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Numerical Solutions Of The Nonlinear Integro-Differential Equations

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Abstract

This paper compares the variational iteration method (VIM) with the Adomian decomposition method (ADM) for solving nonlinear integro- differential equations. From the computational viewpoint, the VIM is more efficient, convenient and easy to use.

Keywords: *Variational iteration method (VIM); Adomian decomposition method (ADM); Integro-differential equations*

1 Introduction

Mathematical modelling of real-life problems usually results in functional equations, like ordinary or partial differential equations, integral and integro- differential equations, stochastic equations. Many mathematical formulation of physical phenomena contain integro-differential equations, these equations arises in many fields like fluid dynamics, biological models and chemical kinetics. Integro-differential equations are usually difficult to solve analytically so it is required to obtain an efficient approximate solution [1].

Nonlinear phenomena are of fundamental importance in various fields of science and engineering. The nonlinear models of real-life problems are still difficult to solve either numerically or theoretically. There has recently been much attention devoted to the search for better and more efficient solution methods for determining a solution, approximate or exact, analytical or numerical, to nonlinear models, [2, 3, 4].

An analytical method called the Adomian decomposition method (ADM) proposed by Adomian [5] aims to solve frontier nonlinear physical problems. It has been applied to a wide class of deterministic and stochastic problems, linear and nonlinear, in physics, biology and chemical reactions etc. For nonlinear models, the method has shown reliable results in supplying analytical approximations that converge rapidly [6].

The variational iteration method (VIM) is a simple and yet powerful method for solving a wide class of nonlinear problems, first envisioned by He [11] (see also [12, 13, 14, 15, 16]). The VIM has successfully been applied to many situations. For example, He [12] solved the classical Blasius' equation using VIM. He [13] used VIM to give approximate solutions for some well-known nonlinear problems. He [14] used VIM to solve autonomous ordinary differential systems. He [15] coupled the iteration method with the perturbation method to solve the well-known Blasius equation. He [16] solved strongly nonlinear equations using VIM. Soliman [17] applied the VIM to solve the KdV-Burger's and Lax's seventh-order KdV equations. The VIM has recently been applied for solving nonlinear coagulation problem with mass loss by Abulwafa et al. [18]. Momani et al. The VIM has been applied for solving nonlinear differential equations of fractional order by Odibat et al. [19]. Bildik et al. [20] used VIM for solving different types of nonlinear partial differential equations. Dehghan and Tatari [21] employed VIM to solve a Fokker-Planck equation. Wazwaz [22] presented a comparative study between the variational iteration method and Adomian decomposition method. Tamer et al. [23] introduced a modification of VIM. Batiha et al. [24] used VIM to solve the generalized Burgers-Huxley equation. Batiha et al. [25] applied VIM to the generalized Huxley equation. Abbasbandy [26] solved one example of the quadratic Riccati differential equation (with constant coefficient) by He's variational iteration method by using Adomian's polynomials. Wang and He [27] applied VIM to solve integro-differential equations. Sweilam [28] used VIM solve both linear and nonlinear boundary value problems for fourth order integro-differential equations.

The objective of this paper is to introduce a comparative study to examine the performance of the VIM and ADM in solving integro-differential equations. The comparison can be realized by using the two simple nonlinear integro-differential equations presented in [29].

2 Variational iteration method

VIM is based on the general Lagrange's multiplier method [31]. The main feature of the method is that the solution of a mathematical problem with linearization assumption is used as initial approximation or trial function. Then a more highly precise approximation at some special point can be obtained. This approximation converges rapidly to an accurate solution [30].

To illustrate the basic concepts of VIM, we consider the following nonlinear differential equation:

$$Lu + Nu = g(x), \tag{1}$$

where L is a linear operator, N is a nonlinear operator, and $g(x)$ is an inhomogeneous term. According to VIM [13, 14, 16, 30], we can construct a correction functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda\{Lu_n(\tau) + N\tilde{u}_n(\tau) - g(\tau)\}\mathrm{d}\tau, \quad n \geq 0, \tag{2}$$

where λ is a general Lagrangian multiplier [31] which can be identified optimally via the variational theory, the subscript n denotes the nth-order approximation, $\tilde{u}_n$ is considered as a restricted variation [13, 14], i.e. $\delta\tilde{u}_n = 0$.

Now, we present the solution of the following integro-differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}x} = f(x) + \int_0^x \Psi(t, u(t), u'(t))\,\mathrm{d}t, \tag{3}$$

where $f(x)$ is the source terme. To do so, we first construct a correction functional,

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)\left[(u_n)_s(s) - f(s) - \int_0^s \Psi(t, \tilde{u}(t), \tilde{u}'(t))\,\mathrm{d}t\right]\mathrm{d}s, \tag{4}$$

where $\tilde{u}_n$ is considered as restricted variations, which means $\tilde{u}_n = 0$. To find the optimal $\lambda(s)$, we proceed as follows:

$$\delta u_{n+1}(x) = \delta u_n(x) + \delta\int_0^x \lambda(s)\left[(u_n)_s(s) - f(s) - \int_0^s \Psi(t, \tilde{u}(t), \tilde{u}'(t))\,\mathrm{d}t\right]\mathrm{d}s, \tag{5}$$

and consequently

$$\delta u_{n+1}(x) = \delta u_n(x) + \delta\int_0^x \lambda(s)(u_n)_s(s)\,\mathrm{d}s, \tag{6}$$

which results in

$$\delta u_{n+1}(x) = \delta u_n(x) + \lambda(s)\delta u_n(x) - \int_0^x \delta u_n(x)\lambda'(s)\,\mathrm{d}s, \tag{7}$$

The stationary conditions can be obtained as follows:

$$\lambda'(s) = 0 \text{ and } 1 + \lambda(s)|_{s=t} = 0. \tag{8}$$

The Lagrange multipliers, therefore, can be identified as

$$\lambda(s) = -1, \tag{9}$$

and the iteration formula is given as

$$u_{n+1}(x) = u_n(x) - \int_0^x \left[(u_n)_s(s) - f(s) - \int_0^s \Psi(t, u(t), u'(t))\,\mathrm{d}t\right]\mathrm{d}s, \tag{10}$$

3 Numerical experiments

In this section, we apply VIM to solve two nonlinear integro-differential equations. The main objective here is to solve these two examples using the VIM given in Section 2 and compare our results with the presented results in [29].

3.1 Example 1.

We first consider the nonlinear integro-differential equation

$$u'(x) = -1 + \int_0^x u^2(t)\,\mathrm{d}t, \tag{11}$$

for $x \in [0,1]$ with the boundary condition $u(0) = 0$.

Using VIM, the iteration formula for Eq. (11) is (see (10)),

$$u_{n+1}(x) = u_n(x) - \int_0^x \left[(u_n)_s(s) + 1 - \int_0^s u^2(t)\,\mathrm{d}t\right]\,\mathrm{d}s. \tag{12}$$

We can take an initial approximation $u_0(x) = -x$.

The first three iterates are easily obtained from (12) and are given by:

$$\begin{aligned}
u_1(x) &= -x + \frac{1}{12}x^4, &(13)\\
u_2(x) &= -x + \frac{1}{12}x^4 - \frac{1}{252}x^7 + \frac{1}{12960}x^{10}, &(14)\\
u_3(x) &= -x + \frac{1}{12}x^4 + \frac{1}{12960}x^{10} - \frac{1}{252}x^7\\
&\quad -\frac{37}{7076160}x^{13} + \frac{109}{914457600}x^{16}\\
&\quad -\frac{1}{558472320}x^{19} + \frac{1}{77598259200}x^{22}. &(15)
\end{aligned}$$

Tables 1 contains a numerical comparison between our solution using 2-iterate VIM and the solution of the same problem presented in [29] using ADM.

3.2 Example 2.

Now, we will find the approximate analytical solution of the integro-differential equation

$$u'(x) = 1 + \int_0^x u(t)\frac{\mathrm{d}u(t)}{\mathrm{d}t}\,\mathrm{d}t, \tag{16}$$

for $x \in [0,1]$ with the boundary condition $u(0) = 0$.

Table 1: A comparison between 2-iterate VIM solutions and ADM solutions for example 1.

x	ADM	VIM
0.0000	0.000000000	0.000000000
0.0938	−0.0937935	−0.0937935
0.2188	−0.2186090	−0.2186091
0.3125	−0.3117060	−0.3117064
0.4062	−0.4039390	−0.4039385
0.5000	−0.4948230	−0.4948226
0.6250	−0.6124310	−0.6124315
0.7188	−0.6969410	−0.6969446
0.8125	−0.7770900	−0.7771007
0.9062	−0.8519340	−0.8519654
1.0000	−0.9204760	−0.9205578

Using VIM, the iteration formula for Eq. (16) is (see (10)),

$$u_{n+1}(x) = u_n(x) - \int_0^x \left[(u_n)_s(s) - 1 - \int_0^s u(t) \frac{\mathrm{d}u(t)}{\mathrm{d}t}\, \mathrm{d}t \right] \mathrm{d}s. \tag{17}$$

We can take an initial approximation $u_0(x) = x$.

The first two iterates are easily obtained from (17) and are given by:

$$u_1(x) = x + \frac{1}{12}x^4, \tag{18}$$

$$u_2(x) = x + \frac{1}{12}x^4 + \frac{1}{84}x^7 + \frac{1}{1440}x^{10} + \frac{1}{67392}x^{13}, \tag{19}$$

Tables 2 contains a numerical comparison between our solution using 3-iterate VIM and the solution of the same problem presented in [29] using ADM.

4 Conclusions

In this paper, the variation iteration method (VIM) has been successfully employed to obtain the approximate analytical solutions of the nonlinear integro-differential equations. The method has been applied directly without using linearization or any restrictive assumptions. More importantly, the VIM reduces the volume of calculations by not requiring the Adomian polynomials, hence the iteration is direct and straightforward. However, ADM requires the use of Adomian polynomials for nonlinear terms, and this needs more work.

Table 2: A comparison between 3-iterate VIM solutions and ADM solutions for example 2.

x	ADM	VIM
0.0000	0.000000000	0.000000000
0.0938	0.0938065	0.0938065
0.2188	0.2189910	0.2189913
0.3125	0.3132980	0.3132982
0.4062	0.4084910	0.4084907
0.5000	0.5053030	0.5053032
0.6250	0.6381770	0.6381768
0.7188	0.7422990	0.7422988
0.8125	0.8518530	0.8518520
0.9062	0.9691440	0.9691418
1.0000	1.0973700	1.0973681

For nonlinear equations that arise frequently to express nonlinear phenomenon, VIM facilitates the computational work and gives the solution rapidly if compared with Adomian method. It may be concluded that VIM is very powerful and efficient in finding analytical as well as numerical solutions for a wide class of linear and nonlinear integro-differential equations. VIM provides more realistic series solutions that converge very rapidly in real physical problems.

5 Open Problem

The work presented in this paper pose interesting new questions to researchers and provides the theoretical framework for studying more complex equations. It would be very interesting to present new modification of VIM to get more accurate solution or to get more rapid convergence. It would also be very interesting for example to present new technique of choosing initial approximation which will improve the accuracy of the solution. Study the convergent concept for the solution presented by VIM and give a proof for this convergence is in a great interest. The chaotic equations are very hot and important subject, so it will be very interesting to use VIM to solve these kind of equations and to see if VIM is suitable and efficient mathematical tool for solving these equations.

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References

[1] N.H. Sweilam, Fourth order integro-differential equations using variational iteration method, *Comp.Math. Appl.*, doi:10.1016/j.camwa.2006.12.055

[2] X.Y. Wang, Exact and explicit solitary wave solutions for the generalized Fisher equation *Phys. Lett. A* 131 (1988) 277-279.

[3] A. Jeffrey, M.N.B. Mohamad, Exact solutions to the KdV-Burgers equation, *Wave Motion* 14 (1991) 369–375.

[4] M. Wadati, The exact solution of the modified Korteweg-de Vries equation. *J. Phys. Soc. Jpn.* 32 (1972) 1681–1687.

[5] Adomian. G, Solving frontier problems of physics: the decomposition method. Boston: *Kluwer Academic*, (1994).

[6] W. Chen and Z. Lu, An algorithm for Adomian decomposition method. *Appl Math Comput*, 159 (2004) 221–235.

[7] M. Senthilvelan, On the extended applications of homogenous balance method, Appl. Math. Comput. 123 (2001) 381–388.

[8] D.S. Li, H. Q. Zhang, Symbolic computation and various exact solutions of potential Kadomstev-Petviashvili equation, *Appl. Math. Comput.* 145 (2003) 351–359.

[9] D.S. Li, H. Q. Zhang, New soliton-like solutions to the potential KadomstevPetviashvili (PKP) equation, *Appl. Math. Comput.* 146 (2003) 381–384.

[10] D. Kaya, S. M. El-Sayed, Numerical soliton-like solutions of the potential Kadomtsev-Petviashvili equation by the decomposition method, *Phys Lett A* 320 (2003) 192-199.

[11] J.H. He, A new approach to nonlinear partial differential equations. *Commun. Nonlin. Sci. Numer. Simul.* 2 (1997) 230-235.

[12] J.H. He, Approximate analytical solution of Blasius' equation. *Commun. Nonlin. Sci. Numer. Simul.* 3 (1998) 260–263.

[13] J.H. He, Variational iteration method–a kind of non-linear analytical technique: some examples. *Int. J. Non-Linear Mech.* 34 (1999) 699–708.

[14] J.H. He, Variational iteration method for autonomous ordinary differential systems. *Appl. Math. Comput.* 114 (2000) 115-123.

[15] J.H. He, A simple perturbation approach to Blasius equation. *Appl. Math. Comput.* 140 (2003) 217–222.

[16] J.H. He, Some asymptotic methods for strongly nonlinear equations. *Int. J. Modern Phys. B* 20 (2006) 1141–1199.

[17] A.A. Soliman, A numerical simulation and explicit solutions of KdV-Burgers' and Lax's seventh-order KdV equations. *Chaos Solitons Fractals* 29(2) (2006) 294–302.

[18] E.M. Abulwafa, M.A. Abdou and A.A. Mahmoud, The solution of nonlinear coagulation problem with mass loss. *Chaos Solitons Fractals* 29(2) (2006) 313–330.

[19] Z.M. Odibat, S. Momani, Application of variational iteration method to nonlinear differential equations of fractional order. *Int. J. Nonlinear Sci. Numer. Simul.* 7 (2006) 27–34.

[20] N. Bildik and A.Konuralp, The use of variational iteration method, differential transform method and Adomian decomposition method for solving different types of nonlinear partial differential equations. *Int. J. Nonlinear. Sci. Numer. Simul.* 7 (2006) 65–70.

[21] M. Dehghan and M. Tatari, The use of He's variational iteration method for solving a Fokker-Planck equation. *Phys. Scr.* 74 (2006) 310–316.

[22] A.M. Wazwaz, A comparison between the variational iteration method and Adomian decomposition method. *J. Comput. Appl. Math.* doi:10.1016/j.cam.2006.07.018.

[23] H. El-Zoheiry, T.A. Abassy and M.A. El-Tawil, Toward a modified variational iteration method. *J. Comput. Appl. Math.* doi:10.1016/j.cam.2006.07.019.

[24] B. Batiha, M.S.M. Noorani, I. Hashim, Application of variational iteration method to the generalized Burgers-Huxley equation. *Chaos Solitons Fractals* doi: 10.1016/j.chaos.2006.06.080, in press

[25] B. Batiha, M.S.M. Noorani, I. Hashim, Numerical simulation of the generalized Huxley equation by He's variational iteration method. *Appl. Math. Comput.* 186 (2007) 1322–1325.

[26] S. Abbasbandy , A new application of He's variational iteration method for quadratic Riccati differential equation by using Adomian's polynomials. *J. Comput. Appl. Math.* doi:10.1016/j.cam.2006.07.012.

[27] S.Q. Wang, J.H. He, Variational Iteration Method for Solving Integro-differential Equations, *Phy. Lett. A* 367 (2007) (3) 188-191

[28] N.H. Sweilam, Fourth order integro-differential equations using variational iteration method. *Comp. Math. Appl.* (2007) 54 (7-8) 1086-1091

[29] S. M. El-Sayed, M.R. Abdel-Aziz, comparison of Adomians decomposition method and wavelet-Galerkin method for solving integro-differential equations. *Appl. Math. Comput.* 136 (2003) 151–159

[30] J.H. He, Non-perturbative methods for strongly nonlinear problems, Berlin: *dissertation.de-Verlag im Internet GmbH*, (2006).

[31] M. Inokuti, H. Sekine and T. Mura, General use of the Lagrange multiplier in nonlinear mathematical physics. In: Nemat-Nassed S, editor. Variational method in the mechanics of solids. *Pergamon Press*, (1978).