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Direct Methods for the Solution of Singular Integral Equations with Finite Number of Different Zeros in Pairwise

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Abstract

We obtain the numerical schemes of collocation methods and mechanical quadratic methods to approximate the solutions of the singular integral equations. The equations are defined on the arbitrary smooth closed contour of the complex plane. Theoretical background for these methods is proved in classical Hölder spaces in the case when singular integral equations have finite number of different zeros in pairwise.

Keywords: *Singular Integral Equations, Collocation Methods, Mechanical Quadratic Methods*

1 Introduction

Singular integral equations with Cauchy kernels (SIE) are used to model many problems in elasticity theory, aerodynamics, mechanics, thermoelasticity, queuing system analysis, etc.[1]-[3]. The general theory of SIE has been widely investigated in the last decades [5]-[7]. It is known that the exact solution for SIE is possible in some particular cases. That is why there is a necessity to elaborate numerical methods for solving of SIE with corresponding theoretical background. The problem for approximate solution of SIE by collocation methods and mechanical quadratic methods has been studied in [6]. The equations are defined on the unit circle.

However, the case when the contour of integration can be an arbitrary smooth closed curve (not unit circle) has not been studied enough. We note that the theoretical background of collocation methods and mechanical quadratic methods was studied in [8]-[11].

In this article we study the collocation methods and mechanical quadratic methods to approximate the solutions of SIE with finite number of different zeros in pairwise.

2 The main definitions and notations

Let Γ be an arbitrary smooth closed contour, bounding a simple-connected region F^+ containing a point $t = 0$, $F^- = C \setminus \{F^+ \cup \Gamma\}$, C is a complex plane.

Let $z = \psi(w)$ be a Riemann function, mapping conformably outside of the unit circle $\{|w| = 1\}$ on the domain F^-, so that $\psi(\infty) = \infty, \psi^{(\prime)}(\infty) = 1$. The class of this contours we denote by Λ.

We denote $H_\beta(\Gamma)$ the complex spaces of functions satisfying on Γ the Hölder condition with some exponent β $(0 < \beta < 1)$ and with norm

$$\|g\|_\beta = \|g\|_C + H(g, \beta), \tag{1}$$

$$H(g,\beta) = \sup_{t' \neq t''} \frac{\left|g(t'') - g(t')\right|}{|t' - t''|^\beta}, t^{'}, t^{''} \in \Gamma.$$

We consider that function $g(t)$ belongs to class $H_\beta^{(q)}(\Gamma)$ $q = 0, 1, \ldots,$ if it has derivatives order q inclusive and $g^{(q)} \in H_\beta(\Gamma)$.

By $H_\beta^{(q)}(\Gamma)$ $q \geq 0$ $H_\beta^{(0)}(\Gamma) = H_\beta(\Gamma)$ we denote the space of q times continuously-differentiable functions. The derivatives of the qth order for these functions are elements of space $H_\beta(\Gamma)$. The norm on $H_\beta^{(q)}(\Gamma)$ is given by the formula:

$$||g||_{\beta,q} = \sum_{k=0}^{q} ||g^{(k)}||_c + H(g^{(q)}, \beta).$$

Let U_n be the Lagrange interpolating polynomial operator constructed on the points $\{t_j\}_{j=0}^{2n}$ for any continuous function on Γ by the formula:

$$(U_n g)(t) = \sum_{s=0}^{2n} g(t_s) \cdot l_s(t),$$
$$l_j(t) = \prod_{k=0, k\neq j}^{2n} \frac{t - t_k}{t_j - t_k} \left(\frac{t_j}{t}\right)^n \equiv \sum_{k=-n}^{n} \Lambda_k^{(j)} t^k, \tag{2}$$
$$t \in \Gamma, \quad j = \overline{0, 2n}.$$

The following quadrature formula holds[8]:

$$\frac{1}{2\pi i} \int_\Gamma g(\tau)\tau^k d\tau \cong \frac{1}{2\pi i} \int_\Gamma U_n(\tau) \cdot g(\tau)\tau^{k-1} d\tau, \tag{3}$$

where $k = -n, \dots, n$.

3 Numerical schemes of methods

We study the singular integral equation

$$(M\varphi \equiv) c(t)\cdot\varphi(t) + \frac{d(t)}{\pi i}\cdot\int_\Gamma \frac{\varphi(\tau)}{\tau - t}d\tau + \frac{1}{2\pi i}\int_\Gamma K(t,\tau)\varphi(\tau)d\tau = f(t),\ t\in\Gamma,\ (4)$$

where $c(t), d(t), f(t)$ and $K(t,\tau)$ are known functions on Γ and $\Gamma\times\Gamma$ respectively, $\varphi(t)$ is unknown function.

We consider that the symbols of equation (4) $a(t) = c(t)+d(t)$, $b(t) = c(t)-d(t)$ on Γ have the finite number of different zeros in pairwise $\alpha_1, \alpha_2, \dots, \alpha_p$, $\beta_1, \beta_2, \dots, \beta_s$ with integer multiplicities $m_1, m_2, \dots m_p$ and $n_1, n_2, \dots, n_s$ respectively and the functions $a(t)$ $b(t)$ permit representation

$$a(t) = R_-(t)\cdot a_0(t), \quad b(t) = R_+(t)\cdot b_0(t)$$

where $a_0(t)\cdot b_0(t) \neq 0$, $t\in\Gamma$ and $R_\pm(t)$ is polynomial functions by t and t^{-1} respectively:

$$R_+(t) = \prod_{j=1}^{p}(t-\alpha_j)^{m_j}, \quad R_-(t) = \prod_{k=1}^{s}\left(\frac{1}{t}-\frac{1}{\beta_k}\right)^{n_k}$$

We suppose

$$l = \max\{m_1, m_2, \dots, m_p; n_1, n_2, \dots, n_s\}. \quad (5)$$

We search for the approximate solution of equation (4) in polynomial form

$$\varphi_n(t) = \sum_{k=-n}^{n}\xi^{(n)}t^k, \quad t\in\Gamma, \quad (6)$$

with unknown coefficients $\xi_k = \xi_k^{(n)}$,$k = -n, \dots, n$.

According to the collocation method the unknown coefficients ξ_k we determine from system of linear algebraic equations (SLAE)

$$\sum_{k=-n}^{n}[a(t_j)\cdot t_j^k\cdot \text{sign}(k) + b(t_j)\cdot t_j^k\text{sign}(-k) +$$

$$+\frac{1}{2\pi i}\int_\Gamma K(t_j,\tau)\tau^k d\tau]\cdot\xi_k = f(t_j), \quad j = 0, \dots, 2n, \quad (7)$$

$\text{sign}(k) = 1, k \geq 0$;sign $(k) = 0, k < 0$.
We approximate the integrals in (7) using the quadrature formula (3).

Thus we obtain the following SLAE from (7)

$$\sum_{k=-n}^{n}[a(t_j)\cdot t_j^k\cdot sign(k)+b(t_j)\cdot t_j^k\cdot sign(-k)+$$
$$+\sum_{r=0}^{2n}t_r\cdot K(t_j,t_r)\cdot\Lambda_{-k}^{(r)}]\xi_k=f(t_j),\quad j=0,\dots,2n, \tag{8}$$

where $\{t_j\}_{j=0}^{2n}$ form a distinct set on Γ. We determine the numbers $\Lambda_j^{(r)}$ from relation (2).

Theorem 1. *Let the following conditions be satisfied:*

1. $\Gamma\in\Lambda$;
2. *the functions* $a_0(t)\neq 0$ *and* $b_0(t)\neq 0$ *belong to the class* $H_\alpha^{(l)}(\Gamma),\alpha\in(0,1]$, *on* Γ, *ind* $b_0^{-1}(t)\cdot a_0(t)=0, t\in\Gamma$; l *is a number from (5).*
3. $K(t,\tau)\in H_\alpha^{(l)}(\Gamma\times\Gamma)$, $f(t)\in H_\alpha^{(l)}(\Gamma)$;
4. *dim Ker* $M=0$;
5. $0<\beta<\alpha$;
6. $t_j(j=0,\dots,2n)$ *form a set of Fejér points on* Γ :
$$t_j=\psi\left(exp\frac{2\pi i}{2n+1}\cdot j\right),j=0,\dots,2n.$$

Then starting from indices $n\geq n_1$ (n_1 *depends from coefficients and right part of equation (4)) the SLAE (7)and SLAE (8) have an unique solution* $\xi_k, k=-n,\dots,n$. *The approximate solutions* $\varphi_n(t)$ *given by formula (6) converge in the norm of space* $H_\beta(\Gamma)$ *to the exact solution* $\varphi(t)$ *of equation (4).*

The following estimation is true

$$||\varphi-\varphi_n||=O\left(\frac{\ln n}{n^{\alpha-\beta}}\right)\textit{for collocation method} \tag{9}$$

$$||\varphi-\varphi_n||=O\left(\frac{\ln^2 n}{n^{\alpha-\beta}}\right)\textit{for mechanical quadratic method.} \tag{10}$$

4 Proof of theorem 1

Let

$$b_0^{-1}(t)\cdot a_0(t)=V_-(t)\cdot V_+(t),\quad t\in\Gamma$$

is canonical form of factorization for function $b_0^{-1}(t)\cdot a_0(t)$.

Using the Riesz operators $P=\frac{1}{2}(I+S)$, $Q=I-P$, (I is an identical operator and S is a singular operator (with Cauchy kernel)) and introducing the notation $R=PR_-+QR_+$, we have

$$c(t)I+d(t)\cdot S=a(t)P+b(t)Q=$$

$$=R_-a_0P+R_+b_0Q=b_0V_+\cdot[(PV_-+QV_+^{-1})R+K_1],$$

$$K_1=QV_-PR+PV_+^{-1}QR+V_-QR_-P+V_+^{-1}PR_+Q.$$

It is simple to verify that the numerical schemes (7), (8) are equivalent to the operator equations

$$U_n[(PV_-+QV_+)R+K_1+K_2]\varphi_n=U_nf_1 \tag{11}$$

and

$$U_n[(PV_-+QV_+)R+K_1]\varphi_n+U_n[b_0^{-1}V_+^{-1}\cdot$$

$$\frac{1}{2\pi i}\int\limits_\Gamma\frac{1}{\tau}U_n^\tau(\tau K(t,\tau))\varphi_n(\tau)d\tau]=U_nf_1 \tag{12}$$

where $K_2(\cdot)=b_0^{-1}V_+^{-1}\cdot\frac{1}{2\pi i}\int\limits_\Gamma K(t,\tau)(\cdot)(\tau)d\tau$, $f_1=b_0^{-1}V_+^{-1}\cdot f$, U_n is a interpolation operator by points $\{t_j\}_{j=0}^{2n}$ (U_n^τ for variable τ):

$$(U_ng)(t)=\sum_{j=0}^{2n}g(t_j)l_j(t).$$

To prove the solvability of SLAE (7) and (8) it is enough to prove the invertibility of operators defined by left parts of equations (11) and (12). The operators are reflected in the space $U_nH_\beta(\Gamma)$. The invertibility of these operators followed from Theorem 2.2 [6] where the convergence of projection methods is established for abstract integral equations. To verify the conditions of theorem we use the results from [8]:

$$\forall\Gamma\in\Lambda\forall g(t)\in H_\alpha(\Gamma),\quad ||U_ng-g||_\beta=O\left(\frac{\ln n}{n^{\alpha-\beta}}\right), \tag{13}$$

$$||U_n||=O(\ln n),\quad U_n:H_\beta\to H_\beta. \tag{14}$$

Furthermore to verify the conditions of theorem 2.2[6] from we use that dim Ker$R = 0$ and operators M and $PV_- + QV_+^{-1}$ are invertible .

We receive the estimations (9) and (10) from (13) and (14).

Remark 2. *If the coefficients and right part from equation (4) belong to* $H_\alpha^{(l+r)}$ *where (r is an integer), then the estimations (9) and (10) will be improved on the value* $O(n^{-r})$.

5 Conclusion

We proved the convergence of collocation and mechanical quadratic methods for approximate solution of singular integral equations.Theoretical background for these methods is proven in classical Hölder spaces in the case when singular integral equations have finite number of different zeros in pairwise. The classical Hölder spaces is Banach nonseparable space. Therefore the approximation in the whole class of functions by norm (1) is impossible. The problem was solved in subspace of classical Hölder spaces.

6 Open Problem

To generalize the results of this article we are going to obtain the theoretical background of collocation methods in Lebesgue spaces and Generalized Hölder spaces in the case when singular integral equations have finite number of different zeros in pairwise. The Generalized Holder spaces are nonseparable. The classical theory of projection methods for L_p does not apply because the norm of projectors (for example, projectors of interpolation), is unbounded in Lebesgue spaces. Thus it is necessary to elaborate the new theory of projection methods.

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