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On Sufficient Conditions and Angular Properties of Starlikeness and Convexity for The Class of Multivalent Bazilevic´ Functions

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Abstract

In this paper we consider a class of analytic and multivalent functions to obtain some sufficient conditions of starlikeness and convexity for the class of Bazelevič function. Moreover we also obtain certain angular properties for functions belonging to this class.

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1 Introduction

Let $A_p(n)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k \qquad (n, p \in \mathbb{N} = \{1, 2, ...\}), \tag{1.1}$$

which are analytic and multivalent in the open unit disk $\mathbb{U} = (z : z \in \mathbb{C} \ , |z| < 1)$. In particular if $p = n = 1$, then $A_1(1) = A$.

A function $f \in \mathcal{A}_p(n)$ is said to be in the subclass $\mathcal{S}(p, n, \alpha)$ of multivalent *starlike* functions of order α in $\mathbb{U}$ if it satisfies the following inequality

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad (z \in \mathbb{U};\ 0 \leqslant \alpha < p;\ p \in \mathbb{N}), \tag{1.2}$$

Let $S^*(p,n,\alpha_1,\alpha_2)$ be the subclass of $A_p(n)$ which satisfies

$$-\frac{\pi\alpha_1}{2} < \arg\frac{zf'(z)}{f(z)} < \frac{\pi\alpha_2}{2} \qquad (z \in U;\ 0 < \alpha_1, \alpha_2 \leqslant p), \tag{1.3}$$

This class is called *strongly starlike* subclass of $A_p(n)$.

On the other hand a function $f \in \mathcal{A}_p(n)$ is said to be a p-valently *Bazelevič* function of type β $(\beta \geq 0)$ and order γ $(0 \leq \gamma < p, p \in \mathbb{N})$ if there exists a function g belonging to the class $\mathcal{S}^*(p,n) := \mathcal{S}^*(p,n,0)$ such that

$$Re\left\{\frac{zf'(z)}{[f(z)]^{1-\beta}\,[g(z)]^{\beta}}\right\} > \gamma \qquad (z \in \mathbb{U};\ 0 \leqslant \gamma < p), \tag{1.4}$$

This class was studied earlier by Irmak et al. [2].

We denote the class of all such functions by $\mathbb{B}(p,\beta,\gamma)$. In particular, when $\beta = 1$, a function $f \in \mathbb{K}(p,\gamma) := \mathbb{B}(p,1,\gamma)$ is said to be p-valently close-to-convex of order γ in the unit disk $\mathbb{U}$. Moreover, $\mathbb{B}(p,0,\gamma) = S^*(p,\gamma)$, when $\beta = 0$.

To prove our main results we need the following lemmas

Lemma 1.1 ([3]): *Let the function $w(z)$ be (nonconstant) analytic in $\mathbb{U}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in \mathbb{U}$, then*

$$z_0 w'(z_0) = k w(z_0), \tag{1.5}$$

where k is real and $k \geq 1$.

Lemma 1.2 ([4]): *Let q(z) be analytic in $\mathbb{U}$ with $q(0) = 1$ and $q(z) \neq 0$ for all $z \in \mathbb{U}$. If there exists two points $z_1, z_2 \in \mathbb{U}$ such that*

$$-\frac{\pi\alpha_1}{2} = arg\ q(z_1) < arg\ q(z) < arg\ q(z_2) = \frac{\pi\alpha_2}{2}, \tag{1.6}$$

for $\alpha_1, \alpha_2 > 0$, and $|\, z \,|<|\, z_1 \,|=|\, z_2 \,|$, then we have

$$\frac{z_1 q'(z_1)}{q(z_1)} = -i\left(\frac{\alpha_1+\alpha_2}{2}\right)m \quad and \quad \frac{z_2 q'(z_2)}{q(z_2)} = i\left(\frac{\alpha_1+\alpha_2}{2}\right)m, \tag{1.7}$$

where

$$m \geqslant \frac{1-|a|}{1-|a|} \quad and \quad a = i\tan\frac{\pi}{4}\left(\frac{\alpha_2-\alpha_1}{\alpha_1+\alpha_2}\right). \tag{1.8}$$

2 Sufficient conditions

With the help of Lemma 1.1, we will prove following theorem:

Theorem 2.3 *Let $z \in \mathbb{U}, \beta \geq 0, 0 \leq \alpha < p, p \in \mathbb{N}$. Suppose that $f(z) \in$*

$\mathcal{A}_p(n)$, and $g(z) \in S^*(p,n)$ satisfy anyone of the following inequalities:

$$\left|\left(\frac{zf'(z)}{[f(z)]^{1-\beta}[g(z)]^{\beta}}\right)\left(\frac{zf''(z)}{f'(z)}-(1-\beta)\frac{zf'(z)}{f(z)}-\beta\frac{zg'(z)}{g(z)}+1\right)\right| < p-\alpha\,, \tag{2.1}$$

$$\left|\frac{\left(\frac{zf''(z)}{f'(z)}-(1-\beta)\frac{zf'(z)}{f(z)}-\beta\frac{zg'(z)}{g(z)}+1\right)}{\left(\frac{zf'(z)}{[f(z)]^{1-\beta}[g(z)]^{\beta}}\right)}\right| < \frac{(p-\alpha)}{(2p-\alpha)^2}\,, \tag{2.2}$$

$$\left|\frac{\left(\frac{zf''(z)}{f'(z)}-(1-\beta)\frac{zf'(z)}{f(z)}-\beta\frac{zg'(z)}{g(z)}+1\right)}{\left(\frac{zf'(z)}{[f(z)]^{1-\beta}[g(z)]^{\beta}}-p\right)}\right| < \frac{1}{(2p-\alpha)}\,, \tag{2.3}$$

$$\left|\left(\frac{zf'(z)}{[f(z)]^{1-\beta}[g(z)]^{\beta}}\right)\frac{\left(\frac{zf''(z)}{f'(z)}-(1-\beta)\frac{zf'(z)}{f(z)}-\beta\frac{zg'(z)}{g(z)}+1\right)}{\left(\frac{zf'(z)}{[f(z)]^{1-\beta}[g(z)]^{\beta}}-p\right)}\right| < 1\,, \tag{2.4}$$

$$\left|\frac{zf''(z)}{f'(z)}-(1-\beta)\frac{zf'(z)}{f(z)}-\beta\frac{zg'(z)}{g(z)}+1+\delta\left(\frac{zf'(z)}{[f(z)]^{1-\beta}[g(z)]^{\beta}}-p\right)\right| < \frac{(p-\alpha)(\delta(2p-\alpha)+1)}{(2p-\alpha)}\,, \tag{2.5}$$

then $f(z) \in \mathcal{B}(p,n,\beta,\alpha)$.

Proof. Let $f(z) \in \mathcal{A}_p(n)$ and $g(z) \in S^*(p,n)$. Define a function $w(z)$ such that

$$\frac{zf'(z)}{[f(z)]^{1-\beta}[g(z)]^{\beta}} = p+(p-\alpha)\,w(z), \quad (z \in \mathbb{U},\ \beta \geq 0,\ 0 \leq \alpha < p,\ p \in \mathbb{N})\,. \tag{2.6}$$

Here $w(z)$ is analytic in $\mathbb{U}$ and $w(0)=0$. Then it follows from the above definition (2.6) that

$$\left(\frac{zf''(z)}{f'(z)}-(1-\beta)\frac{zf'(z)}{f(z)}-\beta\frac{zg'(z)}{g(z)}+1\right) = \frac{(p-\alpha)\,zw'(z)}{p+(p-\alpha)\,w(z)}. \tag{2.7}$$

Hence, from (2.6) and (2.7), we have

$$F_1(z) = \left(\frac{zf'(z)}{[f(z)]^{1-\beta}[g(z)]^{\beta}}\right)\left(\frac{zf''(z)}{f'(z)}-(1-\beta)\frac{zf'(z)}{f(z)}-\beta\frac{zg'(z)}{g(z)}+1\right) = (p-\alpha)\,zw'(z), \tag{2.8}$$

$$F_2(z) = \frac{\left(\frac{zf''(z)}{f'(z)}-(1-\beta)\frac{zf'(z)}{f(z)}-\beta\frac{zg'(z)}{g(z)}+1\right)}{\left(\frac{zf'(z)}{[f(z)]^{1-\beta}[g(z)]^{\beta}}\right)} = \frac{(p-\alpha)\,zw'(z)}{(p+(p-\alpha)\,w(z))^2}, \tag{2.9}$$

$$F_3(z) = \frac{\left(\frac{zf''(z)}{f'(z)} - (1-\beta)\frac{zf'(z)}{f(z)} - \beta\frac{zg'(z)}{g(z)} + 1\right)}{\left(\frac{zf'(z)}{[f(z)]^{1-\beta}[g(z)]^{\beta}} - p\right)} = \frac{zw'(z)}{w(z)}\frac{1}{p + (p-\alpha)\,w(z)}, \tag{2.10}$$

$$F_4(z) = \left(\frac{zf'(z)}{[f(z)]^{1-\beta}\,[g(z)]^{\beta}}\right)\frac{\left(\frac{zf''(z)}{f'(z)} - (1-\beta)\frac{zf'(z)}{f(z)} - \beta\frac{zg'(z)}{g(z)} + 1\right)}{\left(\frac{zf'(z)}{[f(z)]^{1-\beta}[g(z)]^{\beta}} - p\right)} = \frac{zw'(z)}{w(z)}, \tag{2.11}$$

$$\begin{aligned} F_5(z) &= \frac{zf''(z)}{f'(z)} - (1-\beta)\frac{zf'(z)}{f(z)} - \beta\frac{zg'(z)}{g(z)} + 1 + \delta\left(\frac{zf'(z)}{[f(z)]^{1-\beta}\,[g(z)]^{\beta}} - p\right) \\ &= \delta\,(p-\alpha)\,w(z) + \frac{(p-\alpha)\,zw'(z)}{p + (p-\alpha)\,w(z)}. \end{aligned} \tag{2.12}$$

Now from Lemma 1.1, suppose that there exist $z_0 \in \mathbb{U}$ such that

$$\max_{|z|<|z_0|} |w\,(z)| = |w\,(z_0)| = 1$$

Therefore, letting $w(z_0) = e^{i\theta}$ in each of (2.8) - (2.12), we obtain that

$$|F_1(z_0)| = |(p-\alpha)\,zw'(z_0)| = \left|(p-\alpha)\,ke^{i\theta}\right| \geqslant p - \alpha\ , \tag{2.13}$$

$$|F_2(z_0)| = \left|\frac{(p-\alpha)\,z_0w'(z_0)}{[p + (p-\alpha)\,w(z_0)]^2}\right| = \frac{\left|(p-\alpha)\,ke^{i\theta}\right|}{\left|[p + (p-\alpha)\,e^{i\theta}]^2\right|} \geqslant \frac{p-\alpha}{(2p-\alpha)^2}\ , \tag{2.14}$$

$$|F_3(z_0)| = \left|\frac{z_0w'(z_0)}{w(z_0)}\frac{1}{[p + (p-\alpha)\,w(z_0)]}\right| = \left|\frac{ke^{i\theta}}{e^{i\theta}\,[p + (p-\alpha)\,e^{i\theta}]}\right| \geqslant \frac{1}{(2p-\alpha)}\ , \tag{2.15}$$

$$|F_4(z_0)| = \left|\frac{z_0w'(z_0)}{w(z_0)}\right| = |k| \geqslant 1\ , \tag{2.16}$$

$$\begin{aligned} |F_5(z_0)| &= \left|\delta\,(p-\alpha)\,w(z_0) + \frac{(p-\alpha)\,z_0w'(z_0)}{p + (p-\alpha)\,w(z_0)}\right| = \left|\delta\,(p-\alpha)\,e^{i\theta} + \frac{k\,(p-\alpha)\,e^{i\theta}}{p + (p-\alpha)\,e^{i\theta}}\right| \\ &\geqslant \frac{(p-\alpha)\,(\delta\,(2p-\alpha) + 1)}{(2p-\alpha)}\ , \end{aligned} \tag{2.17}$$

which contradicts our assumption (2.1) - (2.5), respectively. Therefore $|\ w(z)\ | < 1$ hold true for all $z \in \mathbb{U}$. Thus from (2.6) we have

$$\left|\frac{zf'(z)}{[f(z)]^{1-\beta}\,[g(z)]^{\beta}} - p\right| = (p-\alpha)\,|w(z)| < (p-\alpha) \qquad (z \in \mathbb{U})\,,$$

which implies that $f(z) \in \mathcal{B}(p, n, \beta, \alpha)$.

3 Corollaries and Consequences

The Theorem 2.1 yields many interesting and important consequences. Some of these are given here. First of all, on setting $\beta = 0$, in Theorem 2.3, we get
Corollary 3.1 If $f(z) \in \mathcal{A}_p(n)$ satisfies anyone of the following inequalities:

$$\left| \left(\frac{zf'(z)}{f(z)} \right) \left(\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + 1 \right) \right| < p - \alpha \quad z \in \mathbb{U}, \tag{3.1}$$

$$\left| \left(\frac{f(z)}{zf'(z)} \right) \left(\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + 1 \right) \right| < \frac{p-\alpha}{(2p-\alpha)^2} \quad z \in \mathbb{U}, \tag{3.2}$$

$$\left| \frac{\left(\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + 1 \right)}{\left(\frac{zf'(z)}{f(z)} - p \right)} \right| < \frac{1}{(2p-\alpha)} \quad z \in \mathbb{U}, \tag{3.3}$$

$$\left| \left(\frac{zf'(z)}{f(z)} \right) \frac{\left(\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + 1 \right)}{\left(\frac{zf'(z)}{f(z)} - p \right)} \right| < 1 \quad z \in \mathbb{U}, \tag{3.4}$$

$$\left| \left(\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + 1 \right) + \delta \left(\frac{zf'(z)}{f(z)} \right) \right| < \frac{(p-\alpha)\left((2p-\alpha)+1\right)}{(\delta 2p - \alpha)} \quad z \in \mathbb{U}, \tag{3.5}$$

then $f(z) \in \mathcal{S}^*(p, n)$.
The first four results (3.1) to (3.4) are also given recently by Prajapat [6].
Putting $\beta = 1$, in Theorem 2.3, we get
Corollary 3.2 If $f(z) \in \mathcal{A}_p(n) and \;\; g(z) \in \mathcal{S}^*(\alpha, p) \; and \; 0 \leq \alpha < p$ satisfies anyone of the following inequalities:

$$\left| \left(\frac{zf'(z)}{g(z)} \right) \left(\frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} + 1 \right) \right| < p - \alpha \quad z \in \mathbb{U}, \tag{3.6}$$

$$\left| \left(\frac{g(z)}{zf'(z)} \right) \left(\frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} + 1 \right) \right| < \frac{p-\alpha}{(2p-\alpha)^2} \quad z \in \mathbb{U}, \tag{3.7}$$

$$\left| \frac{\left(\frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} + 1 \right)}{\left(\frac{zf'(z)}{g(z)} - p \right)} \right| < \frac{1}{(2p-\alpha)} \quad z \in \mathbb{U}, \tag{3.8}$$

$$\left| \left(\frac{zf'(z)}{g(z)} \right) \frac{\left(\frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} + 1 \right)}{\left(\frac{zf'(z)}{g(z)} - p \right)} \right| < 1 \quad z \in \mathbb{U}, \tag{3.9}$$

$$\left| \left(\frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} + 1 + \delta \left(\frac{zf'(z)}{g(z)} \right) \right) \right| < \frac{(p-\alpha)\left(\delta\,(2p-\alpha)+1\right)}{(2p-\alpha)} \quad z \in \mathbb{U}, \tag{3.10}$$

then $f(z) \in \mathbb{K}(p, \gamma)$.
Next putting $n = p = 1, g(z) = z$, in Theorem 2.3, we will have
Corollary 3.3 If $f(z) \in \mathcal{A}$ satisfies anyone of the following inequalities:

$$\left|\left(\frac{z}{f(z)}\right)^{1-\beta} f'(z)\left(\frac{zf''(z)}{f'(z)} - (1-\beta)\frac{zf'(z)}{f(z)} - \beta + 1\right)\right| < 1 - \alpha \qquad z \in \mathbb{U}, \tag{3.11}$$

$$\left|\frac{\left(\frac{zf''(z)}{f'(z)} - (1-\beta)\frac{zf'(z)}{f(z)} - \beta + 1\right)}{\left(\frac{z}{f(z)}\right)^{1-\beta} f'(z)}\right| < \frac{(1-\alpha)}{(2-\alpha)^2} \qquad z \in \mathbb{U}, \tag{3.12}$$

$$\left|\frac{\left(\frac{zf''(z)}{f'(z)} - (1-\beta)\frac{zf'(z)}{f(z)} - \beta + 1\right)}{\left(\frac{z}{f(z)}\right)^{1-\beta} f'(z) - 1}\right| < \frac{1}{(2-\alpha)} \qquad z \in \mathbb{U}, \tag{3.13}$$

$$\left|\left(\frac{z}{f(z)}\right)^{1-\beta} f'(z)\left(\frac{\frac{zf''(z)}{f'(z)} - (1-\beta)\frac{zf'(z)}{f(z)} - \beta + 1}{\left(\frac{z}{f(z)}\right)^{1-\beta} f'(z) - 1}\right)\right| < 1 \qquad z \in \mathbb{U}, \tag{3.14}$$

$$\left|\frac{zf''(z)}{f'(z)} - (1-\beta)\frac{zf'(z)}{f(z)} - \beta + 1 + \delta\left(\left(\frac{z}{f(z)}\right)^{1-\beta} f'(z) - 1\right)\right| < \frac{(1-\alpha)(\delta(2-\alpha)+1)}{(2-\alpha)} \qquad z \in \mathbb{U}, \tag{3.15}$$

then $f(z) \in \mathcal{S}^*(\alpha) := \mathcal{S}^*(1, 1, \alpha)$.

Setting $n = p = \delta = 1$ and $\alpha = 0$ in Corollary 3.2, we easily obtain
Corollary 3.4 If $f(z) \in \mathcal{A}$, $g(z) \in \mathcal{S}^*(1, 1)$ satisfies anyone of the following inequalities:

$$\left|\left(\frac{zf'(z)}{g(z)}\right)\left(\frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} + 1\right)\right| < 1 \qquad z \in \mathbb{U}, \tag{3.16}$$

$$\left|\left(\frac{g(z)}{zf'(z)}\right)\left(\frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} + 1\right)\right| < \frac{1}{4} \qquad z \in \mathbb{U}, \tag{3.17}$$

$$\left|\frac{\frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} + 1}{\frac{zf'(z)}{g(z)} - 1}\right| < \frac{1}{2} \qquad z \in \mathbb{U}, \tag{3.18}$$

$$\left|\left(\frac{zf'(z)}{g(z)}\right)\left(\frac{\frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} + 1}{\frac{zf'(z)}{g(z)} - 1}\right)\right| < 1 \qquad z \in \mathbb{U}, \tag{3.19}$$

$$\left|\frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)}\left(\frac{zf'(z)}{g(z)} - 1\right)\right| < \frac{1}{2} \qquad z \in \mathbb{U}, \tag{3.20}$$

that is, $f(z) \in \mathbb{K}(0)$.
Remark: The first four results of Corollary 3.4 would yield the corresponding known result due to Prajapat [6,Corollary 4] upon setting $g(z) = f(z)$ therein.

4 Argument Properties

Theorem 4.1 *Suppose that*

$$\frac{zf'(z)}{[f(z)]^{1-\beta}[g(z)]^{\beta}} \neq \delta,$$

for $z \in \mathbb{U}, \beta \geq 0$ *and* $0 \leq \delta < p$. *If* $f(z) \in \mathcal{A}_p(n)$ *and* $g(z) \in \mathcal{S}^*(p,n)$ *satisfy*

$$\begin{aligned} -\frac{\pi}{2}\alpha_1 - \tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\alpha_1+\alpha_2)(p-\delta)}{2\gamma}\right) \\ < arg\left\{\left(\frac{zf'(z)}{[f(z)]^{1-\beta}[g(z)]^{\beta}}\right)\right. \\ \left.\left(1+\frac{zf''(z)}{f'(z)}-(1-\beta)\frac{zf'(z)}{f(z)}-\beta\frac{zg'(z)}{g(z)}+\frac{\gamma}{p-\delta}\right)-\frac{\gamma\delta}{p-\delta}\right\} \\ < \frac{\pi}{2}\alpha_2 + \tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\alpha_1+\alpha_2)(p-\delta)}{2\gamma}\right), \end{aligned} \tag{4.1}$$

for $\alpha_1, \alpha_2, \gamma > 0$, *then*

$$-\frac{\pi}{2}\alpha_1 < \arg\left(\frac{zf'(z)}{[f(z)]^{1-\beta}[g(z)]^{\beta}} - \delta\right) < \frac{\pi}{2}\alpha_2. \tag{4.2}$$

Proof: Let $f(z) \in \mathcal{A}_p(n)$ and $g(z) \in \mathcal{S}^*(p,n)$ Define a function q(z) by

$$q(z) = \frac{1}{p-\delta}\left[\frac{zf'(z)}{[f(z)]^{1-\beta}[g(z)]^{\beta}} - \delta\right], \tag{4.3}$$

Then we see that q(z) is analytic in $\mathbb{U}$, $q(0) = 1$ and $q(z) \neq 0$ for all $z \in \mathbb{U}$. it follows from (4.3) that

$$\begin{aligned} \left(\frac{zf'(z)}{[f(z)]^{1-\beta}[g(z)]^{\beta}}\right)\left(1+\frac{zf''(z)}{f'(z)}-(1-\beta)\frac{zf'(z)}{f(z)}-\beta\frac{zg'(z)}{g(z)}+\frac{\gamma}{p-\delta}\right)-\frac{\gamma\delta}{p-\delta}, \\ = (p-\delta)\,zq'(z)+\gamma q(z). \end{aligned} \tag{4.4}$$

Suppose that there exists two points $z_1, z_2 \in \mathbb{U}$ such that condition (1.6) is satisfied, then by Lemma (1.2), we obtain (1.7) under the constraint (1.8). Therefore we have

$$\arg\left(\gamma q(z_1) + (p-\delta)zq'(z_1)\right) = \arg q(z_1) + \arg\left(\gamma + (p-\delta)\frac{z_1q'(z_1)}{q(z_1)},\right)$$

$$= -\frac{\pi}{2}\alpha_1 + \arg\left(\gamma - i\frac{(\alpha_1+\alpha_2)(p-\delta)}{2}m\right),$$
$$= -\frac{\pi}{2}\alpha_1 - \tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(p-\delta)}{2\gamma}m\right),$$
$$\leq -\frac{\pi}{2}\alpha_1 - \tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\alpha_1+\alpha_2)(p-\delta)}{2\gamma}\right)$$

and

$$\arg\left(\gamma q(z_2) + (p-\delta)zq'(z_2)\right) \geq \frac{\pi}{2}\alpha_2 + \tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\alpha_1+\alpha_2)(p-\delta)}{2\gamma}\right),$$

which contradict the assumptions of the theorem. This completes the proof.
Letting $\beta = 1$ in Theorem 4.1, we have

Corollary 4.2 *Suppose that*

$$\frac{zf'(z)}{g(z)} \neq \delta$$

for $(z \in \mathbb{U})$ *and* $0 \leq \delta < p$. *If* $f(z) \in \mathcal{A}_p(n)$ *satisfies*

$$-\frac{\pi}{2}\alpha_1 - \tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\alpha_1+\alpha_2)(p-\delta)}{2\gamma}\right)$$
$$< arg\left\{\left(\frac{zf'(z)}{g(z)}\right)\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} + \frac{\gamma}{p-\delta}\right) - \frac{\gamma\delta}{p-\delta}\right\}$$
$$< \frac{\pi}{2}\alpha_2 + \tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\alpha_1+\alpha_2)(p-\delta)}{2\gamma}\right), \tag{4.5}$$

for $\alpha_1, \alpha_2, \gamma > 0$ *then*

$$-\frac{\pi}{2}\alpha_1 < \arg\left(\frac{zf'(z)}{g(z)} - \delta\right) < \frac{\pi}{2}\alpha_2. \tag{4.6}$$

Letting $\beta = 0$ or $f(z) = g(z)$ in Theorem 4.1, we have

Corollary 4.3 *Suppose that*

$$\frac{zf'(z)}{f(z)} \neq \delta$$

for $(z \in \mathbb{U})$ *and* $0 \leq \delta < p$. *if* $f(z) \in \mathcal{A}_p(n)$ *satisfies*

$$-\frac{\pi}{2}\alpha_1 - \tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\alpha_1+\alpha_2)(p-\delta)}{2\gamma}\right),$$
$$< arg\left\{\left(\frac{zf'(z)}{f(z)}\right)\left(1 + \frac{zf''(z)}{f('z)} - \frac{zf'(z)}{f(z)} + \frac{\gamma}{p-\delta}\right) - \frac{\gamma\delta}{p-\delta}\right\}$$
$$< \frac{\pi}{2}\alpha_2 + \tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\alpha_1+\alpha_2)(p-\delta)}{2\gamma}\right), \tag{4.7}$$

for $\alpha_1, \alpha_2, \gamma > 0$ *then*

$$-\frac{\pi}{2}\alpha_1 < \arg\left(\frac{zf'(z)}{f(z)} - \delta\right) < \frac{\pi}{2}\alpha_2 \tag{4.8}$$

which is also given by Frasin[1, p7, eq(3.7)].

Taking $a = \alpha_1 = \alpha_2 = 1$ in Corollary 4.3, we have

Corollary 4.4 *Suppose that*

$$\frac{zf'(z)}{f(z)} \neq \delta$$

for $(z \in \mathbb{U})$ *and* $0 \leq \delta < p$. *if* $f(z) \in \mathcal{A}_p(n)$ *satisfies*

$$\left|\arg\left\{\frac{z^2 f''(z)}{f(z)} - \left(\frac{zf'(z)}{f(z)}\right)^2 + \left(1 + \frac{\gamma}{p-\delta}\right)\frac{zf'(z)}{f(z)} - \frac{\gamma\delta}{p-\delta}\right\}\right| < \frac{\pi}{2} + \tan^{-1}\left(\frac{p-\delta}{\gamma}\right) \tag{4.9}$$

for $\gamma > 0$, *then* $f(z) \in \mathcal{S}^*(p, n, \delta)$

Taking $a = \alpha_1 = \alpha_2 = 1, p = n = 1$ and $\delta = 0$ in Theorem 4.1, we have

Corollary 4.5 If $f(z) \in \mathcal{A}$ satisfies

$$\left|\arg\left\{\left(\frac{zf'(z)}{[f(z)]^{1-\beta}[g(z)]^{\beta}}\right)\left(1 + \frac{zf''(z)}{f'(z)} - (1-\beta)\frac{zf'(z)}{f(z)} - \beta\frac{zg'(z)}{g(z)} + \gamma\right)\right\}\right| < \frac{\pi}{2}\alpha + \tan^{-1}\frac{\alpha}{\gamma}, \tag{4.10}$$

for $\gamma > 0$, then $f(z) \in \mathbb{B}(1, 1, 0)$

Remark: The Corollary 4.5 would yield the corresponding known result due to Frasin [1, p8, eq(3.12)], upon setting $g(z) = f(z)$, or $\beta = 0$ there in.

5 Open problem

In Corollary 4.5 , if we take $g(z) = z$, $\beta = -\eta$ and $0 < \eta < 1$, then we obtain a class of Non-Bazilevic´ functions (see [6]). Then our open problem is that is it possible to find sufficient conditions and argument properties for Non-Bazilevic´ functions ?

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