

A Note On A Recursive Formula Of The Arf-Kervaire Invariant

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Abstract

A recursive formula has been discussed, which involves the Arf invariant, the special value of the Alexander polynomial and the Jones polynomial, and the Minkowski unit of a knot. Also, we reformulate J. Levine's result on the Kervaire invariant of a homotopy sphere as a recursive formula, which suggests that the smooth structure of spheres would be distinguished from number theoretical reasons.

1 Arf invariant and Legendre symbol

Let p and q be different odd primes such that $p, q \equiv 1 \pmod{4}$. Then the mod 2 linking number, $\text{lk}_2(p, q) \in \mathbb{Z}_2$, of p and q is defined by the equality

$$\left(\frac{q}{p}\right) = (-1)^{\text{lk}_2(p, q)}, \quad (1)$$

where the Legendre symbol (q/p) is defined to be either $+1$ or -1 according as q is or is not a quadratic residue modulo p . Note that the symmetry of the mod 2 linking number, $\text{lk}_2(p, q) = \text{lk}_2(q, p)$, is nothing but the Gauss reciprocity law. However, the equality (1) is originally not a definition but a theorem. In fact, $\text{lk}_2(p, q)$ is defined as the image of the Frobenius automorphism $\sigma_p \in \pi_1(X_q)$ over p in $\text{Gal}(Y_q/X_q) \cong \mathbb{Z}_2$, where $Y_q \rightarrow X_q := \text{Spec}(\mathbb{Z}) - \{q\}$ is the double Etale covering. The mod 2 linking number is not always defined for any primes. For example, $\text{lk}_2(p, 2)$ is ill-defined.

As is indicated in [8] by Morishita, there are many analogies between knots and primes by nature. The purpose of this note is to observe various knot invariants from number theoretical viewpoint and further discuss its higher dimensional analogue.

In [9] Murakami proved a recursive formula (see [4, Theorem 10.6] also) as follows: $V_L(i) = (-\sqrt{2})^{r-1}(-1)^{\text{Arf}(L)}$ for a proper r -component link L , where $\text{Arf}(L)$ denotes the Arf invariant of a link L , $V_L(t)$ is the Jones polynomial of L (see [4] for the definitions) and $i = \sqrt{-1}$. In particular, if L is a knot, then we have

$$V_L(i) = (-1)^{\text{Arf}(L)}. \quad (2)$$

Given a link, the special value of the polynomial invariant often play an important role in studying links. It is well-known (see [4, Chapter 10] again) that for a knot K ,

$$\text{Arf}(K) = \frac{(\Delta_K(-1))^2 - 1}{8} \pmod{2}, \quad (3)$$

where $\Delta_K(t)$ is the Alexander polynomial of a knot K and note that $\Delta_K(-1)$ is odd for any knot K . Moreover, it is easy to see that $V_K(-1) = \Delta_K(-1)$ by checking the skein relations, so we also have

$$\text{Arf}(K) = \frac{(V_K(-1))^2 - 1}{8} \pmod{2}. \quad (4)$$

On one hand in number theory, the 2-nd reciprocity law due to Gauss tells us

$$\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}. \quad (5)$$

Thus we can reformulate (3) and (4) by applying (5) as a recursive formula:

$$\left(\frac{2}{\Delta_K(-1)}\right) = \left(\frac{2}{V_K(-1)}\right) = (-1)^{\text{Arf}(K)}. \quad (6)$$

In addition, Murasugi defined the Minkowski unit $C_p(K)$ of a link for a prime p (see [10]), and he proved that

$$C_2(K) = (-1)^{\text{Arf}(K)}. \quad (7)$$

By (2) and (7) we immediately have $V_K(i) = C_2(K)$. Hence by combining these formulas on knot invariants we have

$$\left(\frac{2}{\Delta_K(-1)}\right) = \left(\frac{2}{V_K(-1)}\right) = V_K(i) = C_2(K) = (-1)^{\text{Arf}(K)}. \quad (8)$$

Remark 1.1. For a knot K , $\Delta_K(1) = \pm 1$ and $\Delta_K(-1)$ possibly takes any odd integer since the Alexander polynomial is symmetric in $\mathbb{Z}[t, t^{-1}]$. The Legendre symbol is defined only for primes but it is extended as the Jacobi symbol (q/p) with same notation, which is defined for any odd p . Note that the 2-nd reciprocity formula with the Jacobi symbol holds in the same formula as (5). Thus it should be regarded that (8) is formulated by using the Jacobi symbol.

2 Kervaire invariant and Legendre symbol

We summarize the well-known facts written in Milnor's book [6] in what follows. Let $f : \mathbb{C}^{n+1}, 0 \rightarrow \mathbb{C}, 0$ be a polynomial function germ with an isolated singular point only at the origin. Then, it is well-known that $K = f^{-1}(0) \cap S_\varepsilon$, the algebraic link, is a $(2n - 1)$ -dimensional closed manifold, where S_ε is the small $(2n + 1)$ -sphere with radius $\varepsilon > 0$. Then we define a map $\phi : S_\varepsilon - K \rightarrow S^1$, $\phi(x) = \frac{f(x)}{|f(x)|}$, which is a locally trivial fibration due to Milnor and the fiber $F_\theta := \phi^{-1}(e^{i\theta})$ is called the Milnor fiber, where note that $F_\theta \simeq S^n \vee \cdots \vee S^n$.

When $n = 1$, K is a link, so $\pi_1(K)$ is not trivial. When $n = 2$, $\pi_1(K) \neq 1$ by Mumford. When $n \geq 3$, K is simply connected by Milnor.

Let us consider the pair (S_ε, K) of manifolds. K is a codimension 2 submanifold of the sphere. In this situation, one can define the Alexander polynomial of K , $\Delta_K(t)$, in the sphere by a characteristic homeomorphism. Milnor proved that K is a homology $(2n - 1)$ -sphere if and only if $\Delta_K(1) = \pm 1$. So, we have when $n \geq 3$, K is homeomorphic to S^{2n-1} if and only if $\Delta_K(1) = \pm 1$ by applying the generalized Poincaré conjecture proven by Smale.

Thus we have a natural question: "Is K diffeomorphic to the sphere?" This has been already solved. When n is even, it is determined by the signature $\sigma(F_0)$ of the intersection form $H_n(F_0) \otimes H_n(F_0) \rightarrow \mathbb{Z}$. For example, when $n = 4$, a polynomial defined by

$$f(z_0, z_1, z_2, z_3, z_4) = z_0^2 + z_1^2 + z_2^2 + z_3^3 + z_4^5$$

gives $\sigma(F_0) = 8$, and hence we can conclude that K is not diffeomorphic to S^7 since the value $\lambda(K) := \sigma(W)/8 \in \mathbb{Z}_2$ such that $\partial W = K$ does not depend on the choices of W and $\lambda(K_1) = \lambda(K_2)$ if K_1 is diffeomorphic to K_2 . Note that $\lambda(S^7) = \sigma(D^8) = 0$.

Next, when n is odd (this is the case in which we are interested), the Kervaire invariant, $c(K) \in \mathbb{Z}_2$, of K determines the smooth structure as follows ([5]): $c(K) = 0$ if and only if K is diffeomorphic to S^{2n-1} . The Kervaire invariant $c(K)$ is the Arf invariant of a quadratic form $H_n(F_0; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$. In [3] J. Levine proved a remarkable result:

$$c(K) = \begin{cases} 0 & (\Delta(-1) \equiv \pm 1 \pmod{8}) \\ 1 & (\Delta(-1) \equiv \pm 3 \pmod{8}) \end{cases}$$

Hence the value of Alexander polynomial at $t = 1$ completely determines the smooth structure of the sphere under the condition that $\Delta_K(1) = \pm 1$. As is in the previous section, we can reformulate Levine's result:

$$\left(\frac{2}{\Delta_K(-1)} \right) = (-1)^{c(K)}. \quad (9)$$

This suggests that the smooth structure of spheres would be distinguished from number theoretical viewpoint.

Remark 2.1. The Alexander polynomial $\Delta(t)$ is characterized by the following two conditions (see [1]): (i) $\Delta(1) = \pm 1$, and (ii) $\Delta(t)$ is a reciprocity polynomial, i.e., there is an integer n such that $\Delta(t) = t^n \Delta(t^{-1})$.

Consider the ring $R = \mathbb{Z}_4[t]/(t^2 - 1)$. Then, R has the following three units; $1, t, 1 + 2t$. Thus, in R the Alexander polynomial $\Delta(t) \in \mathbb{Z}[t, t^{-1}]$ has the form

$$\Delta(t) = \pm(1 + 2b(t - 1)),$$

for some integer b since $\Delta(1) = \pm 1$. Then, $\Delta(-1) = \pm(1 - 4b)$ and hence we have

$$\Delta(-1) \equiv \begin{cases} \pm 1 & (\text{mod } 8) & (b \in 2\mathbb{Z}) \\ \pm 3 & (\text{mod } 8) & (b \notin 2\mathbb{Z}) \end{cases}$$

Thus we see that $b \pmod{2}$ behaves like the Arf-Kervaire invariant. The latter observation was suggested to the author by J. Hillman (see [2, p. 45] also).

Finally, we would like to pose the following questions: What is the number theoretical background or essence in (8) and (9)? Can we define the higher dimensional Jones polynomial $V_K(t)$ of an algebraic link K so that $V_K(-1) = \Delta_K(-1)$? Can we define the Arf invariant in number theoretical context like the Iwasawa polynomial in number theory which just corresponds to the Alexander polynomial in topology?

Acknowledgment: The author wishes to thank Masanori Morishita for his lectures on Arithmetic Topology held at Kinki University, and also Jonathan Hillman, Akio Kawauchi, and Manabu Ozaki for their useful comments. The author was partially supported by the Grant-in-Aid for Scientific Research (No. 18540102), Japan Society for the Promotion of Science.

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