Int. J. Open Problems Compt. Math., Vol. 1, No. 1, June 2008

A Prey-Predator Model with an Alternative Food for the Predator, Harvesting of Both the Species and with A Gestation Period for Interaction

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Abstract

In the classical Lotka - Volterra Prey - Predator model, there is no protection for Prey from the Predator and Predator sustains on the Prey alone. When the Prey population falls below a certain level, the predator would migrate to another region in search of food and return only when the Prey-Population rises to the required level. A population model with time delay was proposed by Kapur, (c.f. Deley differential and integro-differential equations in population dynamics, J.Math.Phy.Sci., 14, 107-29, 1980) and this model motivated the present investigation. In the present investigation we studied a prey - predator model incorporating i) the predator is provided with an alternative food in addition to the prey, ii) both the prey and the predator are harvested proportional to their population sizes and iii) a gestation period for interaction. The model is characterized by a couple of first order integro-differential equations. All the four equilibrium points of the model are identified and stability criteria are discussed. Some threshold results are illustrated.

Keys words: *Equilibrium points, Normal Study State, Normalized Kernels, Prey, Predator, Stability, Threshold Diagrams, and Threshold Results.,*

1 Introduction

Some of the prey-predator models were discussed by Michale Olinnck [4], May [5], Varma [6] Colinvaux [7], Freedman [8], Narayan [9]. A population model with time delay was proposed by Kapur [1]. Volterra formulated a distributed time delay model for prey - predator ecological models. Kapur, [2] discussed the solution in the closed form for that model. Inspired from that, we discussed a more general model by taking an alterative food for the predator and harvesting of both the species. The model is characterized by a couple of first order ordinary delay-differential equations. All the four equilibrium points of the model are identified and stability criteria are discussed. In consonance with the principle of competitive exclusion (Gause [3]) some threshold results are illustrated.

2 Basic Equations

The model equations for a two species Prey-Predator system is given by the following system of first order delay - differential equations employing the following notation:
N_1 and N_2 are the populations of the prey and predator with the natural growth rates a_1 and a_2 respectively, α_{11} is rate of decrease of the prey due to insufficient food, α_{12} is rate of decrease of the prey due to inhibition by the predator, α_{21} is rate of increase of the predator due to successful attacks on the prey, α_{22} is rate of decrease of the predator due to insufficient food other than the prey, k_1 and k_2 are rate of decrease of the prey and predator due to harvesting, $k_3(t-s)$, $k_4(t-s)$ are weight factors to give the influence at time t of N_1, N_2 of time $s(\le t)$ i.e. $k_3(t-s)$, $k_4(t-s)$ are rate of changes of N_1, N_2 after a time interval $(t-s)$.

Further both the variables N_2 and N_2 are non-negative and the model parameters a_1, a_2, α_{11}, α_{12}, α_{21}, α_{22}, k_1, k_2 are assumed to be non-negative constants.

$$\frac{dN_1}{dt} = N_1\left\{a_1(1-k_1) - \alpha_{11}N_1 - \alpha_{12}\int_{-\infty}^{T} k_4(t-s)N_2(s)ds\right\} \tag{2.1}$$

$$\frac{dN_2}{dt} = N_2\left\{a_2(1-k_2) - \alpha_{22}N_2 + \alpha_{21}\int_{-\infty}^{T} k_3(t-s)N_1(s)ds\right\} \tag{2.2}$$

put $t-s=z$, i.e. $s=t-z$ (2.3)

$\therefore k_3(z), k_4(z) \ge 0$, are time delayed so that $\int_0^{\infty} k_3(z)dz = \int_0^{\infty} k_4(z)dz = 1$ (2.4)

are normalized kernels.
Now we rewrite the basic equations as

$$\frac{dN_1}{dt} = N_1\left\{a_1(1-k_1) - \alpha_{11}N_1 - \alpha_{12}\int_0^{\infty} k_4(z)N_2(t-z)dz\right\} \tag{2.5}$$

$$\frac{dN_2}{dt} = N_2 \left\{ a_2(1-k_2) - \alpha_{22}N_2 + \alpha_{21}\int_0^\infty k_3(z)N_1(t-z)dz \right\} \tag{2.6}$$

3 Equilibrium Points

The system under investigation has four equilibrium states:

1. The fully washed out state with the equilibrium points $\overline{N}_1 = 0; \overline{N}_2 = 0$. (3.1)

2. The state in which predator survives and the preys are washed out. The equilibrium point is

$$\overline{N}_1 = 0 \,; \overline{N}_2 = \frac{a_2(1-k_2)}{\alpha_{22}} \tag{3.2}$$

3. The state in which, only the prey survives and the predators are washed out. The equilibrium point is

$$\overline{N}_1 = \frac{a_1(1-k_1)}{\alpha_{11}} \,; \overline{N}_2 = 0 \tag{3.3}$$

4. The co-existence state (**normal study state**). The equilibrium point is

$$\overline{N}_1 = \frac{a_1(1-k_1)\alpha_{22} - a_2(1-k_2)\alpha_{12}}{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}}; \quad \overline{N}_2 = \frac{a_2(1-k_2)\alpha_{11} + a_1(1-k_1)\alpha_{21}}{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}} \tag{3.4}$$

since $\int_0^\infty k_3(z)dz = \int_0^\infty k_4(z)dz = 1$ and this state can exist only when

$$1 < \frac{a_1(1-k_1)\alpha_{11}}{a_2(1-k_2)\alpha_{12}} \tag{3.5}$$

4 The Stability of the Equilibrium States

Let $N = (N_1, N_2)^T = \overline{N} + U$ (4.1)

where U = $(u_1, u_2)^T$ is a small perturbation over the equilibrium state $\overline{N} = (\overline{N}_1, \overline{N}_2)^T$. The basic equations (2.5), (2.6) are quasi-linearized to obtain the equations for the perturbed state. (By trying the solutions $u_1 = c_1 e^{\lambda t}$ *and* $u_2 = c_2 e^{\lambda t}$ for the equations (2.5) and (2.6))

$$\frac{dU}{dt} = A[U] \tag{4.2}$$

where

$$A = \begin{bmatrix} -\alpha_{11}\overline{N}_1 & -\alpha_{12}\overline{N}_1 \int_0^\infty k_4(z)e^{-\lambda z}dz \\ \alpha_{21}\overline{N}_2 \int_0^\infty k_3(z)e^{-\lambda z}dz & -\alpha_{22}\overline{N}_2 \end{bmatrix} \quad (4.3)$$

The characteristic equation for the system is $det\left[A - \lambda I\right] = 0$. (4.4)

The equilibrium state is stable only when the roots of the equation (4.4) are negative in case they are real or have negative real parts in case they are complex.

4.1 Stability of the equilibrium state I

The trajectories for both washed out state are

$$u_1 = u_{10}\ e^{a_1(1-k_1)\mathrm{t}} \quad (4.5)$$

$$u_2 = u_{20}\ e^{a_2(1-k_2)\mathrm{t}} \quad (4.6)$$

The solution curves are illustrated in figures 1 and 2:

Case 1: In this case the predator dominates the prey in natural growth as well as in its initial population strength. i.e. $u_{10} < u_{20}$ and $a_1(1-k_1) < a_2(1-k_2)$ as shown in Fig.1

Case 2: The predator dominates the prey in natural growth rate but its initial strength is less than that of prey. i.e. $u_{10} > u_{20}$ and $a_2(1-k_2) > a_1(1-k_1)$ as illustrated in Fig.2. In this case, the prey out numbers the predator till the time-instant $t = t^* = \dfrac{ln\ \{u_{10}\ /\ u_{20}\}}{\left[a_2(1-k_2)\text{-}a_1(1-k_1)\right]}$ after that the predator out number the prey. (At $t = t^*$, populations of both the species are same, and from (4.5)&(4.6) $u_1 = u_2$)

4.2 Stability of the equilibrium state II

The trajectories for only prey washed out state are:

$u_1 = u_{10}e^{\lambda_1 t}$ and

$$u_2 = \frac{u_{10}a_2(1\text{-}k_2)\alpha_{21}\ k_3^*(\lambda)}{\{\lambda_1 + a_2(1-k_2)\}\alpha_{22}}e^{\lambda_1 t} + \{u_{20} - \frac{u_{10}a_2(1\text{-}k_2)\alpha_{21}\ e^{\lambda_1 t}k_3^*(\lambda)}{\{\lambda_1 + a_2(1-k_2)\}\alpha_{22}}\}e^{-a_2(1-k_2)t} \quad (4.7)$$

where $k_3^*(\lambda)$ is Laplace transformation of $k_3(z)$ and $\int_0^\infty k_3(z)dz = \int_0^\infty k_4(z)dz = 1$

and $\lambda_1 = a_1(1-k_1)\text{-}\dfrac{\alpha_{12}a_2(1-k_2)}{\alpha_{22}}$ (4.8)

The solution curves are illustrated in Fig. 3 & 4.

Case 1: Initially the prey dominates the predator and it continues throughout its growth $u_{10} > u_{20}$ and $a_1(1-k_1) - \frac{\alpha_{12} a_2 (1-k_2)}{\alpha_{22}} > a_2(1-k_2)$ as illustrated in Fig. 3.

Case 2: Initially the predator dominates the prey i.e. $u_{10} < u_{20}$ and $a_1(1-k_1) - \frac{\alpha_{12} a_2 (1-k_2)}{\alpha_{22}} > a_2(1-k_2)$. In this case, the predators out number the prey till the time-instant

$$t = t^* = \frac{1}{\lambda_2 + a_2(1-k_2)} ln \left[\frac{u_{20} - \frac{a_2(1-k_2)\alpha_{21} u_{10} k_3^*(\lambda_2)}{\alpha_{22}\{\lambda_2 + a_2(1-k_2)\}}}{u_{10} - \frac{a_2(1-k_2)\alpha_{21} u_{10} k_3^*(\lambda_2)}{\alpha_{22}\{\lambda_2 + a_2(1-k_2)\}}} \right] \tag{4.9}$$

after that, the prey out number the predator. This is illustrated in Fig. 4

4.3 Stability of the equilibrium state III

The trajectories for only the predator washed out state are:

$$u_1 = -\frac{u_{20} a_1(1-k_1)\alpha_{12}\ k_4^*(\lambda_2)}{\{\lambda_2 + a_1(1-k_1)\}\alpha_{11}} e^{\lambda_2 t} + \{u_{10} + \frac{u_{20} a_1(1-k_1)\alpha_{12}\ e^{\lambda_2 t} k_4^*(\lambda_2)}{\{\lambda_2 + a_1(1-k_1)\}\alpha_{11}}\} e^{-a_1(1-k_1)t} \tag{4.10}$$

$$u_2 = u_{20} e^{\lambda_2 t}, \tag{4.11}$$

where $k_4^*(\lambda)$ is Laplace transformation of $k_4(z)$ and $\lambda_2 = a_2(1-k_2) + \frac{a_1(1-k_1)\alpha_{21}}{\alpha_{11}}$

Case 1: The initial strength of the prey is greater than that of the predator. i. e. $u_{10} > u_{20}$

Initially the prey out number the predator and this continues up to the time instant,

$$t = t^* = \frac{1}{\lambda_2 + a_2(1-k_2)} \left[\frac{u_{10} + \frac{u_{20} a_1(1\text{-}k_1)\alpha_{12}\ k_4^*(\lambda_2)}{\{\lambda_2 + a_1(1-k_1)\}\alpha_{11}}}{u_{20} + \frac{u_{20} a_1(1\text{-}k_1)\alpha_{12}\ k_4^*(\lambda_2)}{\{\lambda_2 + a_1(1-k_1)\}\alpha_{11}}} \right], \tag{4.12}$$

after which the predator out numbers the prey. And also the predator species is noted to be going away from the equilibrium point while the prey-species would become extinct at the instant (t*) of time given by the positive root of the equation

$$e^{\lambda_2 t} + e^{-a_2(1-k_2)t} = \frac{u_{10}\{a_2(1-k_2)\alpha_{11} + a_1(1-k_1)[\alpha_{11} + \alpha_{21}]\}}{u_{20} a_1(1-k_1)\alpha_{12}} \tag{4.13}$$

This is illustrated in Fig. 5

Case 2: The predator dominates the prey in its initial strength. i.e. $u_{10} < u_{20}$

In this case the predator species is noted to be going away from the equilibrium point while the prey-species would become extinct at the instant (t*) of time given by the positive root of the equation

$$e^{\lambda_2 t} + e^{-a_2(1-k_2)t} = \frac{u_{10}\{a_2(1-k_2)\alpha_{11} + a_1(1-k_1)[\alpha_{11}+\alpha_{21}]\}}{u_{20}a_1(1-k_1)\alpha_{12}}$$

As such the state is **unstable**. This is illustrated in Fig. 6

4.4 Stability of the normal study state

The trajectories for normal study state are:

$$u_1 = \left[\frac{u_{10}(\lambda_1+\alpha_{22}\overline{N}_2)\text{-}u_{20}\alpha_{12}\overline{N}_1}{\lambda_1-\lambda_2}\right] e^{\lambda_1 t} + \left[\frac{u_{10}(\lambda_2+\alpha_{22}\overline{N}_2)\text{-}u_{20}\alpha_{12}\overline{N}_1}{\lambda_2-\lambda_1}\right] e^{\lambda_2 t} \quad (4.14)$$

$$u_2 = \left[\frac{u_{20}(\lambda_1+\alpha_{11}\overline{N}_1)\text{-}u_{10}\alpha_{21}\overline{N}_2}{\lambda_1-\lambda_2}\right] e^{\lambda_1 t} + \left[\frac{u_{20}(\lambda_2+\alpha_{11}\overline{N}_1)\text{-}u_{10}\alpha_{21}\overline{N}_2}{\lambda_2-\lambda_1}\right] e^{\lambda_2 t} \quad (4.15)$$

where λ_1 *and* λ_2 are roots of the characteristic equation.

Case 1: Initially the prey dominates the predator and it continues through out its growth i.e. $u_{10} < u_{20}$ In this case the predator always out numbers the prey. It is evident that both the species converging asymptotic to the equilibrium point. Hence this state is **stable.** This is illustrated in Fig. 7.

Case 2: The prey dominates the predator in natural growth rate but its initial strength is less than that of predator. i.e. $u_{10} > u_{20}$ .Initially the prey out number the predator and this continues till the time-instant

$$t = t^* = \frac{1}{\lambda_2+\lambda_1}\, ln\left[\frac{(b_3-a_5)u_{10}+(a_3+b_1)u_{20}}{(b_2-a_6)u_{10}+\ (a_4+b_1)u_{20}}\right] \quad (4.16)$$

where , $a_3 = \lambda_1 + \alpha_{11}\overline{N}_1$; $a_4 = \lambda_2 + \alpha_{11}\overline{N}_1$; $a_5 = \lambda_1 + \alpha_{22}\overline{N}_2$; $a_6 = \lambda_2 + \alpha_{22}\overline{N}_2$;

$b_1 = \lambda_{12}\overline{N}_1$; $b_2 = \lambda_{21}\overline{N}_2$ (4.17)

after which the predator out number the prey. The solution curves are illustrated in Fig. 8.

When $(\alpha_{11}\overline{N}_1 - \alpha_{22}\overline{N}_2)^2 < \quad 4\alpha_{12}\alpha_{21}(1-k)^2\overline{N}_1\overline{N}_2 k_3^*(\lambda)k_4^*(\lambda)$ the roots are complex with negative real part. Hence the equilibrium state is stable. This is illustrated in Fig. 9.

5 Threshold Results

Employing the principle of competitive exclusion (Gauss [3]), the following threshold results are established.

a. When, $\frac{a_1(1-k_1)}{\alpha_{12}} > \frac{a_2(1-k_2)}{\alpha_{22}}$ and $\frac{a_2(1-k_2)}{\alpha_{21}} > \frac{a_1(1-k_1)}{\alpha_{11}}$ (5.1)

Both the species co-exist as shown in Fig. 10

b. When, $\frac{a_1(1-k_1)}{\alpha_{12}} > \frac{a_2(1-k_2)}{\alpha_{22}}$ and $\frac{a_2(1-k_2)}{\alpha_{21}} < \frac{a_1(1-k_1)}{\alpha_{11}}$ (5.2)

Only prey species survives as illustrated in Fig. 11

c. When, $\frac{a_1(1-k_1)}{\alpha_{12}} < \frac{a_2(1-k_2)}{\alpha_{22}}$ and $\frac{a_2(1-k_2)}{\alpha_{21}} > \frac{a_1(1-k_1)}{\alpha_{11}}$ (5.3)

Only predator species survives as illustrated in Fig. 12.

6. Trajectories

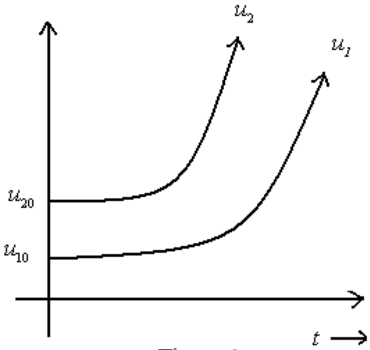

Figure.1

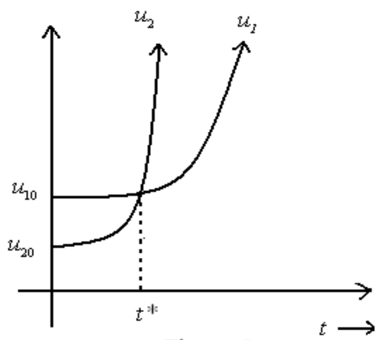

Figure. 2

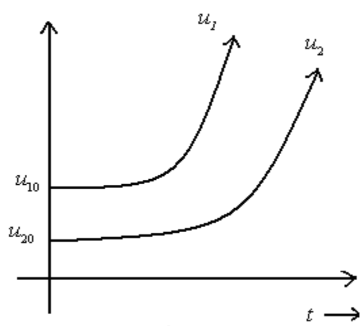

Figure. 3

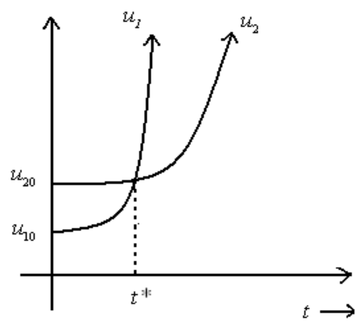

Figure. 4

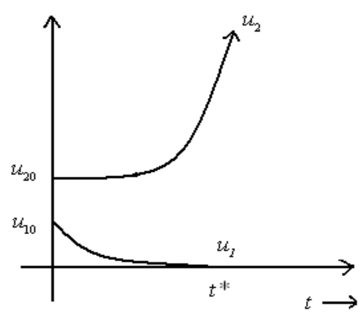

Figure. 5

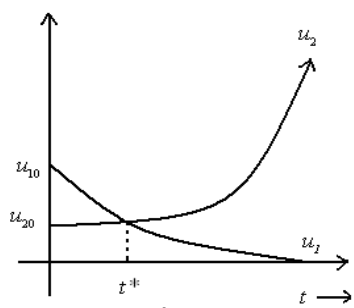

Figure. 6

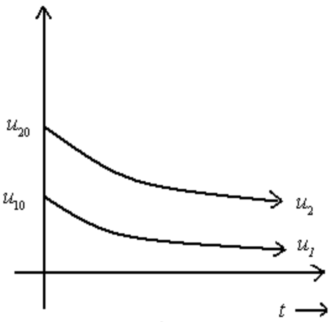

Figure. 7

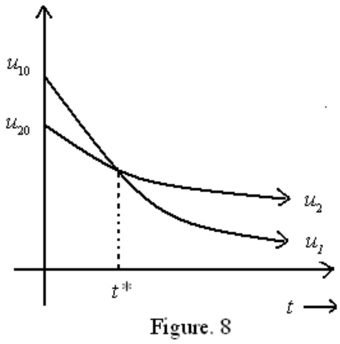

Figure. 8

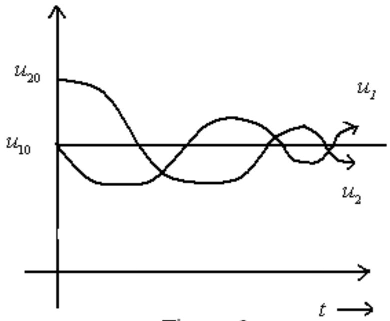

Figure. 9

7 Threshold Diagrams

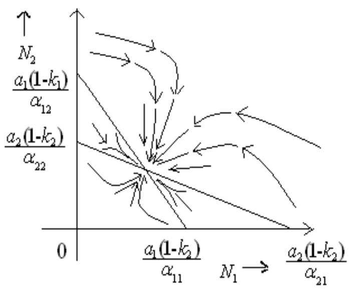

Figure. 10

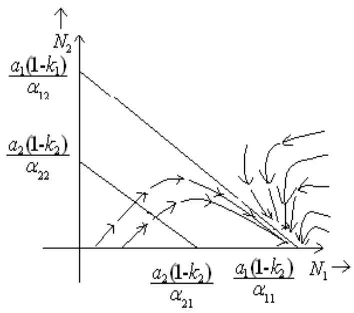

Figure. 11

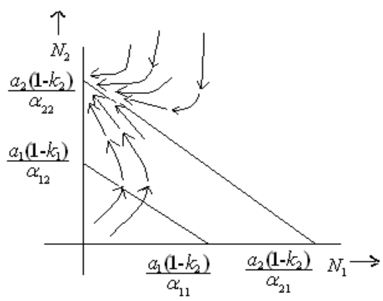

Figure. 12

8 Future Works

In the present paper it is investigated that a Prey-Predator model with harvesting is proportional to the population sizes of the species with gestation period for interaction. There is a scope to study the model with constant harvesting of both species, or harvesting of any of the species. Further cover can be taken for the Prey to protect it from the attacks of the Predator. One can construct Lypunov's function to study the global stability of the model.

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