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The Univalence Conditions For A General Integral Operator

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Abstract

For analytic functions in the open unit disk, J. Becker (Math. Ann. 202(1973)) has given some univalent conditions. In the present paper, some extensions of Becker's type are considered.

Keywords: *Univalence, Schwarz lemma, integral operator.*

1 Introduction

Let $\mathcal{U} = \{z \in \mathcal{C}, |z| < 1\}$

be the open unit disk and $\mathcal{A}$ denote the class of the functions $f(z)$ of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$$

which are analytic in $\mathcal{U}$. Consider

$$\mathcal{S} = \{f \in \mathcal{A} : f \text{ is univalent in } \mathcal{U}\}.$$

Let $\mathcal{A}_2$ be the subclass of $\mathcal{A}$ consisting of functions f(z) of the form

$$f(z) = z + \sum_{k=3}^{\infty} a_k z^k. \tag{1}$$

Let $\mathcal{T}_2$ be the univalent subclass of $\mathcal{A}_2$ which satisfies

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| < 1 \quad (z \in \mathcal{U}). \tag{2}$$

Let $\mathcal{T}_{2,\mu}$ be the subclass of $\mathcal{A}_2$ consisting of functions is of the form (1) which satisfy

$$\left|\frac{z^2 f'(z)}{(f(z))^2} - 1\right| \leq \mu \quad (z \in \mathcal{U}) \tag{3}$$

for some μ $(0 < \mu \leq 1)$. Furthermore, for some real p with $0 < p \leq 2$, we define the subclass $\mathcal{S}(p)$ of $\mathcal{A}$ consisting of all function $f(z)$ which satisfy

$$\left|\left(\frac{z}{f(z)}\right)''\right| \leq p \quad (z \in \mathcal{U}).$$

Singh [8] has shown that if $f(z) \in \mathcal{S}(p)$, then $f(z)$ satisfies

$$\left|\frac{z^2 f'(z)}{(f(z))^2} - 1\right| \leq p|z|^2, \quad (z \in \mathcal{U}). \tag{4}$$

Alfors [1] and Becker [2] have obtained the next univalence criterion:

Theorem 1.1. *Let c be a complex number, $|c| \leq 1$, $c \neq -1$. If $f(z) \in \mathcal{A}$ satisfies*

$$\left|c\,|z|^2 + \left(1 - |z|^2\right)\frac{zf''(z)}{f'(z)}\right| \leq 1$$

for all $z \in \mathcal{U}$, then $f(z) \in \mathcal{S}$.

Furthermore, we need the following theorem given by Pescar [6].

Theorem 1.2. *Let β be a complex number, $\mathbf{Re}\beta > 0$, and c be a complex number, $|c| \leq 1, c \neq -1$. If $f(z) \in \mathcal{A}$ satisfies*

$$\left|c\,|z|^{2\beta} + \left(1 - |z|^{2\beta}\right)\frac{zf''(z)}{\beta f'(z)}\right| \leq 1$$

for all the $z \in \mathcal{U}$, then the function

$$F_\beta(z) = \left[\beta \int_0^z t^{\beta-1} f'(t)dt\right]^{\frac{1}{\beta}}$$

is is in the class $\mathcal{S}$.

Next lemma is well-kwon as the Schwarz lemma (cf. [4]).

The Schwarz Lemma 1.3. *Let the function $f(z)$ be regular in $\mathcal{U}$ with $f(0) = 0$. If $|f(z)| \leq 1 \quad (z \in \mathcal{U}$, then*

$$|f(z)| \leq |z|$$

for all $z \in \mathcal{U}$, where the equality can be hold only if $f(z) = Kz$ and $|K| = 1$.

Breaz and Breaz [3] have considered for $f_i \in \mathcal{A}_2$ $(i = 1, 2, \cdots, n)$ and $\alpha_1, \alpha_2, \cdots, \alpha_n, \beta \in \mathcal{C}$, the integral operator

$$G(z) = \left\{ \beta \int_0^z t^{\beta-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t} \right)^{\alpha_i} dt \right\}^{\frac{1}{\beta}}. \tag{5}$$

2 Main results

Our first result for univalence of $G(z)$ is contained in the following theorem.

Theorem 2.1. *Let the functions $f_i \in \mathcal{S}(p_i)$, for $i \in \{1, ..., n\}$ satisfy the condition (4) with $0 < p_i \leq 2$ and $|f_i(z)| \leq M_i(z \in \mathcal{U})$. If α_i, β, and c are complex numbers such that $\mathbf{Re}\beta > 0$ and*

$$|c| \leq 1 - \frac{1}{\mathbf{Re}\beta} \sum_{i=1}^{n} [(1+p)M + 1] |\alpha_i|, \tag{6}$$

then the function $G(z)$ defined in (5) is in the class $\mathcal{S}$.

Proof. Define a function

$$h(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{f_i(t)}{t} \right)^{\alpha_i} dt,$$

then we have $h(0) = h'(0) - 1 = 0$. Also a simple computation yields

$$h'(z) = \prod_{i=1}^{n} \left(\frac{f_i(z)}{z} \right)^{\alpha_i}$$

and

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^{n} \alpha_i \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right). \tag{7}$$

From the equation (7), we have

$$\left|\frac{zh''(z)}{h'(z)}\right| \le \sum_{i=1}^{n}|\alpha_i|\left(\left|\frac{zf_i'(z)}{f_i(z)}\right|+1\right) = \sum_{i=1}^{n}|\alpha_i|\left(\left|\frac{z^2f_i'(z)}{f_i^2(z)}\right|\left|\frac{f_i(z)}{z}\right|+1\right) \qquad (8)$$

From the hypothesis, we have $|f_i(z)| \le M_i \quad (z \in \mathcal{U}, \quad i = 1, 2, \cdots, n)$, then by Schwarz Lemma, we obtain that

$$|f_i(z)| \le M_i|z| \quad (z \in \mathcal{U}, \quad i = 1, 2, \cdots, n).$$

We apply this result in inequality (8), we obtain

$$\begin{aligned}\left|\frac{zh''(z)}{h'(z)}\right| &\le \sum_{i=1}^{n}|\alpha_i|\left(\left|\frac{z^2f_i'(z)}{(f_i(z))^2}\right|M_i+1\right)\\ &\le \sum_{i=1}^{n}|\alpha_i|\left(\left|\frac{z^2f_i'(z)}{(f_i(z))^2}-1\right|M_i+M_i+1\right)\\ &= \sum_{i=1}^{n}|\alpha_i|\left(p_iM_i\,|z|^2+M_i+1\right)\\ &< \sum_{i=1}^{n}\{(1+p_i)M_i+1\}\,|\alpha_i|.\end{aligned}$$

We have:

$$\left|c\,|z|^{2\beta}+\left(1-|z|^{2\beta}\right)\frac{zh''(z)}{\beta h'(z)}\right| =$$

$$\left|c\,|z|^{2\beta}+\left(1-|z|^{2\beta}\right)\frac{1}{\beta}\sum_{i=1}^{n}\alpha_i\left(\frac{zf_i'(z)}{f_i(z)}-1\right)\right| \le$$

$$|c|+\frac{1}{|\beta|}\cdot\sum_{i=1}^{n}|\alpha_i|\left(\left|\frac{z^2f_i'(z)}{f_i^2(z)}\right|\cdot\frac{|f_i(z)|}{|z|}+1\right).$$

We obtain:

$$\left|c\,|z|^{2\beta}+\left(1-|z|^{2\beta}\right)\frac{zh''(z)}{\beta h'(z)}\right| \le |c|+\frac{1}{|\beta|}\sum_{i=1}^{n}[(1+p_i)M_i+1]\,|\alpha_i| <$$

$$|c|+\frac{1}{\mathbf{Re}\beta}\sum_{i=1}^{n}[(1+p_i)M_i+1]\,|\alpha_i|\,.$$

So, from (6) we have:

$$\left| c\,|z|^{2\beta} + \left(1 - |z|^{2\beta}\right) \frac{zh''(z)}{\beta h'(z)} \right| \leq 1.$$

Applying Theorem 2.1, we obtain that $G(z)$ is univalent.

Corollary 2.2. *Let the functions $f_i \in \mathcal{S}(p_i)$, for $i \in \{1, ..., n\}$ satisfy the condition (4) with $0 < p_i \leq 2$ and $|f_i(z)| \leq M_i (z \in \mathcal{U})$. If α, β, and c are complex numbers such that $\mathbf{Re}\beta > 0$ and*

$$|c| \leq 1 - \frac{1}{\mathbf{Re}\beta} n((p_i + 1)M_i + 1)\,|\alpha|\,,$$

then the function

$$F_{\alpha,\beta}(z) = \left\{ \beta \int_0^z t^{\beta-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t} \right)^{\alpha} dt \right\}^{\frac{1}{\beta}}.$$

is in the class $\mathcal{S}$.

Proof. In Theorem 2.1, we consider $\alpha_1 = \alpha_2 = ... = \alpha_n = \alpha$.

Taking $M = 1$ in Theorem 2.1, we have

Corollary 2.3. *Let the functions $f_i \in \mathcal{S}(p_i)$, for $i \in \{1, ..., n\}$ satisfy the condition (4) with $0 < p_i \leq 2$ and $|f_i(z)| \leq 1 (z \in \mathcal{U})$. If α_i, β, and c are complex numbers such that $\mathbf{Re}\beta > 0$ and*

$$|c| \leq 1 - \frac{1}{\mathbf{Re}\beta} \sum_{i=1}^{n} (p_i + 2)\,|\alpha_i|\,,$$

then the function $G(z)$ defined in (5) is in the class $\mathcal{S}$.

If we take $n = 1$ in Theorem 2.1, then we see

Corollary 2.4. *Let the function $f \in \mathcal{S}(p)$ satisfy the condition (4) and $|f(z)| \leq M (z \in \mathcal{U})$. If α, β, and c are complex numbers such that $\mathbf{Re}\beta > 0$ and*

$$|c| \leq 1 - \frac{1}{\mathbf{Re}\beta} [(1 + p)M + 1]\,|\alpha|\,,$$

then the function

$$G_{\alpha,\beta}(z) = \left\{ \beta \int_0^z t^{\beta-1} \left(\frac{f(t)}{t} \right)^{\alpha} dt \right\}^{\frac{1}{\beta}}.$$

is in the class $\mathcal{S}$.

Letting $M = 1$ in Corollary 2.4, we have

Corollary 2.5. *Let the function* $f \in \mathcal{S}(p)$, *satisfy the condition (4) and* $|f(z)| \leq 1 (z \in \mathcal{U})$. *If* α, β, *and* c *are complex numbers such that* $\mathbf{Re}\beta > 0$ *and*

$$|c| \leq 1 - \frac{1}{\mathbf{Re}\beta}(p+2)\,|\alpha|\,,$$

then the function

$$G_{\alpha,\beta}(z) = \left\{\beta \int_0^z t^{\beta-1}\left(\frac{f(t)}{t}\right)^{\alpha} dt\right\}^{\frac{1}{\beta}}.$$

is in the class $\mathcal{S}$.

Next we derive

Theorem 2.6. *Let the functions* $f_i \in \mathcal{T}_{2,\mu_i}$, *for* $i \in \{1, ..., n\}$ *with* $0 < \mu_i \leq 1$ *satisfy the condition (3) and* $|f_i(z)| \leq M_i$ $(z \in \mathcal{U})$.
If α_i, β, *and* c *are complex numbers such that* $\mathbf{Re}\beta > 0$ *and*

$$|c| \leq 1 - \frac{1}{\mathbf{Re}\beta}\sum_{i=1}^{n}[(1+\mu_i)M_i + 1]\,|\alpha_i|\,, \tag{9}$$

then the function $G(z)$ *defined in (5) is in the class* $\mathcal{S}$.

Proof. Define a function

$$h(z) = \int_0^z \prod_{i=1}^{n}\left(\frac{f_i(t)}{t}\right)^{\alpha_i} dt,$$

then we have $h(0) = h'(0) - 1 = 0$. Also a simple computation yields

$$h'(z) = \prod_{i=1}^{n}\left(\frac{f_i(z)}{z}\right)^{\alpha_i}$$

and

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^{n}\alpha_i\left(\frac{zf_i'(z)}{f_i(z)} - 1\right). \tag{10}$$

From equation (10), we have

$$\left|\frac{zh''(z)}{h'(z)}\right| \leq \sum_{i=1}^{n} |\alpha_i| \left(\left|\frac{zf_i'(z)}{f_i(z)}\right| + 1 \right) = \sum_{i=1}^{n} |\alpha_i| \left(\left|\frac{z^2 f_i'(z)}{f_i^2(z)}\right| \left|\frac{f_i(z)}{z}\right| + 1 \right) \tag{11}$$

From the hypothesis, we have $|f_i(z)| \leq M_i \quad (z \in \mathcal{U}, \quad i = 1, 2, \cdots, n)$, then by Schwarz Lemma, we obtain that

$$|f_i(z)| \leq M_i |z| \quad (z \in \mathcal{U}, \quad i = 1, 2, \cdots, n).$$

We apply this result in inequality (11), we obtain

$$\begin{aligned}
\left|\frac{zh''(z)}{h'(z)}\right| &\leq \sum_{i=1}^{n} |\alpha_i| \left(\left|\frac{z^2 f_i'(z)}{(f_i(z))^2}\right| M_i + 1 \right) \\
&\leq \sum_{i=1}^{n} |\alpha_i| \left(\left|\frac{z^2 f_i'(z)}{(f_i(z))^2} - 1\right| M_i + M_i + 1 \right) \\
&= \sum_{i=1}^{n} |\alpha_i| \left(\mu_i M_i + M_i + 1 \right).
\end{aligned}$$

We have:

$$\left| c\,|z|^{2\beta} + \left(1 - |z|^{2\beta}\right) \frac{zh''(z)}{\beta h'(z)} \right| =$$

$$\left| c\,|z|^{2\beta} + \left(1 - |z|^{2\beta}\right) \frac{1}{\beta} \sum_{i=1}^{n} \alpha_i \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right) \right| \leq$$

$$|c| + \frac{1}{|\beta|} \cdot \sum_{i=1}^{n} |\alpha_i| \left(\left|\frac{z^2 f_i'(z)}{f_i^2(z)}\right| \cdot \frac{|f_i(z)|}{|z|} + 1 \right).$$

We obtain:

$$\left| c\,|z|^{2\beta} + \left(1 - |z|^{2\beta}\right) \frac{zh''(z)}{\beta h'(z)} \right| \leq |c| + \frac{1}{|\beta|} \sum_{i=1}^{n} [(1 + \mu_i) M_i + 1] \, |\alpha_i| <$$

$$|c| + \frac{1}{\mathbf{Re}\beta} \sum_{i=1}^{n} [(1 + \mu_i) M_i + 1] \, |\alpha_i| .$$

So, from (9) we have:

$$\left| c\,|z|^{2\beta} + \left(1 - |z|^{2\beta}\right) \frac{zh''(z)}{\beta h'(z)} \right| \leq 1.$$

Applying Theorem 2.1, we obtain that $G(z)$ is univalent in $\mathcal{U}$.

Corollary 2.7. *Let the functions* $f_i \in \mathcal{T}_{2,\mu}$, *for* $i \in \{1,...,n\}$ *with* $0 < \mu \leq 1$ *satisfy the condition (3) and* $|f_i(z)| \leq M$ $(z \in \mathcal{U})$.

If α, β, *and* c *are complex numbers such that* $\mathbf{Re}\beta > 0$ *and*

$$|c| \leq 1 - \frac{1}{\mathbf{Re}\beta} n((1+\mu)M+1)\,|\alpha|\,,$$

then the function

$$F_{\alpha,\beta}(z) = \left\{\beta \int_0^z t^{\beta-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{\alpha} dt\right\}^{\frac{1}{\beta}}.$$

is in the class $\mathcal{S}$.

Proof. In Theorem 2.6, we consider $\mu_i = \mu, \alpha_1 = \alpha_2 = ... = \alpha_n = \alpha$ and $M_i = M$.

If we make $M = 1$ in Theorem 2.6, then we obtain

Corollary 2.8. *Let the functions* $f_i \in \mathcal{T}_{2,\mu_i}$, *for* $i \in \{1,...,n\}$ *with* $0 < \mu_i \leq 1$ *satisfy the condition (3) and* $|f_i(z)| \leq 1 (z \in \mathcal{U})$.

If α_i, β, *and* c *are complex numbers such that* $\mathbf{Re}\beta > 0$ *and*

$$|c| \leq 1 - \frac{1}{\mathbf{Re}\beta} \sum_{i=1}^{n} (\mu_i + 2)\,|\alpha_i|\,,$$

then the function $G(z)$ *defined in (5) is in the class* $\mathcal{S}$.

Letting $n = 1$ in Theorem 2.6, we have

Corollary 2.9. *Let the function* $f \in \mathcal{T}_{2,\mu}$ *satisfy the condition (3) and* $|f(z)| \leq M(z \in \mathcal{U})$.

If α, β, *and* c *are complex numbers such that* $\mathbf{Re}\beta > 0$ *and*

$$|c| \leq 1 - \frac{1}{\mathbf{Re}\beta} [(1+\mu)M+1]\,|\alpha|$$

then the function

$$G_{\alpha,\beta}(z) = \left\{\beta \int_0^z t^{\beta-1} \left(\frac{f(t)}{t}\right)^{\alpha} dt\right\}^{\frac{1}{\beta}}$$

is in the class $\mathcal{S}$.

Further, taking $M = 1$ in Corollary 2.9, we see

Corollary 2.10. *Let the function* $f \in \mathcal{T}_{2,\mu}$*, satisfy the condition (3) and* $|f(z)| \leq 1 (z \in \mathcal{U})$.

If α, β*, and* c *are complex numbers such that* $\mathbf{Re}\beta > 0$ *and*

$$|c| \leq 1 - \frac{1}{\mathbf{Re}\beta}(\mu + 2)\,|\alpha|\,,$$

then the function

$$G_{\alpha,\beta}(z) = \left\{\beta \int_0^z t^{\beta-1}\left(\frac{f(t)}{t}\right)^{\alpha} dt\right\}^{\frac{1}{\beta}}$$

is in the class $\mathcal{S}$.

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References

[1] L. V. Ahlfors, *Sufficient conditions for quasiconformal extension*, Proc. 1973, Conf. Univ. of Maryland, Ann. Math. Studies **79**(1973), 23-29.

[2] J. Becker, *Lownersche Differential gleichung und Schlichteits-Kriterion*, Math. Ann. **202**(1973), 321-335.

[3] D. Breaz and N. Breaz, *Two integral operators*, Studia Universitatis Babeş-Bolyai, Mathematica, **47**(2002), 13-20.

[4] Z. Nehari, *Conformal mapping*, McGraw Hill Book Comp., New York, (1952) (Dover, Publ. Inc., 1975).

[5] S. Ozaki and M. Nunokawa, *The Schwarzian derivative and univalent functions*, Pro. Amer. Math. Soc. **33**(1972), 392-394.

[6] V. Pescar, *A new generalization of Ahlfor's and Becker's criterion of univalence*, Bull. Malaysian Math. Soc. **19**(1996), 53-54.

[7] V. Pescar, *On the univalence of some integral operators*, J. Indian Acad. Math. **27**(2005), 239-243.

[8] V. Singh, *On a class of univalent functions*, Internat. J. Math. Math. Sci. **23**(2000), 855-857.