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MULTI-PARAMETRIC EXTENSIONS OF NEWMAN'S PHENOMENON

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Abstract

Consider the sequence $(3k)_{k\geq 0}$ written in base two representation and reduce the sum of digits $s_2(3k)$ mod 2. A well-known result of Newman [10] says that the resulting sequence shows an overplus of 0's with respect to 1's. It is also known [3] that, asymptotically speaking, $s_2(7k)_{k\geq 0}$ is more often 0 than 1 or 2. We investigate similar phenomena for the sequence $(7k+i)_{k\geq 0}$ with $0 < i \leq 6$ as well as give a two-parametric family of arithmetic progressions where overplus phenomena can be observed. This paper sharpens and extends results obtained by Drmota and Skałba [3], continuing work presented by Drmota and the author in [4].

1. Introduction

Consider the sequence of numbers $(3k)_{0\leq k\leq K}$ written in the digital base $g = 2$,

$$\mathbf{0},\ \mathbf{11},\ \mathbf{110},\ \mathbf{1001},\ \mathbf{1100},\ \mathbf{1111},\ \mathbf{10010},\ 10101, \ldots$$

Newman [10] showed that, up to any $K \geq 0$, the numbers which contain an even number of 1's (written in boldface) prevail over those which have an odd number. Recently, Drmota and the author [4] proved that for the numbers $(3k+1)_{k\geq 0}$ there holds the opposite. Consider

$$1,\ 100,\ 111,\ \mathbf{1010},\ 1101,\ 10000,\ 10011,\ 10110, \ldots,$$

then there is an overplus of members with odd sum of digits over those with even sum. Again this overplus holds true up to any $K \geq 0$. This sharpens an earlier asymptotical result due to Dumont [5]. A different phenomenon can be observed for the sequence $(3k+2)_{k\geq 0}$,

$$10,\ \mathbf{101},\ 1000,\ 1011,\ 1110,\ \mathbf{10001},\ \mathbf{10100},\ \mathbf{10111}, \ldots$$

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In the latter case, the 'even'-instances and the 'odd'-instances (corresponding to taking the sum of the digits mod 2) are balanced for an infinite number of choices of K (see [4]).

More generally, consider arithmetic progressions of type $(qk+i)_{k\geq 0}$ and the sum of g-ary digits of its members mod a. It is a conjecture that for

$$q \equiv q(a,g) = \kappa(g^a - 1)/(g-1), \qquad \kappa \geq 1, \tag{1}$$

similar overplus phenomena occur. Note, that the original sequence $(3k+i)_{k\geq 0}$ fits (1) with $a = g = 2$ and $\kappa = 1$. Recently, Newman-like phenomena have been verified in the special case $g = 2$ and $2 \leq a \leq 6$ (see [3]), as well as in the case $a = 2$ and $g \geq 3$ with g odd (see [4]). The main purpose of this paper is to investigate the case (1) with $(a,g,q) \equiv (3,g,\kappa(g^2+g+1))$ where $g \geq 2$. Motivated by the above-mentioned example we are also interested in the following questions:

Q1: Regarding the modulus a, do we obtain all possible overplus phenomena by varying i in $(qk+i)_{k\geq 0}$?

Q2: Is there an overplus phenomenon for each $0 \leq i < q$ (up to balance)?

Theorem 3.1 answers Q1 and Q2 in the case $(a,g,q) \equiv (3,2,7)$; so, in particular, Conjecture 1 of [4] is disproved. Next we consider the setting for arbitrary bases g. Again, Newman-like phenomena for $i = 0$ and $i = 1$ can be shown, thus, Theorem 3.2 settles two thirds of Conjecture 2 of [4].

2. Preliminaries

To start with, we recall some notation and basic facts from [3] and [4]. Let $a, g \geq 2$ and $t_k^{(a,g)} = \omega_a^{s_g(k)}$ for $k \geq 0$ be the generalized Thue-Morse sequence, where $\omega_a = \exp(2\pi i/a)$ and $s_g(k)$ denotes the sum of the digits in the g-ary expansion of k. Furthermore, let

$$S_{q,i}^{(a,g)}(n) = \sum_{\substack{0 \leq j < n, \\ j \equiv i \pmod q}} t_j^{(a,g)} \tag{2}$$

denote the summation function in arithmetic progressions. The quantity

$$A_{q,i;m}^{(a,g)}(n) = |\{0 \leq j < n : \quad j \equiv i \pmod q, \quad s_g(j) \equiv m \pmod a\}|$$

counts, how often ω_a^m is realized by going through the members of the generalized Thue-Morse sequence with indices $(qk+i)_{k\geq 0}$. Newman-like phenomena (for short: NLP), generalizing the setting of Section 1, can be described by comparing these numbers $A_{q,i;m}^{(a,g)}(n)$.

Definition 2.1: The triple (a, g, q) is called to *satisfy an* (i, M)*-NLP* if

$$A_{q,i;M}^{(a,g)}(n) > \max_{\substack{0 \le m < a \\ m \ne M}} A_{q,i;m}^{(a,g)}(n) \tag{3}$$

for all but finitely many $n > i$. In particular, if inequality (3) holds for all $n > i$ then (a, g, q) is called to *satisfy a strong* (i, M)*-NLP*.

For example, the triple $(2, 2, 3)$ satisfies a strong $(0, 0)$-NLP [10] as well as a strong $(1, 1)$-NLP [4]. On the other hand, there is no $(2, M)$-NLP for this triple. In general, it is rather difficult to exhibit strong NLP's (see [1, 2, 7, 8]). In any case, establishing (i, M)-NLP's makes use of two fancy properties of the discrete function $S_{q,i}^{(a,g)}(n)$.

We introduce some notation. Set $\zeta_q = \exp(2\pi \mathrm{i}/q)$ and $s = \mathrm{ord}_q(g)$, that is $g^s \equiv 1 \pmod q$. Furthermore, let

$$\eta_l^\varepsilon(k) = \frac{1 - \omega_a^\varepsilon \zeta_q^{\varepsilon l g^k}}{1 - \omega_a \zeta_q^{l g^k}} \qquad \text{and} \qquad \lambda_l(k) = \prod_{j=0}^{k-1} \frac{1 - \omega_a^g \zeta_q^{l g^{j+1}}}{1 - \omega_a \zeta_q^{l g^j}}, \tag{4}$$

where $1 \le l \le q-1$. For any $1 \le \varepsilon \le g-1$ let $l^{(1)}(\varepsilon), l^{(2)}(\varepsilon), \ldots$ be an ordering of the indices l such that

$$|\eta_{l^{(j)}(\varepsilon)}^\varepsilon(0)\lambda_{l^{(j)}(\varepsilon)}(s)| \ge |\eta_{l^{(j+1)}(\varepsilon)}^\varepsilon(0)\lambda_{l^{(j+1)}(\varepsilon)}(s)| \qquad \text{for} \quad 1 \le j \le q-2.$$

Further put

$$L_{\max}(\varepsilon) = \{l^{(1)}(\varepsilon), l^{(2)}(\varepsilon), \ldots, l^{(m)}(\varepsilon)\} \tag{5}$$

where $m = a/\gcd(a, g-1)$. Note that this is only a formal definition for $L_{\max}(\varepsilon)$, which we are going to use later for the special instance $a = 3$, $g \equiv 0$ or $2 \pmod 3$ and $q = \kappa(g^2+g+1)$, in particular $m = 3$. The motivation of setting $m = a/\gcd(a, g-1)$ in the general case comes from our conjecture expressed in the concluding remarks of the paper.

The *recursive structure* of $S_{q,i}^{(a,g)}(n)$ is described by

$$S_{q,i}^{(a,g)}(\varepsilon g^k + n') = S_{q,i}^{(a,g)}(\varepsilon g^k) + \omega_a^\varepsilon S_{q,i-\varepsilon g^k}^{(a,g)}(n') \quad \text{for} \quad n' < \varepsilon g^k. \tag{6}$$

Furthermore, $S_{q,i}^{(a,g)}(n)$ can be *evaluated at multiples of g-powers*. Let $k = k_1 s + k_2$ and $\bar{S}_{q,i}^{(a,g)}(n)$ be the asymptotically leading term of $S_{q,i}^{(a,g)}(n)$. Then

$$S_{q,i}^{(a,g)}(\varepsilon g^k) = \frac{1}{q} \sum_{l=0}^{q-1} \zeta_q^{-li} \left(\eta_l^\varepsilon(0)\lambda_l(s)\right)^{k_1} \eta_l^\varepsilon(k_2)\lambda_l(k_2). \tag{7}$$

The asymptotic leading term is given by

$$\bar{S}_{q,i}^{(a,g)}(\varepsilon g^k) = \frac{1}{q} \sum_{l \in L_{\max}(\varepsilon)} \zeta_q^{-li} \eta_l^\varepsilon(k)\lambda_l(k). \tag{8}$$

The main ingredient of the proofs relies on repeated combination of (6) and (7) resp. (8). Let $n = \varepsilon_1 g^k + \varepsilon_2 g^{k-1} + \ldots$, where $\varepsilon_i \in \{0, \ldots, g-1\}$, $\varepsilon_1 \neq 0$. We first may split off $\varepsilon_1 g^k$ according to the recursive rule (6) and calculate $S_{q,i}^{(a,g)}(\varepsilon_1 g^k)$ with help of (7) (resp. (8)). In the proof of Theorem 3.1 we will use the exact representation (7) and derive *strong* NLP's. For the general case, which involves more parameters (Theorem 3.2), we will use the leading term according to (8). In both cases we estimate the non-expanded tail by means of geometric series.

Apart from the technical part, the main difficulty in proving multi-parameter families of NLP's consists in finding closed-form expressions for the set $L_{\max}(\varepsilon)$, which will be the subject matter of Lemma 5.1.

3. Main Theorems

Theorem 3.1: The triple $(3, 2, 7)$ satisfies a strong $(0, 0)$-NLP, a strong $(1, 1)$-NLP and a strong $(3, 2)$-NLP. There does not hold any $(4, M)$-, $(5, M)$- and $(6, M)$-NLP.

Although numerical simulation suggests that $(3, 2, 7)$ also satisfies a $(2, 1)$-NLP (see Figure 1), we were not able to prove it in a managable amount of work. The expansions emerging from (6) are huge, we could not find any uniform bound (independent of the digits ε_i) for the length of the exact expanded part leading to success. This, of course, is directly related to the particularly large deviations of the line $i = 2$ from the direction indicated by ω_3 (compare with the line $i = 1$).

Theorem 3.2: Let $(g - 1, 3) = (\kappa, 3) = 1$. Then the triple $(3, g, \kappa(g^2 + g + 1)$ satisfies a $(0, 0)$-NLP and a $(1, 1)$-NLP.

Note that the condition $(g - 1, 3) = 1$ is necessary since otherwise there are no NLP's (compare with Theorem 1.4 in [4]).

4. Proof of Theorem 3.1

The plan of the proof is as follows. We first notice, that the condition (3) for an (i, M)-NLP directly translates into a condition on the argument of $S_{7,i}^{(3,2)}(n)$ (see (9)). This allows to derive the statement of Theorem 3.1 for $i \in \{4, 5, 6\}$ by suitably choosing n. For the cases $i \in \{0, 1, 3\}$ we use (6) and (7) to split $S_{7,i}^{(3,2)}(n)$ into a 'head' series and a 'tail' series. The tail of the expansion is estimated in terms of a geometric series. The contribution of this geometric series, however, is negligible in respect to the numerically explicit head of $S_{7,i}^{(3,2)}(n)$, thus finally giving the rest of the statement in Theorem 3.1.

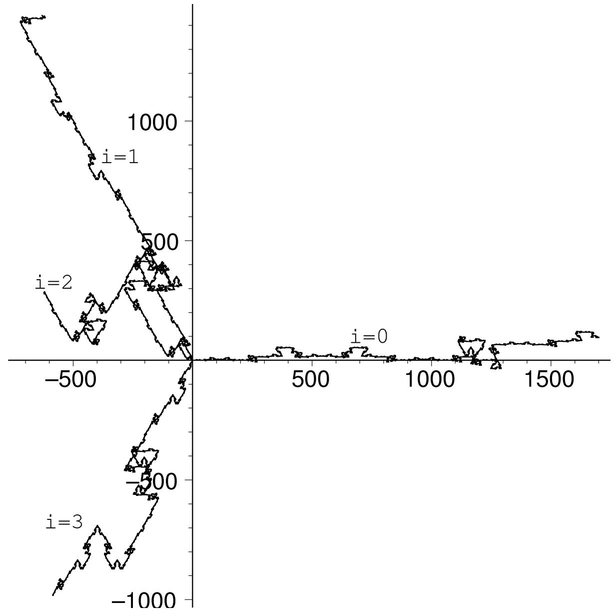

Figure 1: $S_{7,i}^{(3,2)}(n)$ for $i = 0, 1, 2, 3$

The proof basically follows the lines of Drmota and Skalba [3], proof of Proposition 6. However, two modifications have to be noted. First, the value 3/7 given on the second line of p.635 is not correct. As a consequence, the head series of $S_{7,i}^{(3,2)}(n)$ has to assemble more terms in respect to [3]. In the case $i = 0$, for instance, we have to consider all $n = 2^k + \varepsilon_1 2^{k-1} + \varepsilon_2 2^{k-2} + \varepsilon_3 2^{k-3} + n'$ with $k = 3k_1 + k_2$; each quadruple $(\varepsilon_1, \varepsilon_2, \varepsilon_3, k_2)$ corresponds to a separate computational case.

Secondly and more important, our approach is explicit to obtain *strong* NLP's. To begin with, set

$$\alpha_0 = -1,$$
$$\alpha_1 = \frac{1}{2}(5 - \sqrt{21}) = 0.20871\ldots,$$
$$\alpha_2 = \frac{1}{2}(5 + \sqrt{21}) = 4.79128\ldots,$$

which are the eigenvalues obtained in [3], p.634. Moreover, for convenience put

$$\kappa_1(i, k_2) = \zeta_7^{-i}\lambda_1(k_2) + \zeta_7^{-2i}\lambda_2(k_2) + \zeta_7^{-4i}\lambda_4(k_2),$$
$$\kappa_2(i, k_2) = \zeta_7^{-3i}\lambda_3(k_2) + \zeta_7^{-5i}\lambda_5(k_2) + \zeta_7^{-6i}\lambda_6(k_2).$$

Then by (7),

$$\begin{aligned} S_{7,i}^{(3,2)}(2^k) &= \frac{1}{7}\sum_{l=0}^{6}\zeta_7^{-il}\prod_{j=0}^{k-1}\left(1+\omega_3\zeta_7^{l2^j}\right) \\ &= \frac{1}{7}\left(\alpha_0^{k_1}\lambda_0(k_2)+\alpha_1^{k_1}\kappa_1(i,k_2)+\alpha_2^{k_1}\kappa_2(i,k_2)\right) \\ &= \frac{1}{7}\,\alpha_2^{k_1}\kappa_2(i,k_2)+\delta(k_1,k_2,i) \end{aligned}$$

with $|\delta(k_1,k_2,i)| \leq |\delta(4,0,0)| < 0.143671 =: C_1$ if $k_1 > 3$. Now, since

$$S_{7,i}^{(3,2)}(n) = A_{7,i;0}^{(3,2)}(n) + A_{7,i;1}^{(3,2)}(n)\,\omega_3 + A_{7,i;2}^{(3,2)}(n)\omega_3^2$$

and $a_0 + a_1\omega_3 + a_2\omega_3^2 = 0$ if and only if $a_0 = a_1 = a_2$ (for $a_0, a_1, a_2 \in \mathbb{Z}$), condition (3) can plainly be checked by considering $\arg S_{7,i}^{(3,2)}(n)$. Therefore, the triple $(3, 2, 7)$ satisfies an (i, M)-NLP if and only if

$$\arg S_{7,i}^{(3,2)}(n) \in \left((2M-1)\frac{\pi}{3}, (2M+1)\frac{\pi}{3}\right). \tag{9}$$

Since

$$\pi/2 < \arg\kappa_2(4,0) < 2\pi/3 \quad \text{and} \quad 0 < \arg\kappa_2(4,2) < \pi/15,$$

as well as

$$17\pi/10 < \arg\kappa_2(6,1) < 7\pi/4 \quad \text{and} \quad \pi < \arg\kappa_2(6,2) < 4\pi/3,$$

and (9), no NLP exists in the cases $i = 4$ and $i = 6$. If $i = 5$ then for $n = 2^{3k_1+2}$ we have $\arg S_{7,i}^{(3,2)}(n) = -\pi/3$. In other words,

$$A_{7,5;0}^{(3,2)}(2^{3k_1+2}) = A_{7,5;2}^{(3,2)}(2^{3k_1+2}). \tag{10}$$

Actually, these 'balancing' points n aren't rare at all. Relation (10) also remains true for numbers $n = 2^{3k_1+1} + 2^{3k_1+0} + 2^{3k_1-1}$ (apply (6) twice). Consequently, for all n, whose binary expansion is realized by the automaton given in Figure 2, relation (10) holds true, too. The automaton constructs numbers n in the following way. To begin with, a 'head' is constructed which is made up of alternating $1\ldots1$- and $0\ldots0$-blocks, each having length 0 mod 3. Finally, an obligatory 'tail' consisting of 00 is appended.

For the cases $i \in \{0, 1, 3\}$ we first estimate the tail in the expansion of $S_{7,i}^{(3,2)}(n)$ with help of (6). Let

$$C_2 = \frac{1}{7}\max_{\substack{0\leq i<7,\\ 0\leq j<3}}|\kappa_2(i,j)| = \frac{1}{14}\sqrt{106+22\sqrt{21}} = 1.0267045\ldots \tag{11}$$

and $\beta = \alpha_2^{1/3}$. Then, as long as $k - \mu \geq 1$,

$$\begin{aligned} \left|\sum_{\nu=0}^{k-\mu}\eta_\nu S_{7,i_\nu}^{(3,2)}(2^\nu)\right| &\leq C_2\sum_{\nu=0}^{k-\mu}\alpha_2^{\lfloor\nu/3\rfloor} + C_1(k-\mu+1) \\ &\leq C_2\sum_{\nu=0}^{k-\mu}\alpha_2^{\nu/3} \leq \frac{C_2\beta^{k-\mu}}{1-\beta^{-1}} \leq 2.524973\,\beta^{k-\mu}. \end{aligned}$$

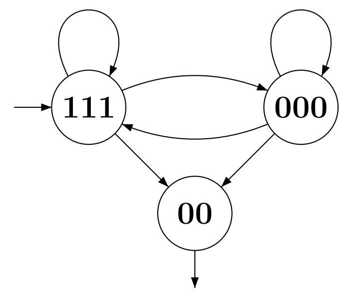

Figure 2: Balancing points for $S_{7,5}^{(3,2)}(n)$

In the sequel we only give indications how to prove that $(3,2,7)$ satisfies a strong $(0,0)$-NLP. The exact calculations are rather involved and we omit the details here[2]. Obviously, it suffices to show that for all $m \equiv m(k) = 2^k + \varepsilon_1 2^{k-1} + \varepsilon_2 2^{k-2} + \varepsilon_3 2^{k-3}$ with $\varepsilon_i \in \{0,1\}$ and

$$\gamma(m) = \arg S_{7,0}^{(3,2)}(m), \qquad c(m) = |S_{7,0}^{(3,2)}(m)| \tag{12}$$

we have

$$\gamma(m) \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right), \tag{13}$$

$$c(m) > 2.524973 \cdot \beta^{k-4} / \sin(\pi/3 - |\gamma(m)|). \tag{14}$$

In total there are $2^3 \cdot 3 = 24$ cases to deal with, we only give an illustration of the algorithm for $k = 3k_1$ and $m = 1 \cdot 2^k + 0 \cdot 2^{k-1} + 1 \cdot 2^{k-2} + 1 \cdot 2^{k-3}$. By (6),

$$\begin{aligned} S_{7,0}^{(3,2)}(m) &= S_{7,0}^{(3,2)}(2^{3k_1}) + \omega_3 S_{7,6}^{(3,2)}(2^{3(k_1-1)+1}) + \omega_3^2 S_{7,4}^{(3,2)}(2^{3(k_1-1)}) \\ &= \frac{1}{7}\, \alpha_2^{k_1} \left(\kappa_2(0,0) + \frac{\omega_3}{\alpha_2} \kappa_2(6,1) + \frac{\omega_3^2}{\alpha_2} \kappa_2(4,0) \right) \\ &\qquad + \delta(k_1,0,0) + \delta(k_1-1,1,6) + \delta(k_1-1,0,4) \\ &= C_3 \alpha_2^{k_1} + \delta(k_1,0,0) + \delta(k_1-1,1,6) + \delta(k_1-1,0,4), \end{aligned}$$

where $C_3 = 0.4850919\ldots + \mathrm{i} \cdot 0.03263216\ldots$ This gives

$$\begin{aligned} |\gamma(m)| &\leq \arcsin \frac{3C_1}{\alpha_2^{k_1}|C_2|} + \arg C_3 \quad \text{and} \\ c(m) &> \alpha_2^{k_1}|C_3| - 3C_1. \end{aligned}$$

It is immediate to verify that (13) and (14) are true for $k_1 \geq 2$. The cases $k_1 = 0, 1$ can be checked by hand.

In the same spirit one proves the statement of Theorem 3.1 for $i = 1$ and $i = 3$, but in contrast, expansions of six leading digits are needed (96 cases).

[2]A simple MAPLE-worksheet has been implemented to do the calculations automatically.

5. Proof of Theorem 3.2

We first may give an outline of the proof. To start with, the set $L_{\max}$ is determined by thoroughly inspecting the factors of $\lambda_l(k)$ (Lemma 5.1). These factors are cyclic since $lg^3 \equiv l \pmod{q}$ for $l \in L_{\max}$. The calculation of $\eta_l^{\varepsilon}(k)$, $\lambda_l(k)$ and $\bar{S}_{q,i}^{(3,g)}(\varepsilon g^k)$ according to (4) and (8) is straightforward (Lemma 5.2–5.4). We finally conclude by expanding $\bar{S}_{q,i}^{(3,g)}(\varepsilon g^k)$ into head and tail series, thus getting the asymptotic result of Theorem 3.2.

In the case $a = 2$ (see [4]) we have $L_{\max}(\varepsilon) = \{l^{(1)}(\varepsilon), l^{(2)}(\varepsilon)\}$ with $l^{(1)}(\varepsilon) = l_1 = \kappa g/2$ and $l^{(2)}(\varepsilon) = l_2 = \kappa(g/2+1)$, where

$$|\eta_{l_1}^{\varepsilon}(0)\lambda_{l_1}(s)| = |\eta_{l_2}^{\varepsilon}(0)\lambda_{l_2}(s)|$$

for all $1 \le \varepsilon \le g-1$.

If $a = 3$ we are also able to explicitly calculate $l^{(1)}(\varepsilon) = l_1$, $l^{(2)}(\varepsilon) = l_2$, $l^{(3)}(\varepsilon) = l_3$, which are not depending on ε. If $g \equiv 2 \pmod 3$ set

$$\begin{aligned} l_1 &= \frac{\kappa}{3}(2g^2+3g+1),\\ l_2 &= \frac{\kappa}{3}(2g^2+3g+4),\\ l_3 &= \frac{\kappa}{3}(2g^2+1). \end{aligned}$$

Note that $g = 2$ implies $(l_3, l_1, l_2) = (3\kappa, 5\kappa, 6\kappa)$, which has been obtained in Lemma 10 of [3]. On the other hand, if $g \equiv 0 \pmod 3$ we denote

$$\begin{aligned} l_1 &= \frac{\kappa}{3}(2g^2+g),\\ l_2 &= \frac{\kappa}{3}(2g^2+g+3),\\ l_3 &= \frac{\kappa}{3}(2g^2+4g+3). \end{aligned}$$

The crucial point in the proof of Theorem 3.2 consists in showing that $L_{\max}(\varepsilon) = \{l_1, l_2, l_3\}$. To begin with, observe that the indices (l_1, l_2, l_3) are permuted to (l_3, l_1, l_2) by multiplication with g and reduction mod q. The following Lemma extends Lemma 5.2 of [4].

Lemma 5.1: Let $z = \exp(\mathrm{i}\varphi)$ and $\delta_\varepsilon(\varphi) = |(1-\omega_3^{\varepsilon} z^{\varepsilon})/(1-\omega_3 z)|$. Further set

$$f_j(\varphi) = \left|\frac{1-\omega_3^{g} z^{g^j}}{1-\omega_3 z^{g^{j-1}}}\right| \quad \text{and} \quad \hat{\varphi}_j = 2\pi l_j/q \quad \text{for} \quad j = 1,2,3.$$

Moreover let $f(\varphi) = f_1(\varphi)f_2(\varphi)f_3(\varphi)$. Then for all $1 \le \varepsilon \le g-1$, all $1 \le l \le q-1$ with $l \ne l_1, l_2, l_3$ and $\hat{\varphi} = 2\pi l/q$ it holds

$$\delta_\varepsilon(\hat{\varphi}_1)f(\hat{\varphi}_1) \ge \delta_\varepsilon(\hat{\varphi}_2)f(\hat{\varphi}_2) \ge \delta_\varepsilon(\hat{\varphi}_3)f(\hat{\varphi}_3) > \delta_\varepsilon(\hat{\varphi})f(\hat{\varphi}).$$

Furthermore,

$$L_{\max}(\varepsilon) = \{l_1, l_2, l_3\}.$$

Proof. The proof exactly follows the lines of the proof of Lemma 10 in [3], we therefore only give the modifications induced by the introduction of the general parameter g. First put $J = [\hat{\varphi}_1, \hat{\varphi}_2]$ and for the rest of the work

$$\varphi_1 = -\frac{\pi(2g+1)}{3(g^2+g+1)}, \quad \varphi_2 = \frac{\pi(g-1)}{3(g^2+g+1)}, \quad \varphi_3 = \frac{\pi(g+2)}{3(g^2+g+1)}.$$

Easy calculations yield

$$\hat{\varphi}_1 = \begin{cases} 4\pi/3 - 2\varphi_3, & \text{if } g \equiv 0 \pmod 3, \\ 4\pi/3 + 2\varphi_2, & \text{if } g \equiv 2 \pmod 3) \end{cases}$$

and

$$\hat{\varphi}_2 = \begin{cases} 4\pi/3 - 2\varphi_2, & \text{if } g \equiv 0 \pmod 3, \\ 4\pi/3 + 2\varphi_3, & \text{if } g \equiv 2 \pmod 3. \end{cases}$$

Let $J_1 = [\xi_1, \xi_2]$ where ξ_1 resp. ξ_2 denotes the smallest resp. largest zero of $f(\varphi)$ in J. Then there are $g-2$ inner local maxima of $f(\varphi)$ on J and

$$\max_{\varphi \in J_1} f(\varphi) < \frac{g}{\sin(\pi/g)} \cdot \max_{\varphi \in J_1} f_1(\varphi).$$

Observe that $f_1(\varphi)$ is increasing on J_1 in case of $g \equiv 0 \pmod 3$ and decreasing on J_1 if $g \equiv 2 \pmod 3$. Hence

$$\max_{\varphi \in J_1} f_1(\varphi) < \max_{\varphi \in J} \left| \frac{\sin(\pi g/3 + \varphi g/2)}{\sin(\pi/3 + \varphi/2)} \right| = \frac{3\sqrt{3}}{2\pi^2} g + \frac{3\sqrt{3}}{\pi} - 1 + \mathcal{O}(g^{-1})$$

and $\max_{\varphi \in J_1} f(\varphi) = \frac{3\sqrt{3}}{2\pi^3} g^3 + \mathcal{O}(g^2)$. If $g \equiv 0 \pmod 3$ we have

$$\begin{aligned} f(\hat{\varphi}_1) =& \frac{81\sqrt{3}}{16\pi^3} g^3 + \left(\frac{27}{4\pi^2} + \frac{243\sqrt{3}}{32\pi^3} \right) g^2 + \left(\frac{1701\sqrt{3}}{64\pi^3} - \frac{3\sqrt{3}}{16\pi} + \frac{27}{4\pi^2} \right) g \\ &+ \frac{1539\sqrt{3}}{128\pi^3} - \frac{3\sqrt{3}}{32\pi} + \frac{27}{\pi^2} - \frac{3}{4} + \mathcal{O}(g^{-1}). \end{aligned}$$

On the other hand, if $g \equiv 2 \pmod 3$ the same growth can be observed,

$$\begin{aligned} f(\hat{\varphi}_1) =& \frac{81\sqrt{3}}{16\pi^3} g^3 + \frac{243\sqrt{3}}{32\pi^3} g^2 + \left(\frac{1701\sqrt{3}}{64\pi^3} - \frac{27\sqrt{3}}{16\pi} \right) g \\ &+ \frac{1539\sqrt{3}}{128\pi^3} - \frac{27\sqrt{3}}{32\pi} + \frac{1}{2} + \mathcal{O}(g^{-1}). \end{aligned}$$

Since $\frac{3\sqrt{3}}{2\pi^3} < \frac{81\sqrt{3}}{16\pi^3}$ and by estimating the tail-terms we therefore see that $\max_{\varphi \in J} f(\varphi) = f(\hat{\varphi}_1)$ for all $g \geq 3$. It is also simple to see, that the factor $\delta_\varepsilon(\varphi)$ does not change this behaviour. The inequality chain now follows directly from monotonicity considerations of $\delta_\varepsilon(\varphi)$ on $[0, 2\pi]$.

In order to conclude with $L_{\max}(\varepsilon) = \{l_1, l_2, l_3\}$, we adopt a set-splitting argument similar to [3, Lemma10]. Partition $\{0, 1, \ldots, s-1\}$ into the sets M_1, $M_2 = M_1 + 1$, $M_3 = M_1 + 2$, M_4, $M_5 = M_4 + 1$, $M_6 = M_4 - 1$, M_7 where M_1 consists of all j with $\arg \zeta_q^{l2^j} \in (\hat{\varphi}_1, \hat{\varphi}_2)$ and M_4 of all those $j \notin M_2$ with $\arg \zeta_q^{l2^j} \in (\hat{\varphi}_3, \hat{\varphi}_1)$. Then

$$\begin{aligned} |\eta_l^\varepsilon(0)\lambda_l(s)| &= \delta_\varepsilon(2\pi l/q) \cdot \prod_{j=0}^{s-1} f_{j+1}(2\pi l/q) \\ &= \delta_\varepsilon(2\pi l/q) \cdot \prod_{j \in M_1 \cup M_2 \cup M_3} \delta_1(2\pi l/q) f_{j+1}(2\pi l/q) \cdot \\ &\qquad \prod_{j \in M_4 \cup M_5 \cup M_6} \delta_1(2\pi l/q) f_{j+1}(2\pi l/q) \cdot \\ &\qquad \prod_{j \in M_7} \delta_1(2\pi l/q) f_{j+1}(2\pi l/q), \end{aligned}$$

where we used the fact that $\delta_1(\varphi) = 1$. Now, replace exactly one of the $\delta_1(2\pi l/q)$ terms by the leading $\delta_\varepsilon(2\pi l/q)$. Then the first part of Lemma 5.1 implies

$$\left|\eta_{l_1}^\varepsilon(0)\lambda_{l_1}(s)\right| \geq \left|\eta_{l_2}^\varepsilon(0)\lambda_{l_2}(s)\right| \geq \left|\eta_{l_3}^\varepsilon(0)\lambda_{l_3}(s)\right| \geq |\eta_l^\varepsilon(0)\lambda_l(s)|$$

for all $l \neq l_1, l_2, l_3$ and all $1 \leq \varepsilon \leq g-1$. In other words, $L_{\max}(\varepsilon) = \{l_1, l_2, l_3\}$.

□

It is now a direct calculation from (4) to obtain expressions for $\lambda_l(k)$ with $l \in L_{\max}$. We restrict our proof to the case $g \equiv 2 \pmod 3$, the line of proof similarly applies when $g \equiv 0 \pmod 3$. For sake of shortness put

$$\Lambda = \frac{1}{2}\left(-(\sqrt{3}\cot\varphi_1 + 1)(\sqrt{3}\cot\varphi_2 + 1)(\sqrt{3}\cot\varphi_3 + 1)\right)^{1/3}$$

and

$$\varrho_{j,m} = \frac{\sin(2\pi/3 + \pi l_j/q)}{\sin(\pi/3 + \pi l_m/q)}, \quad \text{for} \quad j, m \in \{1, 2, 3\}.$$

Remember that all quantities are only depending on g.

Lemma 5.2:

1. Let $k \equiv 0 \pmod 3$. Then
$$\lambda_{l_1}(k) = \lambda_{l_2}(k) = \lambda_{l_3}(k) = \Lambda^k.$$
2. Let $k \equiv 1 \pmod 3$. Then
$$\begin{aligned} \lambda_{l_1}(k) &= \omega_3^{1/2}\Lambda^{k-1}\rho_{3,1}\exp\left(\mathrm{i}\pi(g+1)/(g^2+g+1)\right), \\ \lambda_{l_2}(k) &= \omega_3^{1/2}\Lambda^{k-1}\rho_{1,2}\exp\left(-\mathrm{i}\pi g/(g^2+g+1)\right), \\ \lambda_{l_3}(k) &= \omega_3^{1/2}\Lambda^{k-1}\rho_{2,3}\exp\left(-\mathrm{i}\pi/(g^2+g+1)\right). \end{aligned}$$

3. Let $k \equiv 2 \pmod 3$. Then

$$\begin{aligned}
\lambda_{l_1}(k) &= \frac{\omega_3}{2}\Lambda^{k-2}\rho_{2,1}\left(\sqrt{3}\cot\varphi_3+1\right)\exp\left(\mathrm{i}\pi g/(g^2+g+1)\right),\\
\lambda_{l_2}(k) &= \frac{\omega_3}{2}\Lambda^{k-2}\rho_{3,2}\left(\sqrt{3}\cot\varphi_1+1\right)\exp\left(\mathrm{i}\pi/(g^2+g+1)\right),\\
\lambda_{l_3}(k) &= \frac{\omega_3}{2}\Lambda^{k-2}\rho_{1,3}\left(\sqrt{3}\cot\varphi_2+1\right)\exp\left(-\mathrm{i}\pi(g+1)/(g^2+g+1)\right).
\end{aligned}$$

Moreover,

$$\eta_l^\varepsilon(k) = \left(\omega_3\zeta_q^{lg^k}\right)^{\varepsilon/2-1/2} U_{\varepsilon-1}\left(\cos\left(\frac{1}{2}\arg\omega_3\zeta_q^{lg^k}\right)\right),$$

where $U_{\varepsilon-1}(\cos\varphi) = \sin\varepsilon\varphi/\sin\varphi$ denotes the ε-th Chebyshev polynomial of the second kind.

Again a straightforward calculation gives

Lemma 5.3:

1. If $l = l_1$, $k \equiv 0 \pmod 3$ or $l = l_2$, $k \equiv 1 \pmod 3$ or $l = l_3$, $k \equiv 2 \pmod 3$, then
$$\eta_l^\varepsilon(k) = \exp(\mathrm{i}\pi(\varepsilon-1)\varphi_1)\cdot U_{\varepsilon-1}(\cos\varphi_1).$$
2. If $l = l_1$, $k \equiv 1 \pmod 3$ or $l = l_2$, $k \equiv 2 \pmod 3$ or $l = l_3$, $k \equiv 0 \pmod 3$, then
$$\eta_l^\varepsilon(k) = \exp(\mathrm{i}\pi(\varepsilon-1)\varphi_3)\cdot U_{\varepsilon-1}(\cos\varphi_3).$$
3. If $l = l_1$, $k \equiv 2 \pmod 3$ or $l = l_2$, $k \equiv 0 \pmod 3$ or $l = l_3$, $k \equiv 1 \pmod 3$, then
$$\eta_l^\varepsilon(k) = \exp(\mathrm{i}\pi(\varepsilon-1)\varphi_2)\cdot U_{\varepsilon-1}(\cos\varphi_2).$$

We now use (8) in order to calculate the leading term of the expansion.

Lemma 5.4: If $k \equiv 0 \pmod 3$ then

$$\begin{aligned}
\bar{S}_{q,i}^{(3,g)}(\varepsilon g^k) = \frac{\Lambda^k}{q}\,\omega_3^i \Bigg(&\exp\left(\mathrm{i}\varphi_1(\varepsilon-2i-1)\right)\frac{\sin\varepsilon\varphi_1}{\sin\varphi_1}\\
&+\exp\left(\mathrm{i}\varphi_2(\varepsilon-2i-1)\right)\frac{\sin\varepsilon\varphi_2}{\sin\varphi_2}\\
&+\exp\left(\mathrm{i}\varphi_3(\varepsilon-2i-1)\right)\frac{\sin\varepsilon\varphi_3}{\sin\varphi_3}\Bigg).
\end{aligned}$$

If $k \equiv 1 \pmod 3$ then

$$\begin{aligned}
\bar{S}_{q,i}^{(3,g)}(\varepsilon g^k) = \frac{\Lambda^{k-1}}{q}\,\omega_3^{i+1/2}\Bigg(&\exp\left(\mathrm{i}(\varepsilon\varphi_3-(2i+1)\varphi_1)\right)\frac{\sin\varepsilon\varphi_3}{\sin\varphi_3}\cdot\frac{\sin(\varphi_3+\pi/3)}{\sin\varphi_1}\\
&+\exp\left(\mathrm{i}(\varepsilon\varphi_1-(2i+1)\varphi_2)\right)\frac{\sin\varepsilon\varphi_1}{\sin\varphi_1}\cdot\frac{\sin(\varphi_1+\pi/3)}{\sin\varphi_2}\\
&+\exp\left(\mathrm{i}(\varepsilon\varphi_2-(2i+1)\varphi_3)\right)\frac{\sin\varepsilon\varphi_2}{\sin\varphi_2}\cdot\frac{\sin(\varphi_2+\pi/3)}{\sin\varphi_3}\Bigg).
\end{aligned}$$

If $k \equiv 2 \pmod 3$ then

$$\begin{aligned}
&\bar{S}_{q,i}^{(3,g)}(\varepsilon g^k) = \\
&= \frac{\Lambda^{k-2}}{2q}\,\omega_3^{i+1}\Bigg(\exp\left(\mathrm{i}(\varepsilon\varphi_2-(2i+1)\varphi_1)\right)\frac{\sin\varepsilon\varphi_2}{\sin\varphi_2}\cdot\frac{\sin(\varphi_2+\pi/3)}{\sin\varphi_1}(\sqrt{3}\cot\varphi_3+1) \\
&\qquad + \exp\left(\mathrm{i}(\varepsilon\varphi_3-(2i+1)\varphi_2)\right)\frac{\sin\varepsilon\varphi_3}{\sin\varphi_3}\cdot\frac{\sin(\varphi_3+\pi/3)}{\sin\varphi_2}(\sqrt{3}\cot\varphi_1+1) \\
&\qquad + \exp\left(\mathrm{i}(\varepsilon\varphi_1-(2i+1)\varphi_3)\right)\frac{\sin\varepsilon\varphi_1}{\sin\varphi_1}\cdot\frac{\sin(\varphi_1+\pi/3)}{\sin\varphi_3}(\sqrt{3}\cot\varphi_2+1)\Bigg).
\end{aligned}$$

Similar to (11) we note the immediate

Corollary 5.5:

$$|\sum_{\nu=0}^{k-\mu}\bar{S}_{q,i}^{(3,g)}(\varepsilon_\nu g^\nu)| < \frac{7g}{2q}\cdot\frac{\Lambda}{\Lambda-1}\,\Lambda^{k-\mu}.$$

Proof. Since

$$\Lambda = \frac{3\sqrt{3}}{8\pi}\sqrt[3]{4}(2g+1)+\mathcal{O}(g^{-1})$$

and the absolute values of the right hand sides in Lemma 5.4 are all monotone increasing in ε, we have $|q\bar{S}_{q,i}^{(3,g)}(\varepsilon g^k)/\Lambda^k| < 7g/2$. This upper bound is obtained when we take absolute values of the summands in Lemma 5.4 and expand into a series in g. □

For reasons of simplicity we now restrict us to one special case in order to see which sort of calculations have to be carried out in general. Let $i=0$, $k \equiv 0 \pmod 3$, $g \equiv 2 \pmod 3$ and $m = \varepsilon_1 g^k + \varepsilon_2 g^{k-1}$. We have

$$\bar{S}_{q,0}^{(3,g)}(m) = \bar{S}_{q,0}^{(3,g)}(\varepsilon_1 g^k) + \omega_3^{\varepsilon_1}\bar{S}_{q,-\varepsilon_1 g^k}^{(3,g)}(\varepsilon_2 g^{k-1}) = \frac{\Lambda^k}{q}(P_1+P_2),$$

where

$$\begin{aligned}
P_1 &= \exp\left(\mathrm{i}\varphi_1(\varepsilon_1-1)\right)\frac{\sin\varepsilon_1\varphi_1}{\sin\varphi_1} + \exp\left(\mathrm{i}\varphi_2(\varepsilon_1-1)\right)\frac{\sin\varepsilon_1\varphi_2}{\sin\varphi_2} + \exp\left(\mathrm{i}\varphi_3(\varepsilon_1-1)\right)\frac{\sin\varepsilon_1\varphi_3}{\sin\varphi_3}, \\
P_2 &= \frac{\omega_3^{\varepsilon_1+1}}{2\Lambda^3}\Bigg(\exp\Big(\mathrm{i}(\varepsilon_2\varphi_2+(2\varepsilon_1 g^k-1)\varphi_1)\Big)\frac{\sin\varepsilon_2\varphi_2}{\sin\varphi_2}\cdot\frac{\sin(\varphi_2+\pi/3)}{\sin\varphi_1}(\sqrt{3}\cot\varphi_3+1) \\
&\quad + \exp\Big(\mathrm{i}(\varepsilon_2\varphi_3+(2\varepsilon_1 g^k-1)\varphi_2)\Big)\frac{\sin\varepsilon_2\varphi_3}{\sin\varphi_3}\cdot\frac{\sin(\varphi_3+\pi/3)}{\sin\varphi_2}(\sqrt{3}\cot\varphi_1+1) \\
&\quad + \exp\Big(\mathrm{i}(\varepsilon_2\varphi_1+(2\varepsilon_1 g^k-1)\varphi_3)\Big)\frac{\sin\varepsilon_2\varphi_1}{\sin\varphi_1}\cdot\frac{\sin(\varphi_1+\pi/3)}{\sin\varphi_3}(\sqrt{3}\cot\varphi_2+1)\Bigg).
\end{aligned}$$

We are first concerned with bounding the absolute value of P_2. As before, we take absolute values of the summands and observe that the maximum is attained for $\varepsilon_2 = g-1$. This gives

$$|P_2| \le \frac{4\sqrt{3}\pi}{9g}$$

and shows that the behavior of $\bar{S}^{(3,g)}_{q,0}(m)$ is determined by P_1, because of

$$|P_1| = \frac{15\sqrt{3}}{4\pi}g + \frac{15\sqrt{3}}{8\pi} - \frac{3}{4} + \mathcal{O}(g^{-1}).$$

For the geometric tail we have by Corollary 5.5,

$$\frac{q}{\Lambda^k}\left|\sum_{\nu=0}^{k-2} \bar{S}^{(3,g)}_{q,i}(\varepsilon_\nu g^\nu)\right| = \frac{7\sqrt{3}\pi}{9}\ \sqrt[3]{2} + \mathcal{O}(g^{-1}).$$

Instead of $\arg \bar{S}^{(3,g)}_{q,0}(m)$ it suffices to consider $\arg P_1$. Again, it is an easy observation, that $\arg P_1 \in (0, \pi/3)$ and $\arg P_1$ is maximal for $\varepsilon_1 = g-1$. Thus, for sufficiently large g we have $\arg \bar{S}^{(3,g)}_{q,i}(n) \in (-\pi/3, \pi/3)$ for all $n = \varepsilon_1 g^{3k_1} + \varepsilon_2 g^{3k_1-1} + n'$. The exact calculations for small g involve case distinctions on the digits ε_1 and ε_2 which can be done with a computer.

The same approach succeeds in case of $i = 1$. Analogously, we write

$$\bar{S}^{(3,g)}_{q,1}(m) = \bar{S}^{(3,g)}_{q,1}(\varepsilon_1 g^k) + \omega_3^{\varepsilon_1}\bar{S}^{(3,g)}_{q,1-\varepsilon_1 g^k}(\varepsilon_2 g^{k-1}) = \frac{\Lambda^k}{q}(T_1 + T_2),$$

where $m = \varepsilon_1 g^k + \varepsilon_2 g^{k-1}$ and estimate T_1 and T_2 with help of Lemma 5.4. This finishes the proof of Theorem 3.2.

6. Final remarks

The factor $\delta_\varepsilon(\varphi)$ in Lemma 5.1 does affect the behavior of the $L_{\max}(\varepsilon)$ if $a \geq 4$. As a consequence, $\bar{S}^{(a,g)}_{q,i}(n)$ cannot be 'uniformly' expanded, as there is no closed-form expression for $L_{\max}(\varepsilon)$ for all $1 \leq \varepsilon \leq g-1$; much more work has to be done in order to handle all possible digital expansions. However, in the case $\varepsilon = 1$ and $g \equiv r \pmod a$ we conjecture that $L_{\max}(1) = \{\hat{l}g^j \bmod q\}$, where

$$\hat{l} = \frac{\kappa}{a}\sum_{j=0}^{a-1} b^{(a)}_{r,j} g^j \quad \text{with} \quad b^{(a)}_{r,j} = g(a-1-j) + j \pmod a.$$

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