$. Claim 1 implies that A is the unique subgraph of $A \cup \{e, e'\}$ isomorphic to A and the argument of Claim 2 shows that $\mathcal{B}$ loses.

Let us count the total number r_4 of rounds. If $n = m = l + 1$, then

$$r_4 \le 2(m-1) + 9 + 12 + 2 \le 3(n - m + k) + 2m + 15,$$

as $k \ge 8$. Otherwise,

$$r_4 \le 2l + 9 + 5(m - l - 1) + k(n - l) + 2 \le k(n - m + 3) + 2m + 15,$$

where we used the inequality $l \ge m - 3$. This finishes the proof of Theorem 1 as $L_{\mathrm{sym}}(K_{m,n}) \le r_4 - 1$.

3. Concluding Remarks

It is not hard to convert our strategy into an algorithm which has running time $O(mn)$ per one move of $\mathcal{A}$.

Unfortunately, not for every pair (m, n) an integer k satisfying (2) can be found. For example, if $n > 2^m$, then any subgraph of $K_{m,n}$ contains two vertices in V' with the same neighbourhood in V, which ruins our strategy. The value of n can range up to

$$\max_{k\geq 8} \frac{(m-2k-3)^k}{(k+1)!\, m} = (1.531... + o(1))^m.$$

The optimal choice is to take the smallest integer $k \geq 8$ satisfying (2). If m is large, then the following formulae can be routinely verified. If $\log n/\log m < 7 - o(1)$, then $k = 8$. If $7 - o(1) \leq \log n/\log m = O(1)$, then

$$\frac{\log n}{\log m} + 1 < k < \frac{\log n}{\log m} + 3.$$

If $O(\log m) < \log n = o(m)$, then

$$k = (1 + o(1))\, \frac{\log n}{\log m - \log\log n}.$$

If $(1 + o(1))^m < n < (1.531... + o(1))^m$, then $k = (x_0 + o(1))\, m$, where x_0 is the smallest positive root of equation

$$((1-2x)\mathrm{e}/x)^x = \mathrm{e}^{\log n/m}.$$

The present best known bounds on $L_{\mathrm{sym}}(K_{n,n})$ for odd n are given by (1) and (4). Unfortunately, it seems that a minor modification of our method cannot give any considerable improvement on (4) as any graph H with no non-trivial automorphism has at least $(1 - o(1))\, v(H)$ edges. Indeed, if we fix some large constant C, then H has $O(1)$ components with less than C vertices (as no two can be isomorphic), while the remaining components span at least $(v(H) - O(1))\, \frac{c-1}{c}$ edges. Also, one meets difficulties when trying to improve on (1): it is not hard to see that $\mathcal{A}$ can ensure that within first $\frac{n+1}{2}$ rounds there was a position when no blue-red isomorphism could be generated by an involutary automorphism of $K_{n,n}$. Hence, $\mathcal{B}$ must go beyond copycat if he wants to survive for more than $(\frac{1}{2} + o(1))\, n$ rounds.

In fact, we do not even have any solid conjecture what the value of

$$\lim_{n\to\infty} \frac{L_{\mathrm{sym}}(K_{2n+1,2n+1})}{n}$$

is (if the limit exists).

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