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Starlikeness and Convexity for Analytic Functions Concerned With Jack's Lemma

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Abstract

There are many results for sufficient conditions of functions $f(z)$ which are analytic in the open unit disc $\mathbb{U}$ to be starlike and convex in $\mathbb{U}$. The object of the present paper is to derive some interesting sufficient conditions for $f(z)$ to be starlike of order α and convex of order α in $\mathbb{U}$ concerned with Jack's lemma. Some examples for our results are also considered with the help of Mathematica 5.2.

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1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ that are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, so that $f(0) = f'(0) - 1 = 0$.

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of univalent functions $f(z)$ in $\mathbb{U}$. Let $\mathcal{S}^*(\alpha)$ be the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U})$$

for some $0 \leqq \alpha < 1$. A function $f(z) \in \mathcal{S}^*(\alpha)$ is sais to be starlike of order α in $\mathbb{U}$. We denote by $\mathcal{S}^* = \mathcal{S}^*(0)$.
Also, let $\mathcal{K}(\alpha)$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy

$$\text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (z \in \mathbb{U})$$

for some $0 \leqq \alpha < 1$. A function $f(z)$ in $\mathcal{K}(\alpha)$ is said to be convex of order α in $\mathbb{U}$. We say that $\mathcal{K} = \mathcal{K}(0)$. From the definitions for $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$, we know

that $f(z) \in \mathcal{K}(\alpha)$ if and only if $zf'(z) \in \mathcal{S}^*(\alpha)$.
Let $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$. Then $f(z)$ is said to be subordinate to $g(z)$ if there exists an analytic function $w(z)$ in $\mathbb{U}$ satisfying $w(0) = 0$, $|w(z)| < 1$ $(z \in \mathbb{U})$ and $f(z) = g(w(z))$. We denote this subordination by

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}).$$

The basic tool in proving our results is the following lemma due to Jack [1] (also, due to Miller and Mocanu [2]).

Lemma 1 *Let $w(z)$ be analytic in $\mathbb{U}$ with $w(0) = 0$. Then if $|w(z)|$ attains its maximum value on the circle $|z| = r$ at a point $z_0 \in \mathbb{U}$, then we have $z_0 w'(z_0) = kw(z_0)$, where $k \geqq 1$ is a real number.*

2 Main results

Applying Lemma 1, we drive the following result.

Theorem 1 *If $f(z) \in \mathcal{A}$ satisfies*

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{\alpha+1}{2(\alpha-1)} \qquad (z \in \mathbb{U})$$

for some α $(2 \leqq \alpha < 3)$, or

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{5\alpha-1}{2(\alpha+1)} \qquad (z \in \mathbb{U})$$

for some α $(1 < \alpha \leqq 2)$, then

$$\frac{zf'(z)}{f(z)} \prec \frac{\alpha(1-z)}{\alpha - z} \qquad (z \in \mathbb{U})$$

and

$$\left|\frac{zf'(z)}{f(z)} - \frac{\alpha}{\alpha+1}\right| < \frac{\alpha}{\alpha+1} \qquad (z \in \mathbb{U}).$$

This implies that $f(z) \in \mathcal{S}^*$ and $\displaystyle\int_0^z \frac{f(t)}{t} dt \in \mathcal{K}$.

Proof. Let us define the function $w(z)$ by

$$\frac{zf'(z)}{f(z)} = \frac{\alpha(1-w(z))}{\alpha - w(z)} \qquad (w(z) \neq \alpha).$$

Clearly, $w(z)$ is analytic in $\mathbb{U}$ and $w(0) = 0$. We want to prove that $|w(z)| < 1$ in $\mathbb{U}$. Since

$$1 + \frac{zf''(z)}{f'(z)} = \frac{\alpha(1 - w(z))}{\alpha - w(z)} - \frac{zw'(z)}{1 - w(z)} + \frac{zw'(z)}{\alpha - w(z)},$$

we see that

$$\begin{aligned}\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) &= \operatorname{Re}\left(\frac{\alpha(1 - w(z))}{\alpha - w(z)} - \frac{zw'(z)}{1 - w(z)} + \frac{zw'(z)}{\alpha - w(z)}\right) \\ &< \frac{\alpha + 1}{2(\alpha - 1)} \qquad (z \in \mathbb{U})\end{aligned}$$

for $2 \leqq \alpha < 3$, and

$$\begin{aligned}\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) &= \operatorname{Re}\left(\frac{\alpha(1 - w(z))}{\alpha - w(z)} - \frac{zw'(z)}{1 - w(z)} + \frac{zw'(z)}{\alpha - w(z)}\right) \\ &< \frac{5\alpha - 1}{2(\alpha + 1)} \qquad (z \in \mathbb{U})\end{aligned}$$

for $1 < \alpha \leqq 2$. If there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leqq |z_0|} |w(z)| = |w(z_0)| = 1,$$

then Lemma 1 gives us that $w(z_0) = e^{i\theta}$ and $z_0 w'(z_0) = k w(z_0)$, $k \geqq 1$. Thus we have

$$\begin{aligned}1 + \frac{z_0 f''(z_0)}{f'(z_0)} &= \frac{\alpha(1 - w(z_0))}{\alpha - w(z_0)} - \frac{z_0 w'(z_0)}{1 - w(z_0)} + \frac{z_0 w'(z_0)}{\alpha - w(z_0)} \\ &= \alpha + \alpha(1 - \alpha + k)\frac{1}{\alpha - e^{i\theta}} - \frac{k}{1 - e^{i\theta}}.\end{aligned}$$

If follows that

$$\begin{aligned}\operatorname{Re}\left(\frac{1}{\alpha - w(z_0)}\right) &= \operatorname{Re}\left(\frac{1}{\alpha - e^{i\theta}}\right) \\ &= \frac{1}{2\alpha} + \frac{\alpha^2 - 1}{2\alpha(1 + \alpha^2 - 2\cos\theta)}\end{aligned}$$

and

$$\begin{aligned} \operatorname{Re}\left(\frac{1}{1-w(z_0)}\right) &= \operatorname{Re}\left(\frac{1}{1-e^{i\theta}}\right) \\ &= \frac{1}{2}. \end{aligned}$$

Therefore, we have

$$\operatorname{Re}\left(1+\frac{z_0 f''(z_0)}{f'(z_0)}\right) = \frac{1+\alpha}{2} + \frac{(\alpha^2-1)(1-\alpha+k)}{2(1+\alpha^2-2\alpha\cos\theta)}.$$

This implies that, for $2 \leqq \alpha < 3$,

$$\begin{aligned} \operatorname{Re}\left(1+\frac{z_0 f''(z_0)}{f'(z_0)}\right) &\geqq \frac{1+\alpha}{2} + \frac{(\alpha+1)(1-\alpha+k)}{2(\alpha-1)} \\ &\geqq \frac{1+\alpha}{2} + \frac{(\alpha+1)(2-\alpha)}{2(\alpha-1)} \\ &= \frac{\alpha+1}{2(\alpha-1)} \end{aligned}$$

and, for $1 < \alpha \leqq 2$,

$$\begin{aligned} \operatorname{Re}\left(1+\frac{z_0 f''(z_0)}{f'(z_0)}\right) &\geqq \frac{1+\alpha}{2} + \frac{(\alpha-1)(1-\alpha+k)}{2(\alpha+1)} \\ &\geqq \frac{1+\alpha}{2} + \frac{(\alpha-1)(2-\alpha)}{2(\alpha+1)} \\ &= \frac{5\alpha-1}{2(\alpha+1)}. \end{aligned}$$

This contradicts the condition in the theorem. Therefore, there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = 1$ for all $z \in \mathbb{U}$, that is, that

$$\frac{zf'(z)}{f(z)} \prec \frac{\alpha(1-z)}{\alpha-z} \qquad (z \in \mathbb{U}).$$

Furthermore, since

$$w(z) = \frac{\alpha\left(\dfrac{zf'(z)}{f(z)} - 1\right)}{\dfrac{zf'(z)}{f(z)} - \alpha} \qquad (z \in \mathbb{U})$$

and $|w(z)| < 1$ $(z \in \mathbb{U})$, we conclude that

$$\left| \frac{zf'(z)}{f(z)} - \frac{\alpha}{\alpha+1} \right| < \frac{\alpha}{\alpha+1} \qquad (z \in \mathbb{U}),$$

which implies that $f(z) \in \mathcal{S}^*$. Furthermore, we see that $f(z) \in \mathcal{S}^*$ if and only if $\int_0^z \frac{f(t)}{t} dt \in \mathcal{K}$. □

Thaking $\alpha = 2$ in the theorem, we have following corollary due to R. Singh and S. Singh [3].

Corollary 1 *If $f(z) \in \mathcal{A}$ satisfies*

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{3}{2} \qquad (z \in \mathbb{U}),$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{2(1-z)}{2-z} \qquad (z \in \mathbb{U})$$

and

$$\left| \frac{zf'(z)}{f(z)} - \frac{2}{3} \right| < \frac{3}{2} \qquad (z \in \mathbb{U}).$$

With Theorem 1, we give the following example.

Example 1 For $2 \leqq \alpha < 3$, weconsider the function $f(z)$ given by

$$f(z) = \frac{\alpha-1}{2}\left(1 - (1-z)^{\frac{2}{\alpha-1}}\right) \qquad (z \in \mathbb{U}).$$

If follows that

$$\frac{zf'(z)}{f(z)} = \frac{2z(1-z)^{\frac{3-\alpha}{\alpha-1}}}{(\alpha-1)\left(1-(1-z)^{\frac{2}{\alpha-1}}\right)} \qquad (z \in \mathbb{U})$$

and

$$\begin{aligned} \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) &= \mathrm{Re}\left(\frac{\alpha-1-2z}{(\alpha-1)(1-z)}\right) \\ &= \mathrm{Re}\left(\frac{2}{\alpha-1} - \frac{3-\alpha}{(\alpha-1)(1-z)}\right) \\ &< \frac{\alpha+1}{2(\alpha-1)} \qquad (z \in \mathbb{U}). \end{aligned}$$

Therefore, the function $f(z)$ satisfies the condition in Theorem 1. If we define the function $w(z)$ by

$$\frac{zf'(z)}{f(z)} = \frac{\alpha(1-w(z))}{\alpha - w(z)} \qquad (w(z) \neq \alpha),$$

then we see that $w(z)$ is analytic in $\mathbb{U}$, $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$ with Mathematica 5.2. This implies that

$$\frac{zf'(z)}{f(z)} \prec \frac{\alpha(1-z)}{\alpha - z} \qquad (z \in \mathbb{U}).$$

For $1 < \alpha \leqq 2$, we consider

$$f(z) = \frac{\alpha+1}{2(2\alpha-1)}\left(1-(1-z)^{\frac{2(2\alpha-1)}{\alpha+1}}\right) \qquad (z \in \mathbb{U}).$$

Then we have that

$$\frac{zf'(z)}{f(z)} = \frac{2(2\alpha-1)z(1-z)^{\frac{3(\alpha-1)}{\alpha+1}}}{(\alpha+1)\left(1-(1-z)^{\frac{2(2\alpha-1)}{\alpha+1}}\right)}$$

and

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) = \operatorname{Re}\left(\frac{\alpha+1-2(2\alpha-1)z}{(\alpha+1)(1-z)}\right) < \frac{5\alpha-1}{2(\alpha+1)} \qquad (z \in \mathbb{U}).$$

Thus, the function $f(z)$ satisfies the condition in Theorem 1. Define the function $w(z)$ by

$$\frac{zf'(z)}{f(z)} = \frac{\alpha(1-w(z))}{\alpha - w(z)} \qquad (w(z) \neq \alpha).$$

Then $w(z)$ is analytic in $\mathbb{U}$, $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$ with Mathematica 5.2. Therefour, we have that

$$\frac{zf'(z)}{f(z)} \prec \frac{\alpha(1-z)}{\alpha - z} \qquad (z \in \mathbb{U}).$$

In particular, if we take $\alpha = 2$ in this example, then $f(z)$ becomes

$$f(z) = z - \frac{1}{2}z^2 \in \mathcal{S}^*,$$

where $\mathcal{S}^*$ denotes the class of all starlike function in $\mathbb{U}$.

Theorem 2 *If $f(z) \in \mathcal{A}$ satisfies*

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > -\frac{\alpha+1}{2\alpha(\alpha-1)} \qquad (z \in \mathbb{U}) \tag{2.1}$$

for some α $(\alpha \leqq -1)$*, or*

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \frac{3\alpha+1}{2\alpha(\alpha+1)} \qquad (z \in \mathbb{U}) \tag{2.2}$$

for some α $(\alpha > 1)$*, then*

$$\frac{f(z)}{zf'(z)} \prec \frac{\alpha(1-z)}{\alpha-z} \qquad (z \in \mathbb{U})$$

and

$$f(z) \in \mathcal{S}^*\left(\frac{\alpha+1}{2\alpha}\right).$$

This implies that $\displaystyle\int_0^z \frac{f(t)}{t}dt \in \mathcal{K}\left(\frac{\alpha+1}{2\alpha}\right)$.

Proof. Let us define the function $w(z)$ by

$$\frac{f(z)}{zf'(z)} = \frac{\alpha(1-w(z))}{\alpha-w(z)} \qquad (w(z) \neq \alpha). \tag{2.3}$$

Then, we have that $w(z)$ is analytic in $\mathbb{U}$ and $w(0)=0$. We want to prove that $|w(z)|<1$ in $\mathbb{U}$. Differentiating (2.3) in both side logarithmically and simplifying, we obtain

$$1+\frac{zf''(z)}{f'(z)} = \frac{\alpha-w(z)}{\alpha(1-w(z))} + \frac{zw'(z)}{1-w(z)} - \frac{zw'(z)}{\alpha-w(z)},$$

and, hence

$$\begin{aligned}\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) &= \operatorname{Re}\left(\frac{\alpha-w(z)}{\alpha(1-w(z))} + \frac{zw'(z)}{1-w(z)} - \frac{zw'(z)}{\alpha-w(z)}\right)\\ &> -\frac{\alpha+1}{2\alpha(\alpha-1)} \qquad (z \in \mathbb{U})\end{aligned}$$

for $\alpha \leqq -1$, or

$$\begin{aligned}\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) &= \operatorname{Re}\left(\frac{\alpha-w(z)}{\alpha(1-w(z))} + \frac{zw'(z)}{1-w(z)} - \frac{zw'(z)}{\alpha-w(z)}\right)\\ &> \frac{3\alpha+1}{2\alpha(\alpha+1)} \qquad (z \in \mathbb{U})\end{aligned}$$

for $\alpha > 1$. If there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leqq |z_0|} |w(z)| = |w(z_0)| = 1,$$

then Lemma 1 gives us that $w(z_0) = e^{i\theta}$ and $z_0 w'(z_0) = k w(z_0),\ k \geqq 1$. Thus we have

$$\begin{aligned} 1 + \frac{z_0 f''(z_0)}{f'(z_0)} &= \frac{\alpha - w(z_0)}{\alpha(1 - w(z_0))} + \frac{z_0 w'(z_0)}{1 - w(z_0)} - \frac{z_0 w'(z_0)}{\alpha - w(z_0)} \\ &= \frac{1}{\alpha} + \frac{\alpha - 1}{\alpha(1 - e^{i\theta})} + \frac{k}{1 - e^{i\theta}} - \frac{k\alpha}{\alpha - e^{i\theta}}. \end{aligned}$$

Therefore, we have

$$\operatorname{Re}\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right) = \frac{1}{2} + \frac{1}{2\alpha} - \frac{k(\alpha^2 - 1)}{2(1 + \alpha^2 - 2\alpha\cos\theta)}.$$

This implies that, for $\alpha \leqq -1$,

$$\begin{aligned} \operatorname{Re}\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right) &\leqq \frac{1}{2} + \frac{1}{2\alpha} - \frac{k(\alpha + 1)}{2(\alpha - 1)} \\ &\leqq \frac{1}{2} + \frac{1}{2\alpha} - \frac{\alpha + 1}{2(\alpha - 1)} \\ &= -\frac{\alpha + 1}{2\alpha(\alpha - 1)}. \end{aligned}$$

and, for $\alpha > 1$,

$$\begin{aligned} \operatorname{Re}\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right) &\leqq \frac{1}{2} + \frac{1}{2\alpha} - \frac{k(\alpha - 1)}{2(\alpha + 1)} \\ &\leqq \frac{1}{2} + \frac{1}{2\alpha} - \frac{\alpha - 1}{2(\alpha + 1)} \\ &= \frac{3\alpha + 1}{2\alpha(\alpha + 1)}. \end{aligned}$$

This contradicts the condition in the theorem. Therefore, there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = 1$. This means that $|w(z)| < 1$ for all $z \in \mathbb{U}$, this is, that

$$\frac{f(z)}{z f'(z)} \prec \frac{\alpha(1 - z)}{\alpha - z} \qquad (z \in \mathbb{U}).$$

Furthermore, since

$$w(z) = \frac{\alpha\left(1 - \dfrac{zf'(z)}{f(z)}\right)}{1 - \alpha\dfrac{zf'(z)}{f(z)}} \qquad (z \in \mathbb{U})$$

and $|w(z)| < 1$ $(z \in \mathbb{U})$, we conclude that

$$f(z) \in \mathcal{S}^*\left(\frac{\alpha+1}{2\alpha}\right).$$

Noting that $f(z) \in \mathcal{S}^*(\alpha)$ if and only if $\displaystyle\int_0^z \frac{f(t)}{t}dt \in \mathcal{K}(\alpha)$, we complete the proof of the theorem. □

For Theorem 2, we give the following example.

Example 2 For $\alpha > 1$, we take

$$f(z) = \frac{\alpha(\alpha+1)}{-\alpha^2+2\alpha+1}\left(1 - (1-z)^{\frac{-\alpha^2+2\alpha+1}{\alpha(\alpha+1)}}\right) \qquad (z \in \mathbb{U}).$$

Then, $f(z)$ satisfies

$$\frac{zf'(z)}{f(z)} = \frac{(-\alpha^2+2\alpha+1)z}{\alpha(\alpha+1)(1-z)^{\frac{2\alpha^2-\alpha-1}{\alpha(\alpha+1)}}\left(1-(1-z)^{\frac{-\alpha^2+2\alpha+1}{\alpha(\alpha+1)}}\right)}$$

and

$$\begin{aligned} \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) &= \operatorname{Re}\left(\frac{\alpha(\alpha+1) + (\alpha^2-2\alpha-1)z}{\alpha(\alpha+1)(1-z)}\right) \\ &> \frac{3\alpha+1}{2\alpha(\alpha+1)} \qquad (z \in \mathbb{U}). \end{aligned}$$

Therefore, $f(z)$ satisfies the condition of Theorem 2. Let us define the function $w(z)$ by

$$\frac{f(z)}{zf'(z)} = \frac{\alpha(1-w(z))}{\alpha - w(z)} \qquad (w(z) \neq \alpha).$$

Then $w(z)$ is analytic in $\mathbb{U}$, $w(0) = 0$ and $|w(z)| < 1$ $z \in \mathbb{U}$ with Mathematica 5.2. It follows that

$$\frac{f(z)}{zf'(z)} \prec \frac{\alpha(1-z)}{\alpha - z} \qquad (z \in \mathbb{U}).$$

Furthermore, for $\alpha \leqq -1$, we consider the following function

$$f(z) = -\frac{\alpha(\alpha-1)}{\alpha^2+1}\left(1-(1-z)^{-\frac{\alpha^2+1}{\alpha(\alpha-1)}}\right).$$

Note that

$$\frac{zf'(z)}{f(z)} = \frac{-(\alpha^2+1)z}{\alpha(\alpha-1)(1-z)^{\frac{2\alpha^2-\alpha+1}{\alpha(\alpha-1)}}\left(1-(1-z)^{-\frac{\alpha^2+1}{\alpha(\alpha-1)}}\right)}$$

and

$$\begin{aligned}\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) &= \mathrm{Re}\left(\frac{\alpha(\alpha-1)+(\alpha^2+1)z}{\alpha(\alpha-1)(1-z)}\right)\\ &> \frac{\alpha+1}{2\alpha(\alpha-1)} \qquad (z \in \mathbb{U}).\end{aligned}$$

This implies that $f(z)$ satisfies the condition of Theorem 2. Definning the function $w(z)$ by

$$\frac{f(z)}{zf'(z)} = \frac{\alpha(1-w(z))}{\alpha-w(z)} \qquad (w(z) \neq \alpha),$$

we see that $w(z)$ is analytic in $\mathbb{U}$, $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$ with Mathematica 5.2. Thus we have that

$$\frac{f(z)}{zf'(z)} \prec \frac{\alpha(1-z)}{\alpha-z} \qquad (z \in \mathbb{U}).$$

Making $\alpha = -1$ for $f(z)$, we have

$$f(z) = \frac{z}{1-z} \in \mathcal{K}.$$

3 Open Question

As we say in Example 1, we need to use Mathematica 5.2 to check that $|w(z)| < 1$ $(z \in \mathbb{U})$ for

$$\frac{zf'(z)}{f(z)} = \frac{\alpha(1-w(z))}{\alpha-w(z)} \qquad (w(z) \neq \alpha).$$

Because, it is not so easy to calculate the fact that $|w(z)| < 1$ $(z \in \mathbb{U})$ in this case. If $\alpha = 2$ in Example 1, then we see that $|w(z)| = |z| < 1$.

Also, in Example 2, we use Mathematica 5.2 to see that $|w(z)| < 1$ $(z \in \mathbb{U})$. If $\alpha = -1$ in Example 2, then we know that $|w(z)| = |z| < 1$. Thus we have to leave our open questions to prove $|w(z)| < 1$ $(z \in \mathbb{U})$ without Mathematica 5.2. Can we prove that $|w(z)| < 1$ for all $z \in \mathbb{U}$ without Mathematica 5.2 in Example 1 and Example 2?

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