NON-CANONICAL EXTENSIONS OF ERDŐS-GINZBURG-ZIV THEOREM[1]

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Abstract

In 1961, Erdős-Ginzburg-Ziv proved that for a given natural number $n \geq 1$ and a sequence $a_1, a_2, \cdots, a_{2n-1}$ of integers (not necessarily distinct), there exist $1 \leq i_1 < i_2 < \cdots < i_n \leq 2n-1$ such that $a_{i_1} + a_{i_2} + \cdots + a_{i_n}$ is divisible by n. Moreover, the constant $2n-1$ is tight. By now, there are many canonical generalizations of this theorem. In this paper, we shall prove some non-canonical generalizations of this theorem.

1. Introduction and Preliminaries

Additive number theory, graph theory and factorization theory provide inexhaustible sources for combinatorial problems in finite abelian groups (cf. [27], [28], [13], [11], [29] and [3]). Among them zero sum problems have been of growing interest. Starting points of recent research in this area were the Theorem of Erdős-Ginzburg-Ziv (EGZ Theorem, for short) and a question of H. Davenport on an invariant which today carries his name.

We shall denote the cyclic group of order n by $\mathbf{Z}_n$. A sequence $S = \{a_i\}_{i=1}^{\ell}$ of length ℓ in $\mathbf{Z}_n$, we mean $a_i \in \mathbf{Z}_n$ and a_i's are not necessarily distinct, unless otherwise specified. Also, throughout this paper, writting $n \geq 1$, we mean n is an arbitrary natural number and writting p, we mean an arbitrary prime number.

We shall define some terminologies as follows. A sequence S is called zero sequence if its sum is zero. A sequence S is called zero-free sequence if it contains no zero subsequence. A sequence S is called minimal zero sequence if S is a zero sequence; but any proper subsequence is zero-free. Now, we shall restate the EGZ theorem, using the above terminologies, as follows.

EGZ Theorem. (cf. [12]) *Given a sequence S in $\mathbf{Z}_n$ of length $2n-1$, one can extract*

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a zero subsequence of length n *in* $\mathbf{Z}_n$.

We shall state the following known result which will be useful for our further discussion.

Cauchy-Davenport inequality. *Let* A *and* B *be two non-empty subsets of* $\mathbf{Z}_p$. *Then*

$$|A+B| \geq \min\{p, |A|+|B|-1\}$$

where $A+B=\{x=a+b \in \mathbf{Z}_p : a, b \in \mathbf{Z}_p\}$ *and* $|K|$ *denotes the cardinality of the subset* K *of* $\mathbf{Z}_p$.

This was first proved by Cauchy (cf. [9]) in 1813 and was rediscovered by Davenport (cf. [10]) in 1947.

Corollary 1.1 *Let* $A_1, A_2, \cdots, A_h$ *be non-empty subsets of* $\mathbf{Z}_p$. *Then,*

$$|A_1+A_2+\cdots+A_h| \geq \min\{p, \sum_{i=1}^{h}|A_i|-h+1\}.$$

By today there are several extensions of Erdős-Ginzburg-Ziv theorem are known. All the known extensions are natural and we call them canonical extensions. In this paper, we shall prove several non-canonical extensions of EGZ theorem.

2. Canonical extensions of EGZ Theorem

In this section, we shall survey the results which are natural generalization or extensions of EGZ Theorem and we call them as C-Extensions. The first natural generalization of EGZ in $\mathbf{Z}$ is the following due to Olson.

C-Extension 1. (Olson, 1969, [30]) *Suppose* $m \geq k \geq 2$ *are integers such that* $k|m$. *Let* $a_1, a_2, \cdots, a_{m+k-1}$ *be a sequence of integers. Then there exists a non-empty subset* I *of* $\{1, 2, \cdots, m+k-1\}$, *such that* $|I|=m$ *and* $\sum_{i\in I} a_i \equiv 0 \pmod{k}$.

If one view EGZ theorem as a statement over the solvable group $\mathbf{Z}_n$, then one can ask for the same in any finite group. Indeed, Olson proved that

C-Extension 2. (Olson, 1976, [31]) *Let* $g_1, g_2, \cdots, g_{2n-1}$ *be a sequence of length* $2n-1$ *in a finite group (but not necessarily Abelian)* G *of order* n. *Then there exists elements* $g_{i_1}, g_{i_2}, \cdots, g_{i_n}$ *from the given sequence satisfies* $g_{i_1}+g_{i_2}+\cdots+g_{i_n}=0$ *in* G.

Conjecture 2.1 (Olson, 1976, [31]) *The same conclusion holds in C-Extension 2, together with* $i_1 < i_2 < \cdots < i_n$.

In this direction, W. D. Gao (cf. [16]) proved in 1996 that if G is a non-cyclic solvable group and $s = [11n/6] - 1$, then for any sequence $a_1, a_2, \cdots, a_s$ in G, we have $i_1, i_2, \cdots, i_n$ distinct such that $a_{i_1} + \cdots + a_{i_n} = 0$ in G. Other than this result, we know nothing about Conjecture 2.1.

In C-Extension 2, $2n - 1$ may not be tight except for the group $G \cong \mathbf{Z}_n$. Indeed, if G is an abliean group (additively written) of order n, then Gao (cf. [17]) proved that $n + D(G) - 1$ is the right constant in place of $2n - 1$ where $D(G)$ (is the Davenport Constant), which is the least positive integer such that given any sequence S in G of length $\ell(S)$ with $\ell(S) \geq D(G)$, there exists a zero subsequence T of S in G. One can easily see that when $G \cong \mathbf{Z}_n$, we have $D(\mathbf{Z}_n) = n$.

There is yet another generalization of EGZ theorem as follows. EGZ theorem proves the existence of one zero subsequence of length n, whenever we consider a sequence in $\mathbf{Z}_n$ of length $2n - 1$.

C-Extension 3 (W. D. Gao, 1997, [19]) *If* $a_1, a_2, \cdots, a_{2n-1}$ *be a sequence in* $\mathbf{Z}_n$, *then there exists at least* n *number of subsequences of length* n *having its sum 'a' for any given* $a \in \mathbf{Z}_n^*$ *provided no element occurs more than* n *times in the sequence. More over, there exists at least* $n+1$ *number of zero subsequences of length* n *in* $\mathbf{Z}_n$ *unless only two elements* x *and* y *occur* n *and* $n-1$ *times respectively in that sequence.*

Indeed, chronologically, H. B. Mann (cf. [26]) in 1967 proved the existence of one subsequence of length p whose sum is g for a given $g \in \mathbf{Z}_p$, whenever we consider a sequence in $\mathbf{Z}_p$ of length $2p - 1$. In 1996, W. D. Gao (cf. [20]) proved that C-Extension 3 for $n = p$ prime. (Indeed, Sury (cf. [34]) gave a different proof of this fact).

One should mention the result of Bialostocki and Dierker which is a stronger version of EGZ theorem;

C-Extension 4'. (Bialostocki and Dierker, 1992, [5]) *If* $S = \{a_i\}$ *is a sequence in* $\mathbf{Z}_p$ *of length* $2p - 1$, *then there are* p *indices* $1 \leq i_i < i_2 < \cdots < i_p \leq 2p - 1$ *such that*

$$a_{i_1} + a_{i_2} + \cdots + a_{i_p} \equiv 0 \pmod{p}.$$

Moreover, if for two indices j, k *we have* $a_j \not\equiv a_k \pmod{p}$, *then we can choose* $i_1, i_2, \cdots, i_p$ *such that not both* j *and* k *are among them in that zero of length* p.

Indeed, we shall prove better result than C-Extension 4' as follows.

C-Extension 4. *If $S = \{a_i\}$ is a sequence in $\mathbf{Z}_p$ of length $2p-1$, then there are p indices $1 \le i_i < i_2 < \cdots < i_p \le 2p-1$ such that*

$$a_{i_1} + a_{i_2} + \cdots + a_{i_p} \equiv 0 \pmod{p}. \tag{1}$$

Moreover, if $s \ge 2$ distinct elements of $\mathbf{Z}_p$, say, $a_1, a_2, \cdots, a_s$ are in S, then we can choose the $i_1, i_2, \cdots, i_p$ such that only one of the indices from $1, 2, \cdots, s$ appears among $i_1, i_2, \cdots, i_p$ and satisfying (1).

Proof. If one of the element of S is repeated more than p times, then the result follows trivially. Assume that none of the elements of S appears more than $p-1$ times. Let $A_1 = \{a_1, a_2, \cdots, a_s\} \subset \mathbf{Z}_p$ and the remaining a_i's be distributed into non-empty $p-1$ incongruent classes modulo p, say, $A_2, A_3, \cdots, A_{p-1}$. Then by Cauchy-Davenport inequality, we have

$$|A_1 + A_2 + \cdots + A_p| \ge \min\{p, \sum_i |A_i| - p + 1\} = \min\{p, 2p-1-p+1\} = p$$

$$\Longrightarrow A_1 + A_2 + \cdots + A_p = \mathbf{Z}_p.$$

Thus the result follows. □

3. Non-Canonical extensions of EGZ Theorem

In this section, we prove non-canonical generalizations of EGZ Theorem and we call them as N-C Extensions.

N-C Extension 1. *Let S be a sequence in $\mathbf{Z}_n$ of length at least n. Let $h = h(S) = \max_{a \in S} g(a)$ where $g(a)$ denote the number of times $a \in \mathbf{Z}_n$ appearing in S. Then there is a zero subsequence of length less than or equal to h.*

This theorem was proved by Gao and Yang (cf. [24]) in 1997. We shall prove that this theorem indeed implies EGZ Theorem.

Theorem 3.1 *N-C Extension 1 implies EGZ theorem.*

Before going to the proof of Theorem 3.1, we prove the following lemma.

Lemma 3.1.1 *Let S be a sequence in $\mathbf{Z}_n$ of length $2n-1$. Suppose there is an element $a \in \mathbf{Z}_n$ such that a is appearing in S at least $[n/2]$ times, then there is a zero subsequence of S of length n in $\mathbf{Z}_n$.*

Proof. Let S be a sequence in $\mathbf{Z}_n$ of length $2n-1$. Suppose S consists of an element $a \in \mathbf{Z}_n$ which is repeated $s \geq [n/2]$ number of times. If $s \geq n$, then the result is obvious. Let us assume that $s \leq n-1$.

Consider the translated sequence $S-a$ in which 0 is repeated s number of times. The length of the subsequence T_1 of $S-a$ which consists of all the non-zero elements of $S-a$ is $2n-1-s \geq n$. Since $D(\mathbf{Z}_n) = n$, the sequence T_1 contains a zero subsequence say T_2. Let the length of T_2 be t_2. Clearly, $2 \leq t_2 \leq n$. Choose T_2 such that it has the maximal length t_2. Also, note that if $s + t_2 \geq n$, then we can extract a zero of length n in $S-a$ which in turn produces a zero subsequence of length n in S. Thus we can assume that $s + t_2 < n$. Note that $[n/2]+1 \leq t_2 \leq n$. If not, that is, $t_2 \leq [n/2]$. Then after omitting T_2 from T_1, the length of the sequence $T_1 \backslash T_2$ is at least n and hence it contains a zero subsequence say T_3 with length t_3. Clearly, $t_3 \leq t_2 \leq [n/2]$, which implies, $t_2 + t_3 \leq n$, which contradicts to the maximality of T_2. Hence $[n/2]+1 \leq t_2 \leq n$ is true. Since we have at least $[n/2]$ zeros out side T_2, by adding appropriate number of zeros to T_2, we get a zero sequence of length n in $S-a$ which in turn produces a zero sequence of length n in S. $\square$

Proof of Theorem 3.1. Let S be a sequence in $\mathbf{Z}_n$ of length $2n-1$. Suppose S consists of an element $a \in \mathbf{Z}_n$ which is repeated maximum number of s times. If $s \geq n$, then nothing to prove.

Case (i) ($[n/2] \leq s \leq n-1$)

This case is covered by Lemma 3.1.1.

Case (ii) ($2 \leq s \leq [n/2]-1$)

Consider the translated sequence $S-a$ inwhich 0 is repeated s times. The length of the subsequence T_1 of $S-a$ which consists of all the non-zero elements of $S-a$ is $2n-1-s \geq n+n-[n/2]$. Since $D(\mathbf{Z}_n) = n$, there exists a zero subsequence of T_2 of length t. Choose T_2 having the maximum length. Then, it follows that (apply the same argument given in the begining of the proof of Lemma 3.1) $[n/2]+1 \leq t \leq n$.

Claim. $s + t > n$.

Suppose not, that is, $s+t < n$. Now delete the subsequence T_2 from T_1. Then the length of the deleted sequence, say T_3, is $2n-1-s-t \geq 2n-1-(n-1) = n+1$. By N-C Extension 1, there exists a zero subsequence of length less than or equal to s in T_3. Therefore there exists a subsequence T_4 of T_3 such that the length of T_4, say t_1 is less than or equal to s. Since T_2 is maximal with respect to its length less than or equal to n, it is clear that $t+t_1 > n$. If $2 \leq t_1 \leq s \leq [n/2]-1$, then $n-t_1+1 \leq t \leq n-1$. Since $t_1 \leq s$, $n-t_1+1 \geq n-s+1$ which implies that $s+t > n$ which is a contradiction to the assumption. This proves the claim.

Since $s+t>n$ and $t\le n$, we can add appropriate number of zeros to T_2 so as to get a zero sequence of length n. $\square$

Before going into the further discussions, we shall prove the following theorem.

Theorem 3.2. *Let n and k be positive integers such that $1\le k<(n+2)/3$. Then, the following statments are equivalent.*

(I) Let S be a minimal zero sequence of $\mathbf{Z}_n$ of length $n-k+1$. Then there exists $a\in\mathbf{Z}_n$ such that a appears in S at least $n-2k+2$ times.

(II) Let S be a zero-free sequence in $\mathbf{Z}_n$ of length $n-k$. Then one element of S is repeated at least $n-2k+1$ times.

(III) Let S be a sequence of $\mathbf{Z}_n$ of length $2n-k-1$. Suppose S does not have a zero subsequence of length n. Then there exist $a\neq b\in\mathbf{Z}_n$ such that a and b appear in S at least $n-2k+1$ times.

Remark. The statement (I) was proved by the author in 2001 (cf. [35]). The statement (II) was proved by Bovey, Erdős and Niven in 1975 (cf. [6]). Also, all the three statements are valid for $n-2k\ge 1$. But the equivalence is valid only for the range $1\le k<(n+2)/3$.

Proof. (I) $\Longleftrightarrow$ (II)

Assume that (I) is true. Consider a zero-free sequence $S=\{a_i\}$ in $\mathbf{Z}_n$ of length $n-k$. Let $a_{n-k+1}=-\sum_{i=1}^{n-k}a_i$ and S_1 be the sequence consisting of all the elements a_i together with a_{n-k+1}. Then, S_1 is a minimal sequence of length $n-k+1$ in $\mathbf{Z}_n$. For, if any proper subsequence, say, T of S together with a_{n-k+1} is a zero subsequence of S_1, then the deleted sequence $S\backslash T$ is a zero subsequence of S which is a contradiction. Hence S_1 is a minimal zero sequence of length $n-k+1$ in $\mathbf{Z}_n$. Now, by the statement (I), there exists $a\in\mathbf{Z}_n$ such that a is repeated in S_1 at least $n-2k+2$ times. Thus, the element a is repeated in S at least $n-2k+1$ times.

Assume that (II) is true. Consider a minimal zero sequence $S=\{a_i\}$ in $\mathbf{Z}_n$ of length $n-k+1$. Let S_1 be the sequence obtained from S by deleting the element a_{n-k+1}. Clearly, S_1 is a zero-free sequence in $\mathbf{Z}_n$ of length $n-k$. Therefore, by the statement (II), there exists $a\in\mathbf{Z}_n$ such that a is repeated in S_1 at least $n-2k+1$ times. Now, let S_2 be a zero sequence in $\mathbf{Z}_n$ of length $n-k$, obtained from S by deleting the element $a\neq a_{n-k+1}$ (if $a=a_{n-k+1}$, then nothing to prove). Again by the statement (II), there exists an element $b\in\mathbf{Z}_n$ such that b is repeated $n-2k+1$ times in S_2. If $a\neq b$, then the length of S would be at least $2n-4k+2\le n-2k+1$. This forces that $n\le 2k-1$. That is, $k\ge(n+1)/2$, which is a contradition to the assumption that $k\le(n+2)/3$. Therefore, $a=b$ and hence S has an element $a\in\mathbf{Z}_n$ which is repeated in S at least $n-2k+2$ times.

(II) $\Longleftrightarrow$ (III)

Assume that (II) is true. Let S be a sequence in $\mathbf{Z}_n$ of length $2n-k-1$ satisfying the hypothesis. Suppose a is an element of S which appears maximum number of, say h, times in S. Consider the translated sequence $S-a$. Let S_1 be the subsequence of $S-a$ such that it consists of all non-zero elements of $S-a$.

Claim. There exists zero-free subsequence T of S_1 of length $n-k$.

Assume the contrary. Suppose every subsequence T of S_1 of length $n-k$ has a zero subsequence. Let M be one such zero subsequence of S_1. Choose M such that M has the maximal length. Since every subsequence of length $n-k$ of S_1 has a zero subsequence, it is clear that

$$2n-k-1-h-|M| \leq n-k-1 \Longrightarrow |M| \geq n-h.$$

If $|M| \leq n$, then by adding appropriate number of zeros to M to get a zero subsequence of length n in $S-a$ which in turn produces a zero subsequence of length n in S (this is because we have h zeros out side S_1). This is a contradiction to the assumption. Therefore, $|M| > n$. In this case, by N-C Extension 1, we can get a zero subsequence of length $\leq h$, in M. Therefore, inductively, deleting the zero subsequences $M_1, M_2, \cdots, M_r$ from M so that we can make $n-h \leq |M'| \leq n$ where M' is the sequence obtained from M after deleting those sequences M_i's. This is a contradiction as before. This contradiction implies that there is a zero-free subsequence T of S_1 of length $n-k$. Therefore, by the statement (II), T has an element which is repeated at least $n-2k+1$ times. Since 0 appears in $S-a$ maximum number of times, $h \geq n-2k+1$. Therefore two distinct elements of $\mathbf{Z}_n$ in $S-a$ which appears at least $n-2k+1$ times. Hence S has two distinct elements of $\mathbf{Z}_n$ such that both appears in S at least $n-2k+1$ times.

Assume that (III) is true. Let S be a zero-free sequence of $\mathbf{Z}_n$ of length $n-k$. Let

$$S_1 : S, \underbrace{0, 0, \cdots, 0}_{n-1 \text{ times}}$$

be sequence in $\mathbf{Z}_n$ of length $2n-k-1$. Clearly S_1 does not contain a zero subsequence of length n. Therefore by the statement (III), we know, S_1 consists of two distinct elements of $\mathbf{Z}_n$ such that both appears $n-2k+1$ times. Since 0 appears $n-1$ times, it is clear that S consists of one element of $\mathbf{Z}_n$ which is repeated at least $n-2k+1$ times. $\square$

Remark 1. The statements (II) and (III) are equivalent for all n and k such that $n-2k \geq 1$. Also, in the statement (III), there is a moreover part. That is, we can prove that, in the conclusion of the statement (III), S can consists of at most $k+1$ distinct residue classes modulo n. This is because of the following. In [4], it is proved that if any sequence R in $\mathbf{Z}_n$ of length $2n-m+1$ consists of m distinct residue classes modulo n, then R contain a zero subsequence of length n. Since the length of the given sequence S is $2n-k-1$ and S doesn't have any zero subsequence of length n, it follows that S can contain at most $k+1$ distinct residue classes modulo n.

Theorem 3.3 *Let k be an integer with $1 \leq k \leq \frac{1}{2}(n-[n/2]+1)$. Then, the statement (III) in Theorem 3.2 implies EGZ Theorem.*

Proof. Let S be a sequence in $\mathbf{Z}_n$ of length $2n-1$. Let T be a subsequence of length $2n-1-k$ where $1 \leq k \leq \frac{1}{2}(n-[n/2]+1)$. Either T has a zero subsequence of length n or it doesn't have. If T has such a zero subsequence, then nothing to prove. If T doesn't have any zero subsequence of length n, then by the statement (III) in Theorem 3.2, T consists of two distinct elements of $\mathbf{Z}_n$ each appearing $n-2k+1$ times. Since k lies in $1 \leq k \leq \frac{1}{2}(n-[n/2]+1)$, we get $n-2k+1 \geq [n/2]$. Therefore, T has one element of $\mathbf{Z}_n$ repeating at least $[n/2]$ times. Then by Lemma 3.1, we get the required zero subsequence of length n. □

N-C Extension 2. *Let n and k be positive integers such that $1 \leq k < (n+2)/3$. Let S be a minimal zero sequence of $\mathbf{Z}_n$ of length $n-k+1$. Then there exists $a \in \mathbf{Z}_n$ such that a appears in S at least $n-2k+2$ times.*

Proof. This is nothing but (I) of Theorem 3.2. Since in Theorem 3.2, it has been proved that (I) $\iff$ (II) $\iff$ (III) and by Theorem 3.3, we get the result. □

For the simillar reasons, the following two statements are also true.

N-C Extension 3. *Let n and k be positive integers such that $n-2k \geq 1$. Let S be a zero-free sequence in $\mathbf{Z}_n$ of length $n-k$. Then one element of S is repeated at least $n-2k+1$ times.*

N-C Extension 4. *Let n and k be positive integers such that $n-2k \geq 1$. Let S be a sequence of $\mathbf{Z}_n$ of length $2n-k-1$. Suppose S does not have a zero subsequence of length n. Then there exist $a \neq b \in \mathbf{Z}_n$ such that a and b appear in S at least $n-2k+1$ times.*

The statement (III) in Theorem 3.2 is the generalization of the following results.

Corollary 3.2.1 *Any sequence S in $\mathbf{Z}_n$ of length $2n-2$ having no zero subsequence of length n consists of two distinct elements in $\mathbf{Z}_n$ each appearing exactly $n-1$ times.*

Proof. Put $k=1$ in the statement (III) in Theorem 3.2, we get the result. □

Corollary 3.2.1 was first proved by Yuster and Peterson (cf. [32]) and also proved by Bialostocki and Dierker (cf. [5]).

When $k = 2$ in Theorem 3.2, we get any sequence in $\mathbf{Z}_n$ of length $2n - 3$ which does not have any zero subsequence of length n will have two distinct elements of $\mathbf{Z}_n$ each appearing at least $n - 3$ times. Indeed, a better result was proved by C. Flores and O. Ordaz (cf. [14]) as follows.

A Result of Flores and Ordaz. (cf. [14]) *Suppose S is any sequence in $\mathbf{Z}_n$ of length $2n-3$ such that S has no zero subsequence of length n. Then there exists $a, b \in \mathbf{Z}_n$ such that $\mathbf{Z}_n$ is generated by $b - a$ and a appearing $n - 1$ times in S and one of the following conditions hold;*

(i) b appearing exactly $n - 2$ times.

(ii) b appearing exactly $n-3$ times in S and also, $2b-a$ appearing exactly once in S.

Remark 2 (i) By putting $k = 1$ in the statement (II) of Theorem 3.2, we get, if S is a zero-free sequence in $\mathbf{Z}_n$ of length $n - 1$, then S consists of only one element $a \in \mathbf{Z}_n$ which is appearing $n - 1$ times. Since S is zero-free, it is clear that the order of a has to be n.

(ii) Now, we put $k = 2$ in the statement (II) in Theorem 3.2. If a zero-free sequence S in $\mathbf{Z}_n$ of length $n - 2$, we get less information about the structure of S. Using the following result of Hamidoune, we see in the following Proposition 3.4, in this case, the structure of S.

A Result of Hamidoune. (See for instance, Lemma 2.3 in [14]) *Let S be a zero-free sequence in $\mathbf{Z}_n$ of length at least $n-2$. Also assume that S consists of at most 2 distinct residue modulo n. Then the length of S is at most $n-1$ and will have one of the following form;*

(i) $S : \underbrace{a, a, \cdots, a}_{r \text{ times}}$ *where* $r \leq n - 1$.

(ii) $S : \underbrace{a, a, \cdots, a}_{n-2 \text{ times}}, 2a.$

Using this, we shall prove the following Proposition.

Proposition 3.4 *Let S be a zero-free sequence in $\mathbf{Z}_n$ of length $n-2$. Then S consists of an element $a \in \mathbf{Z}_n$ which is repeated either $n - 2$ times or $n - 3$ times and $2a$ appearing exactly once.*

Proof. By putting $k = 2$ in the statement Theorem 3.2, we get the sequence S has one element $a \in \mathbf{Z}_n$ which is repeated at least $n - 3$ times. If a is repeated $n - 2$ times, then nothing to prove. If a is repeated $n-3$ times exactly, then S can consist only of two distinct residue classes modulo n. Then by above mentioned result of Hamidoune, it follows that the second residue has to be $2a$ and the result follows. $\square$

At this juncture, we should mention a beautiful result of Gao and Geroldinger (cf. [22]) as follows.

A result of Gao and Geroldinger. *Let $n \geq 4$ be a natural number. Let S be a zero-free sequence in $\mathbf{Z}_n$ of length at least $(n+3)/2$. Then, there exists $a \in \mathbf{Z}_n$ of order n and a appears in S at least $n/6 + 13/12$ times.*

Using this result and the above techniques, we can prove the following theorem (we skip the proof here).

Theorem 3.5 *Let S be a sequence in $\mathbf{Z}_n$ of length at least $n-1+(n+3)/2$. Suppose S does not have any zero subsequence of length n. Then there exist $a \neq b \in \mathbf{Z}_n$ such that both appears in S at least $n/6 + 13/12$ times and either a or b is of order n.*

Theorem 3.6 *Let n, k be positive integers such that $n - 2k \geq 1$. Let S be a sequence in $\mathbf{Z}_n$ of length $2n-k-1$ such that at most one element in S appears more than $n-2k$ times. Then there exists a zero subsequence of S of length n.*

Proof. Suppose not, that is S does not have zero subsequence of length n. Then by the statement (III) in Theorem 3.2 we get a contradiction and hence the result. $\square$

We give a short proof of the following known theorem.

Theorem 3.7 *Let S be a sequence in $\mathbf{Z}_n$ of length $2n-1$. The sequence S has exactly one zero subsequence of length n. in $\mathbf{Z}_n$ if and only if there exists a and b in $\mathbf{Z}_n$ such that a appears n times and b appears $n-1$ times in S.*

Proof. The converse is easy. We shall prove the other implication. Let

$$S : a_1, a_2, \cdots, a_{2n-1}$$

be the given sequence. By EGZ theorem, there is a zero subsequence of length n. We let $a_1 + a_2 + \cdots + a_n \equiv 0 \pmod{n}$, if necessary by renaming the indices. Let S_1 be a subsequence of S of length $2n-2$ such that a_1 does not appear in S_1. Clearly by hypothesis, S_1 does not have any zero subsequence of length n. Therefore by Corollary 3.2.1, S_1 consists of two distinct elements a and b of $\mathbf{Z}_n$ each appearing $n-1$ times. Now, let us consider a subsequence S_2 of S such that $a_n = a$ does not appear in S_2. Clearly, S_2 has length $2n-2$ and it cannot have a zero subsequence of length n, by the hypothesis. Therefore, it forces that a_1 has to be a and hence the theorem. $\square$

4. Concluding Remarks

In this section, we shall discuss about the analoguous situation of the non-canonical extensions of EGZ Theorem for the group $\mathbf{Z}_p \oplus \mathbf{Z}_p$. For the detail history and status of the following conjectures, one may refer to [35].

There is no result so far known which is an analogue of N-C Extension 1 for the group $\mathbf{Z}_p \oplus \mathbf{Z}_p$. The analogue of EGZ Theorem for the group $\mathbf{Z}_p \oplus \mathbf{Z}_p$ is the following conjecture of Kemnitz.

Conjecture 4.1 (Kemnitz, 1983, [25]) *Let S be a sequence in $\mathbf{Z}_p \oplus \mathbf{Z}_p$ of length $4p-3$. Then there exists a zero subsequence T of S of length p.*

Conjecture 4.1 is yet to be resolved. For the analogue of Corollary 3.2.1, we have the following conjecture of Gao.

Conjecture 4.2 (W. D. Gao, 2000, [21]) *Let S be sequence in $\mathbf{Z}_p \oplus \mathbf{Z}_p$ of length $4p-4$. Suppose that S does not contain any zero subsequences of length p. Then S consists of four distinct elements a, b, c and d in $\mathbf{Z}_p \oplus \mathbf{Z}_p$ each of them appearing in S exactly $p-1$ times.*

Conjecture 4.2 is also open. The analogue of Remark 2 (i) is the following conjecture of van Emde Boas.

Conjecture 4.3 (van Emde Boas, 1969, [36]) *Let S be a sequence in $\mathbf{Z}_p \oplus \mathbf{Z}_p$ of length $3p-3$. If S does not contain any zero subsequences of length at most p, then S consists of three distinct elements a, b and c of $\mathbf{Z}_p \oplus \mathbf{Z}_p$ each appearing exactly $p-1$ times.*

Of course, Conjecture 4.3 is also open till now. The analogue of the statement (I) in Theorem 3.2 when $k = 1$ is the following conjecture of Gao and Geroldinger.

Conjecture 4.4 (Gao and Geroldinger, 1998, [23]) *If S is a minimal zero sequence in $\mathbf{Z}_p \oplus \mathbf{Z}_p$ of length $2p-1$, then there exists $a \in \mathbf{Z}_p \oplus \mathbf{Z}_p$ such that a is appearing in S at least $p-1$ times.*

As we have seen in the last section, it natural to ask for the inter relationships between these four conjectures.

Conjecture 4.2 implies Conjecture 4.1 was proved by Gao (cf. [21]). Conjecture 4.4 implies Conjecture 4.3 was proved by Gao and Geroldinger (cf. [23]). Conjecture 4.2

implies Conjecture 4.3 was proved by the author (cf. [35]). Also, in [35] it has been proved the following partial implication.

Theorem 4.4 (Thangadurai, 2001, [35]) *Assume Conjecture 4.3. Let S be a sequence of the form*

$$\underbrace{a, a, \cdots, a,}_{s \text{ times}} a_1, a_2, \cdots, a_{4p-4-s}$$

where $a, a_i \in \mathbf{Z}_p \oplus \mathbf{Z}_p$ for all $i = 1, 2, \cdots, 4p-4-s$ and $s > \left[\frac{p-3}{2}\right]$. Suppose S does not contain a zero subsequence of length p. Then S consists of four distinct elements a, b, c and d in $\mathbf{Z}_p \oplus \mathbf{Z}_p$ each of them appearing exactly $p-1$ times. In other words, S satisfies Conjecture 4.2.

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