

Goursat Functions for a Problem of an Isotropic Plate With a Curvilinear Hole

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Abstract

Here, we used a rational mapping function with complex constants in order to study the effect of complex constants. Also, a complex variable method have been applied to deduce exact expressions for Goursat functions for the first and second fundamental problems of an infinite plate weakened by a hole having arbitrary shape. The edge of the hole is conformally mapped on the domain outside a unit circle by means of the rational mapping function. Moreover, the interesting cases when the hole takes different shapes have been considered besides applying computer work to determine strong and weak points of stress and strain components.

Keywords: Conformal mapping; Plane elasticity; Complex variable method; Curvilinear hole

1 Introduction

For many years, contact and mixed problems in the theory of elasticity has been recognized as a rich and challenging subject for study, see Atkin and Fox (1990). These problems can be established from the initial value problems or from the boundary value problems, or from the mixed problems, see Colton and Kress (1983) and Abdou (2003). Also, many different methods are established for solving the contact and mixed problems in elastic and thermoelastic problems, the books edited by Noda et al. (2003), Hetnarski (2004), Parkus (1976) and Popov (1988) contain many different methods to solve the problems in the theory of elasticity in one, two and three dimensions.

It is known that, see Muskhelishvili (1953), the first and second fundamental problems in the plane theory of elasticity are equivalent to finding two analytic functions $\phi_1(z)$ and $\psi_1(z)$ of one complex argument $z = x + iy$. These functions, Goursat functions, satisfy the boundary conditions,

$$k \phi_1(t) - t \overline{\phi_1'(t)} - \overline{\psi_1(t)} = f(t), \quad (1.1)$$

where t denotes the affix of a point on the boundary. In terms of $z = c\omega(\zeta)$, $c > 0$, $\omega'(\zeta)$ does not vanish or become infinite for $|\zeta| > 1$, the infinite region outside a closed contour is conformally mapped outside a unit circle ν .

For $k = -1$ and $f(t)$ is a given function of stress in (1.1), we have the boundary condition for the first fundamental problem (or in other words the stress boundary value problem);

$$\phi_1(t) + t \overline{\phi_1'(t)} + \overline{\psi_1(t)} = f_1(t), \quad f_1(t) = -f(t). \tag{1.2}$$

While for $k = \chi = \frac{\lambda+3\mu}{\lambda+\mu} > 1$; λ, μ are the Lamé's constants and $f(t) = 2\mu g(t)$ is a given function of displacement, we have the principal formula for the second fundamental problem (or the displacement boundary value problem);

$$\chi \phi_1(t) - t \overline{\phi_1'(t)} - \overline{\psi_1(t)} = 2\mu g(t). \tag{1.3}$$

The stress problem is ordered the first because any displacement for a body is resulted after a stress effect on this body.

In the absence of body forces, it is known from Muskhelishvili (1953) that stress components, in the plane theory of elasticity, take the form

$$\begin{aligned} \sigma_{xx} + \sigma_{yy} &= 4 \operatorname{Re} \{ \phi'(z) \}, \\ \sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} &= 2 [\bar{z} \phi''(z) + \psi'(z)], \end{aligned} \tag{1.4}$$

where the complex functions of potentials $\phi_1(z)$ and $\psi_1(z)$, take the form

$$\begin{aligned} \phi_1(z) &= -\frac{X+iY}{2\pi(1+\chi)} \ln \zeta + c\Gamma\zeta + \phi(\zeta), \\ \psi_1(z) &= \frac{\chi(X-iY)}{2\pi(1+\chi)} \ln \zeta + c\Gamma^*\zeta + \psi(\zeta), \end{aligned} \tag{1.5}$$

where X, Y are the components of the resultant vector of all external forces acting on the boundary and Γ, Γ^* are complex constants. Generally, the two complex potential functions $\phi(\zeta), \psi(\zeta)$ are single-valued analytic functions within the region outside the unit circle and $\phi(\infty) = 0, \psi(\infty) = 0$.

Muskhelishvili (1953) used the transformation

$$z = c(\zeta + m\zeta^{-1}), \quad (c > 0, m, \text{ real numbers}), \tag{1.6}$$

for solving the problem of stretching of an infinite plate weakened by an elliptic hole. Sokolonikoff (1985) used the same rational mapping of Eq.(1.6) to solve the problem of elliptical ring, where the Laurant's theorem is used. This transformation, of (1.6), conformally maps the infinite domain bounded internally by an ellipse into the domain outside the unit circle $|\zeta| = 1$ in the ζ -plane. The application of the Hilbert problem is used by Muskhelishvili (1953) to discuss the case of a stretched infinite plate weakened by a circular cut.

Kalandiya (1975) used the transformation mapping (1.6) to solve the torsion problem of an elastic beam, in two dimensions, in the theory of elasticity.

Also the same author used the same rational mapping (1.6) to solve the first fundamental problem in the theory of elasticity by using Laurant's method.

England (1980) considered an infinite plate which is weakened by a hypotrochoid hole, conformally mapped into a unit circle $|\zeta| = 1$ by the transformation mapping

$$z = c \left(\zeta + m\zeta^{-n} \right) \quad , (c > 0, 0 \leq m < \frac{1}{n}), \quad (1.7)$$

where $z'(\zeta)$ does not vanish or become infinite outside the unit circle ν and he solved the boundary value problems of the first fundamental problem. In the papers, El-Sirafy and Abdou (1984), Abdou and Khar- El din (1994), Abdou and Badr (1999), Abdou and Khamis (2000), Abdou (2002) and Abdou et al. (2002), many rational mappings are used to solve the first and second fundamental problems of an infinite plate with a curvilinear hole, using Cauchy complex variable method, where the two complex Goursat functions are obtained. Moreover, in all previous works the coefficients of the rational mappings were real.

In this paper the complex variable method will be applied to solve the first and second fundamental problems for the same previous domain of the infinite plate with a general curvilinear hole C conformally mapped on the domain outside a unit circle ν by the rational function

$$z = l\zeta + m\zeta^{-n} \quad , (|l| > 0, 0 \leq \left| \frac{m}{l} \right| < \frac{1}{n}), \quad (1.8)$$

where $l = l_1 + il_2$, $|l| > 0$, $m = m_1 + im_2$, $n = 1, 2, \dots, p$; and $\left| \frac{m}{l} \right|$ is a parameter restricted such that $z'(\zeta)$ does not vanish or become infinite outside the unit circle ν . The interesting cases when the shape of the hole with the rotating axis takes different shapes are included.

2 The rational mapping

Whereas, our present mapping function deals with famous shapes of tunnels, then it is useful to use it in studying stresses and strains around tunnels. In underground engineering the tunnel is assumed to be driven in a homogeneous, isotropic, linear elastic and pre-stressed geometrical situation. Also, the tunnel is considered to be deep enough such that the stress distribution before excavation is homogeneous. Excavating underground openings in soils and rocks are done for several purposes and in multi-sizes. At least, excavation of the opening will cause the soil or rock to deform elastically.

Worth mentioning, that excavation in soil or rock is a complicated, dangerous, and expensive process. The mechanics of this can be very complex. However, the use of conformal mapping that allows us to study stresses and

strains around a unit circle makes it useful for engineers and easier for mathematicians.

The physical interest of the mapping (1.8) comes from its different shapes of holes it treats where we find from Fig's. 1-6 the following notations;

- The number of the holes corners is subjected to n 's values. They are given by $n + 1$.
- The complex constant m works on circling the shape from its situation in the case of real m and the circling angle is given by $\theta = \tan^{-1} \frac{m_2}{m_1}$ ($m = m_1 + im_2$). Positive values of θ means that the circling will be in the positive direction i.e. in the anticlockwise direction and for negative values the circling will be in the negative direction i.e. in clockwise direction.
- The complex constant l works on expanding the corners of the hole shape.

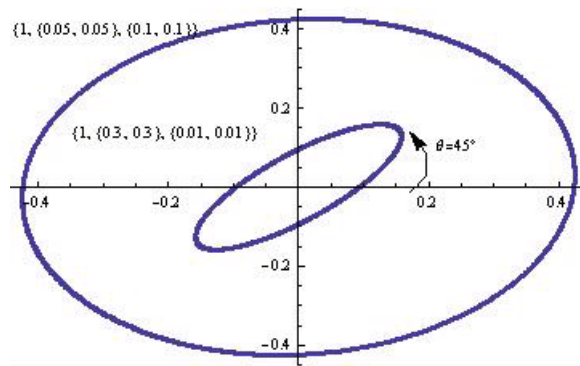


Fig. 1 ($z = l\zeta + m\zeta^{-1}$)

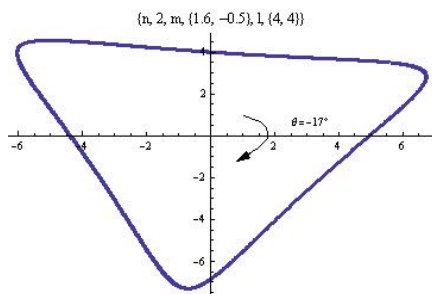


Fig. 2 ($z = l\zeta + m\zeta^{-2}$)

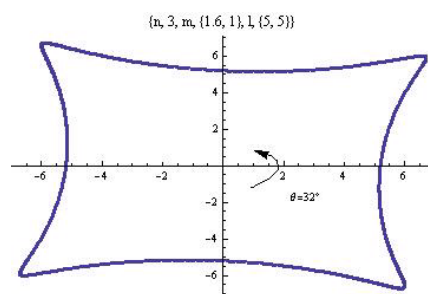


Fig. 3 ($l\zeta + m\zeta^{-3}$)

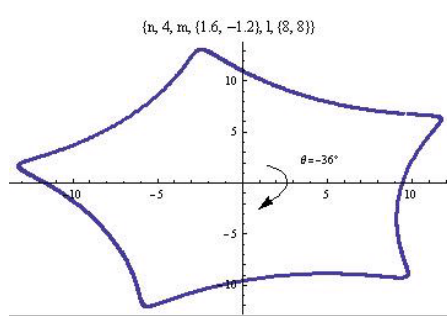


Fig. 4 ($z = l\zeta + m\zeta^{-4}$)

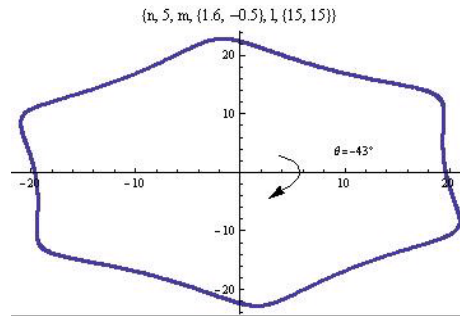


Fig. 5 ($z = l\zeta + m\zeta^{-5}$)

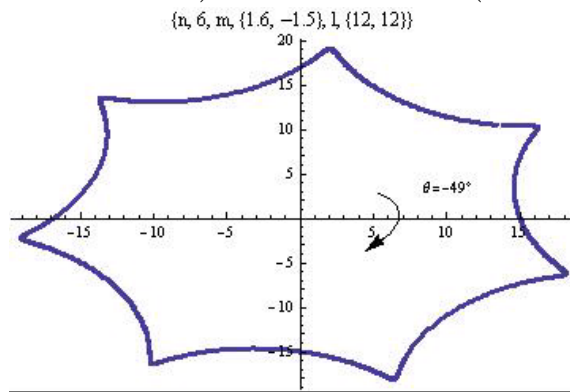


Fig. 6 ($z = l\zeta + m\zeta^{-6}$)

3 Method of solution

In this section, we will use the transformation mapping (1.8) in the boundary condition (1.1), then we will apply the complex variable method with the residue theorems to obtain a closed expression for Goursat functions. Therefore, the expression $\omega(\zeta) / \omega'(\zeta^{-1})$ will be written in the form

$$\frac{\omega(\zeta)}{\omega'(\zeta^{-1})} = \alpha(\zeta) + \overline{\beta(\zeta)}, \tag{3.1}$$

where

$$\alpha(\zeta) = \frac{h}{\zeta^n}, \quad h = \frac{ml}{|l|^2}, \tag{3.2}$$

and $\beta(\zeta)$ is a regular function for $|\zeta| > 1$.

Using (3.1) in the boundary condition (1.1) and on $\zeta = \sigma$, for generality, we get

$$k\phi(\sigma) - \alpha(\sigma) \overline{\phi'(\sigma)} - \overline{\psi_*(\sigma)} = f_*(\sigma), \tag{3.3}$$

where

$$\psi_*(\sigma) = \psi(\sigma) + \beta(\sigma) \phi'(\sigma), \tag{3.4a}$$

$$f_*(\sigma) = F(\sigma) - lk\Gamma\sigma + \frac{\bar{l}\Gamma^*}{\sigma} - \frac{h(X-iY)}{2\pi(1+\chi)} \left(\frac{1}{\sigma^{n-1}}\right) + N(0)\alpha(\sigma) + N(\sigma)\beta(\sigma), \tag{3.4b}$$

$$N(\sigma) = \bar{l}\Gamma - \frac{X-iY}{2\pi(1+\chi)}\sigma, \tag{3.4c}$$

and

$$F(\sigma) = f(t). \tag{3.4d}$$

The function $F(\sigma)$ with its derivatives must satisfy the Hölder condition. Multiplying both sides of (3.3) by $(1/2\pi i) 1/(\sigma - \zeta)$ and integrating with respect to σ on ν , one has

$$k\phi(\zeta) + \frac{1}{2\pi i} \int_{\nu} \frac{\alpha(\sigma) \overline{\phi'(\sigma)} d\sigma}{\sigma - \zeta} = \frac{\bar{l}\Gamma^*}{\zeta} + \frac{h}{\zeta^n} N(\zeta) - A(\zeta), \tag{3.5}$$

where

$$A(\zeta) = -\frac{1}{2\pi i} \sum_{\eta=0}^{\infty} \frac{1}{\zeta^{\eta+1}} \int_{\nu} \sigma^{\eta} F(\sigma) d\sigma, \quad |\zeta| > 1.$$

Using (3.2), we have

$$\frac{1}{2\pi i} \int_{\nu} \frac{\alpha(\sigma) \overline{\phi'(\sigma)} d\sigma}{\sigma - \zeta} = -\frac{h}{(n-1)!} \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{j! \overline{\phi^{n-j}(0)}}{\zeta^{1+j}}, \tag{3.6}$$

where $\overline{\phi^{n-j}(0)}$ is a complex constant to be determined.

Substituting from (3.6) into (3.5), we have

$$-k\phi(\zeta) = A(\zeta) - \frac{\bar{l}\Gamma^*}{\zeta} - \frac{h}{(n-1)!} \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{j! \overline{\phi^{n-j}(0)}}{\zeta^{1+j}} - \frac{h}{\zeta^n} N(0). \tag{3.7}$$

Differentiating (3.7) with respect to ζ , and using the result in (3.6), yields

$$\overline{\phi^{n-j}(0)} = -\frac{1}{k} \overline{A^{n-j}(0)}, \quad j = 0, 2, \dots, n-1. \tag{3.8}$$

Also from (3.3), $\psi(\zeta)$ can be determined in the form

$$\psi(\zeta) = \frac{\bar{l}k\bar{\Gamma}}{\zeta} - \frac{\omega(\zeta^{-1})}{\omega'(\zeta)} \phi_*(\zeta) + \frac{\bar{h}}{(n-1)!} \sum_{j=0}^{n-1} \binom{n-1}{j} j! \overline{\phi^{n-j}(0)} \zeta^{1+j} - \overline{hN(0)} \zeta^n + B(\zeta) - B, \tag{3.9}$$

where

$$\phi_*(\zeta) = \left[\phi'(\zeta) + \overline{N(\zeta)} \right], \quad (3.10a)$$

$$B(\zeta) = \frac{1}{2\pi i} \int_{\nu} \frac{\overline{F(\sigma)}}{\sigma - \zeta} d\sigma, \quad (3.10b)$$

and

$$B = \frac{1}{2\pi i} \int_{\nu} \frac{\overline{F(\sigma)}}{\sigma} d\sigma. \quad (3.10c)$$

4 Special cases

- 1 To employ the conformal mapping on the interior of the unit circle we replace ζ by $\frac{1}{\zeta}$ in (1.8) and then the resulting mapping function $z = \omega(\zeta) = l\zeta^{-1} + m\zeta^n$ will produce the following forms of Goursat functions;

$$-k\phi(\zeta) = A(\zeta^{-1}) - \bar{l}\bar{\Gamma}^*\zeta - \frac{h}{(n-1)!} \sum_{j=0}^{n-1} \binom{n-1}{j} j! \overline{\phi^{n-j}(0)} \zeta^{1+j} - h\zeta^n N(0), \quad (4.1)$$

$$\begin{aligned} \psi(\zeta) = \bar{l}k\bar{\Gamma}\zeta - \frac{\omega(\zeta^{-1})}{\omega'(\zeta)} \phi_*(\zeta^{-1}) + \frac{\bar{h}}{(n-1)!} \sum_{j=0}^{n-1} \binom{n-1}{j} j! \phi^{n-j}(0) \zeta^{-(1+j)} - \overline{hN(0)} \zeta^{-n} \\ + B(\zeta^{-1}) - B. \end{aligned} \quad (4.2)$$

2. In the first special case of Eq's (4.1), (4.2) which maps the hole into the interior of a unit circle if we consider the reality of the constants for the first fundamental problem, we get

$$\phi(\zeta) = A(\zeta^{-1}) - l\bar{\Gamma}^*\zeta - \frac{h}{(n-1)!} \sum_{j=0}^{n-1} \binom{n-1}{j} j! \overline{\phi^{n-j}(0)} \zeta^{1+j} - h\zeta^n N(0), \quad (4.3)$$

$$\begin{aligned} \psi(\zeta) = -l\bar{\Gamma}\zeta - \frac{\omega(\zeta^{-1})}{\omega'(\zeta)} \phi_*(\zeta^{-1}) + \frac{h}{(n-1)!} \sum_{j=0}^{n-1} \binom{n-1}{j} j! \phi^{n-j}(0) \zeta^{-(1+j)} - \overline{hN(0)} \zeta^{-n} \\ + B(\zeta^{-1}) - B. \end{aligned} \quad (4.4)$$

These two expressions of Goursat functions are equivalent to those derived by England (1980).

3. Also in the first special case of Eq's (4.1), (4.2) if we consider the reality of the constants with letting $n = 1$, we have

$$-k\phi(\zeta) = A(\zeta^{-1}) - l\bar{\Gamma}^*\zeta - h\overline{\phi'(0)}\zeta - h\zeta N(0), \tag{4.5}$$

$$\psi(\zeta) = kl\bar{\Gamma}\zeta - \frac{\omega(\zeta^{-1})}{\omega'(\zeta)}\phi_*(\zeta^{-1}) + h\phi'(0)\zeta^{-1} - h\overline{N(0)}\zeta^{-1} + B(\zeta^{-1}) - B. \tag{4.6}$$

The above two expressions of Goursat functions are equivalent to those derived by Muskhelishvili (1953).

4. Now in our recent mapping function (1.8) if we consider the reality of the constants with letting $n = 1$, we have

$$-k\phi(\zeta) = A(\zeta) - \frac{l\bar{\Gamma}^*}{\zeta} - \frac{h\overline{\phi'(0)}}{\zeta} - \frac{h}{\zeta}N(0), \tag{4.7}$$

$$\psi(\zeta) = \frac{lk\bar{\Gamma}}{\zeta} - \frac{\omega(\zeta^{-1})}{\omega'(\zeta)}\phi_*(\zeta) + h\phi'(0)\zeta - h\overline{N(0)}\zeta + B(\zeta) - B. \tag{4.8}$$

If we substitute $n = 0$ in the expressions derived by El-Sirafy and Abdou (1984), we get exactly the same above expressionsin of Goursat functions because of using the same method (complex varieable method).

5 Applications

5.1 Curvilinear hole for an infinite plate subjected to a uniform tensile stress :

For $k = -1$, $\Gamma = \frac{P}{4}$, $\Gamma^* = -\frac{1}{2}P \exp(-2i\theta)$ and $X = Y = f = 0$, we have an infinite plate stretched at infinity by the application of a uniform tensile stress of intensity P , making an angle θ with x -axis. The plate is weakened by a curvilinear hole C which is free from stress.

The functions (3.7), (3.9) take the form

$$\phi(\zeta) = \frac{\bar{l}P}{2} \left(\frac{1}{\zeta} e^{2i\theta} - \frac{h}{2\zeta^n} \right), \tag{5.1}$$

$$\psi(\zeta) = -\frac{\bar{l}P}{4\zeta} - \frac{\omega(\zeta^{-1})}{\omega'(\zeta)} \left[\phi'(\zeta) + \frac{lP}{4} \right] - \frac{lP\bar{h}}{4} \zeta^n. \quad (5.2)$$

And for $l = i$, $m = 1 + i$, $P = 0.25$, $h = -1 + i$, $n = 1$, stress components are obtained as

$$\begin{aligned} \sigma_{xx} = & -\frac{1}{8}X_2^+ - \frac{1}{8} [X_3^+ \cos \theta - X_3^- \sin \theta] - \frac{1}{16} (\sin 2\theta - 1) \\ & + \frac{1}{8 [(X_2^+)^2 + (1 + X_2^-)^2]} [X_3^+ ([\sin \theta + X_1^-] X_2^+ + [X_1^- - \cos \theta] [1 + X_2^-])] \\ & + X_3^- ([\cos \theta - X_1^-] X_2^+ + [\sin \theta + X_1^-] [1 + X_2^-]) \\ & + \frac{1}{16 [(2 (\sin 4\theta - X_2^-) - 1)^2 + 4 (X_2^+ - \cos 4\theta)^2]} [\\ & (3 + X_2^-) (2 (7 \cos 2\theta - X_4^- - 1) (X_2^+ - \cos 4\theta) \\ & - (2 (\sin 4\theta - X_2^-) - 1) (7 \sin 2\theta + X_4^+ + 1)) \\ & + X_2^+ ((7 \cos 2\theta - X_4^- - 1) (2 (\sin 4\theta - X_2^-) - 1) \\ & + 2 (7 \sin 2\theta + X_4^+ + 1) (X_2^+ - \cos 4\theta))], \end{aligned} \quad (5.3)$$

$$\begin{aligned} \sigma_{yy} = & -\frac{1}{8}X_2^+ + \frac{1}{8} [X_3^+ \cos \theta - X_3^- \sin \theta] + \frac{1}{16} (\sin 2\theta - 1) \\ & - \frac{1}{8 [(X_2^+)^2 + (1 + X_2^-)^2]} [X_3^+ ([\sin \theta + X_1^-] X_2^+ + [X_1^- - \cos \theta] [1 + X_2^-])] \\ & + X_3^- ([\cos \theta - X_1^-] X_2^+ + [\sin \theta + X_1^-] [1 + X_2^-]) \\ & - \frac{1}{16 [(2 (\sin 4\theta - X_2^-) - 1)^2 + 4 (X_2^+ - \cos 4\theta)^2]} [\\ & (3 + X_2^-) (2 (7 \cos 2\theta - X_4^- - 1) (X_2^+ - \cos 4\theta) \\ & - (2 (\sin 4\theta - X_2^-) - 1) (7 \sin 2\theta + X_4^+ + 1)) \\ & + X_2^+ ((7 \cos 2\theta - X_4^- - 1) (2 (\sin 4\theta - X_2^-) - 1) \\ & + 2 (7 \sin 2\theta + X_4^+ + 1) (X_2^+ - \cos 4\theta))], \end{aligned} \quad (5.4)$$

$$\begin{aligned}
 \sigma_{xy} = & -\frac{1}{8} [X_3^- \cos \theta + X_3^+ \sin \theta] + \frac{1}{16} (\cos 2\theta + 1) \\
 & + \frac{1}{8 [(X_2^+)^2 + (1 + X_2^-)^2]} [X_3^- ([\sin \theta + X_1^-] X_2^+ + [X_1^- - \cos \theta] [1 + X_2^-]) \\
 & - X_3^+ ([\sin \theta + X_1^-] [1 + X_2^-] - [X_1^- - \cos \theta] X_2^+)] \\
 & + \frac{1}{16 [(2 (\sin 4\theta - X_2^-) - 1)^2 + 4 (X_2^+ - \cos 4\theta)^2]} [\\
 & (3 + X_2^-) ((7 \cos 2\theta - X_4^- - 1) (2 (\sin 4\theta - X_2^-) - 1) \\
 & + 2 (7 \sin 2\theta + X_4^+ + 1) (X_2^+ - \cos 4\theta)) \\
 & - X_2^+ (2 (7 \cos 2\theta - X_4^- - 1) (X_2^+ - \cos 4\theta) \\
 & - (2 (\sin 4\theta - X_2^-) - 1) (7 \sin 2\theta + X_4^+ + 1))]. \tag{5.5}
 \end{aligned}$$

where

$$X_n^\pm = \sin n\theta \pm \cos n\theta. \tag{5.6}$$

The above relations are illustrated in Fig's. 7,8,9,10 and 11, which determine stress components.

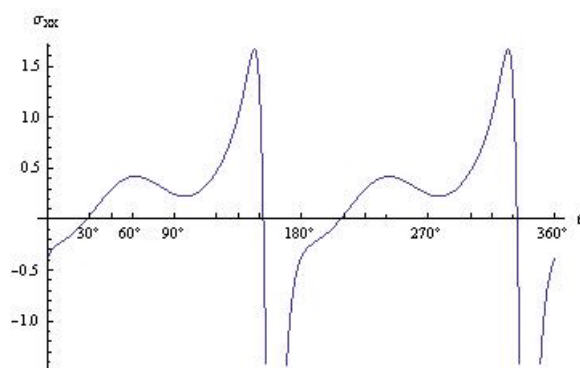


Fig. 7

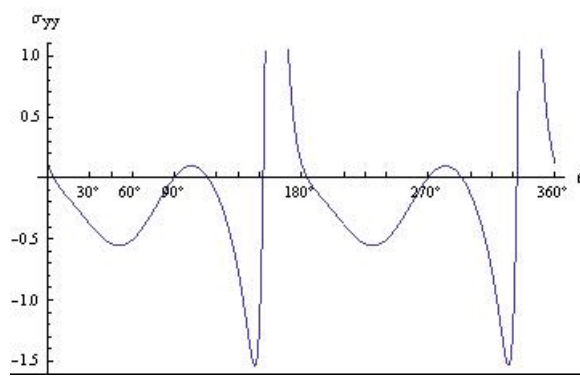


Fig.8

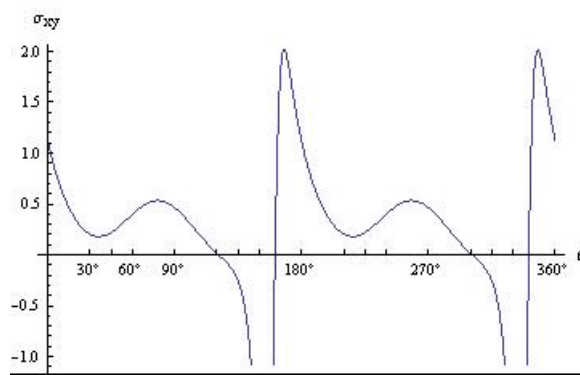


Fig. 9

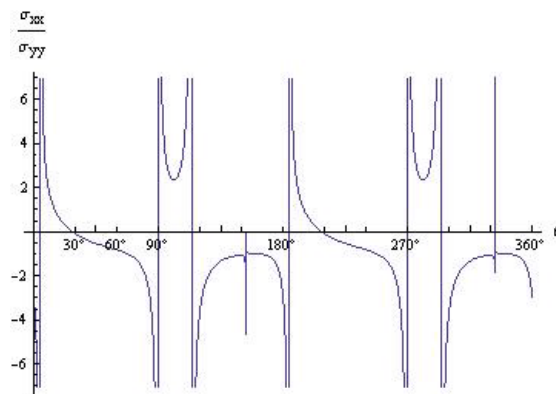


Fig. 10

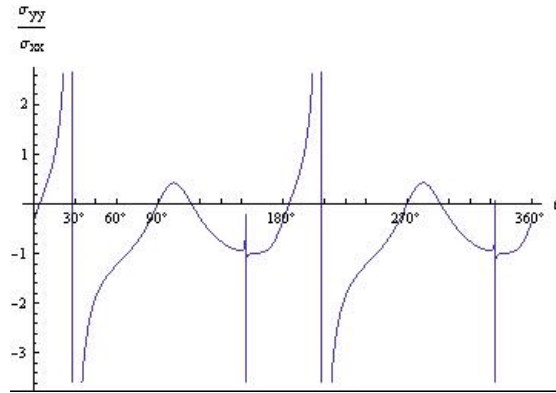


Fig. 11

5.2 Uni-Directional tension of an infinite plate with a rigid curvilinear center :

For $k = \chi$, $\Gamma = \frac{P}{4}$, $\Gamma^* = -\frac{1}{2}P \exp(-2i\theta)$ and $X = Y = 0$, $f = 2iP\varepsilon$, the two complex functions (3.7), (3.9) take the form

$$-\chi\phi(\zeta) = \frac{\bar{l}P}{2} \left(\frac{1}{\zeta} e^{2i\theta} - \frac{h}{2\zeta^n} \right), \tag{5.7}$$

$$\psi(\zeta) = \frac{\bar{l}\chi P}{4\zeta} - \frac{\omega(\zeta^{-1})}{\omega'(\zeta)} \left[\phi'(\zeta) + \frac{lP}{4} \right] - \frac{lP\bar{h}}{4}\zeta^n + 2iP\varepsilon. \tag{5.8}$$

Therefore we have the case of uni-directional tension of an infinite plate with a rigid curvilinear center.

The constant ε can be determined from the condition that the resultant moment of the forces acting on the curvilinear center from the surrounding material must vanish, i.e.

$$M = \text{Re} \left\{ \int \left[\psi(\zeta) - \frac{lP}{2} e^{-2i\theta} \zeta \right] \omega'(\zeta) d\zeta \right\} = 0. \tag{5.9}$$

Hence,

$$\begin{aligned}
\varepsilon = & \frac{1}{4(m_1 \sin(n+1)\theta - m_2 \cos(n+1)\theta)} \left[\right. \\
& \frac{1}{2} (z(h, m) \cos(n+1)\theta - \tilde{z}(h, m) \sin(n+1)\theta) \\
& + \frac{1}{n} \left(\frac{\chi l_1}{2} + \frac{l_1}{\chi} + \frac{n z(h, m)}{2\chi} - l_1 \right) \cos \theta \\
& + \frac{1}{n} \left(\frac{\chi l_2}{2} - \frac{l_2}{\chi} + \frac{n \tilde{z}(h, m)}{2\chi} - l_2 \right) \sin \theta \\
& - \left(\frac{\chi m_1}{2} - \frac{z(h, l)}{2\chi} - m_1 \right) \cos(n+2)\theta \\
& \left. - \left(\frac{\chi m_2}{2} - \frac{\tilde{z}(h, l)}{2\chi} - m_2 \right) \sin(n+2)\theta \right], \tag{5.10}
\end{aligned}$$

where

$$z(h, x) = h_1 x_1 + h_2 x_2 \quad \& \quad \tilde{z}(h, x) = h_2 x_1 - h_1 x_2, \quad \text{for any } x = x_1 + ix_2. \tag{5.11}$$

Also, for $l = i$, $m = 1 + i$, $P = 0.25$, $h = -1 + i$, $n = 1$, $\chi = 2$, stress components are obtained as

$$\begin{aligned}
\sigma_{xx} = & \frac{1}{16} X_2^+ - \frac{1}{16} [X_3^- \sin \theta - X_3^+ \cos \theta] + \frac{1}{16} (1 + 2 \sin 2\theta) \\
& - \frac{1}{16 \left[(X_2^+)^2 + (1 + X_2^-)^2 \right]} [X_3^+ ([\sin \theta + X_1^-] X_2^+ + [X_1^- - \cos \theta] [1 + X_2^-])] \\
& + X_3^- ([\sin \theta + X_1^-] [1 + X_2^-] - [X_1^- - \cos \theta] X_2^+) \\
& + \frac{1}{32 \left[(2(\sin 4\theta - X_2^-) - 1)^2 + 4(X_2^+ - \cos 4\theta)^2 \right]} \left[\right. \\
& X_2^- (2(-7 \cos 2\theta + X_4^- + 1)(X_2^+ - \cos 4\theta) \\
& + (2(\sin 4\theta - X_2^-) - 1)(7 \sin 2\theta + X_4^+ + 1)) \\
& + X_2^+ ((-7 \cos 2\theta + X_4^- + 1)(2(\sin 4\theta - X_2^-) - 1) \\
& \left. - 2(7 \sin 2\theta + X_4^+ + 1)(X_2^+ - \cos 4\theta)) \right], \tag{5.12}
\end{aligned}$$

$$\begin{aligned}
\sigma_{yy} = & \frac{1}{16} X_2^+ + \frac{1}{16} [X_3^- \sin \theta - X_3^+ \cos \theta] - \frac{1}{16} (1 + 2 \sin 2\theta) \\
& + \frac{1}{16 [(X_2^+)^2 + (1 + X_2^-)^2]} [X_3^+ ([\sin \theta + X_1^-] X_2^+ + [X_1^- - \cos \theta] [1 + X_2^-])] \\
& + X_3^- ([\sin \theta + X_1^-] [1 + X_2^-] - [X_1^- - \cos \theta] X_2^+) \\
& - \frac{1}{32 [(2 (\sin 4\theta - X_2^-) - 1)^2 + 4 (X_2^+ - \cos 4\theta)^2]} [
\end{aligned}$$

$$\begin{aligned}
& X_2^- (2 (-7 \cos 2\theta + X_4^- + 1) (X_2^+ - \cos 4\theta) \\
& + (2 (\sin 4\theta - X_2^-) - 1) (7 \sin 2\theta + X_4^+ + 1)) \\
& + X_2^+ ((-7 \cos 2\theta + X_4^- + 1) (2 (\sin 4\theta - X_2^-) - 1) \\
& - 2 (7 \sin 2\theta + X_4^+ + 1) (X_2^+ - \cos 4\theta)), \tag{5.13}
\end{aligned}$$

$$\begin{aligned}
\sigma_{xy} = & \frac{1}{16} [X_3^- \cos \theta + X_3^+ \sin \theta] + \frac{1}{16} (1 - 2 \cos 2\theta) \\
& - \frac{1}{16 [(X_2^+)^2 + (1 + X_2^-)^2]} [X_3^- ([\sin \theta + X_1^-] X_2^+ + [X_1^- - \cos \theta] [1 + X_2^-])] \\
& - X_3^+ ([\sin \theta + X_1^-] [1 + X_2^-] - [X_1^- - \cos \theta] X_2^+) \\
& + \frac{1}{32 [(2 (\sin 4\theta - X_2^-) - 1)^2 + 4 (X_2^+ - \cos 4\theta)^2]} [\\
& X_2^- ((-7 \cos 2\theta + X_4^- + 1) (2 (\sin 4\theta - X_2^-) - 1) \\
& - 2 (7 \sin 2\theta + X_4^+ + 1) (X_2^+ - \cos 4\theta)) \\
& - X_2^+ (2 (-7 \cos 2\theta + X_4^- + 1) (X_2^+ - \cos 4\theta) \\
& + (2 (\sin 4\theta - X_2^-) - 1) (7 \sin 2\theta + X_4^+ + 1))]. \tag{5.14}
\end{aligned}$$

The above relations are illustrated in Fig's. 12, 13,14,15 and 16, which determine stress components

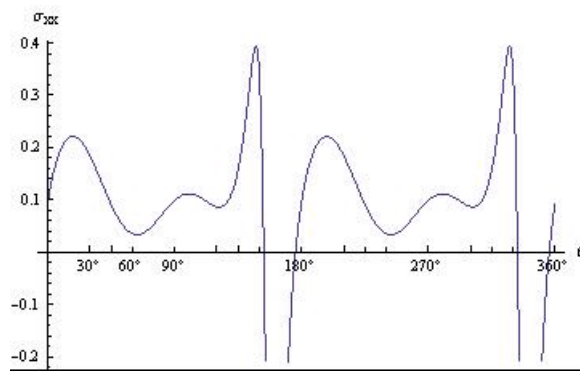


Fig. 12

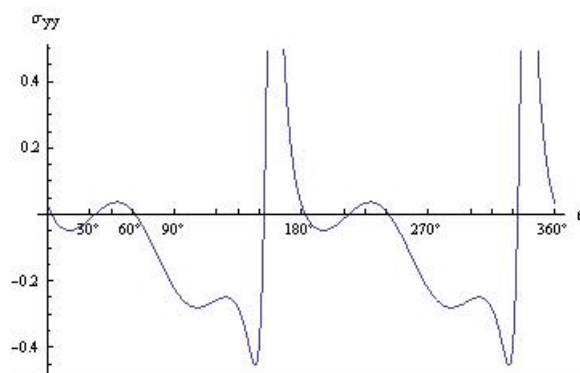


Fig. 13

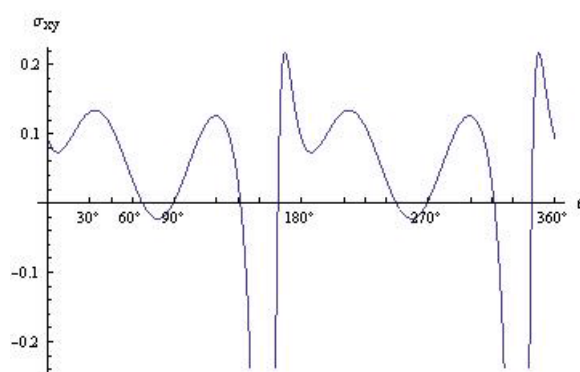


Fig. 14

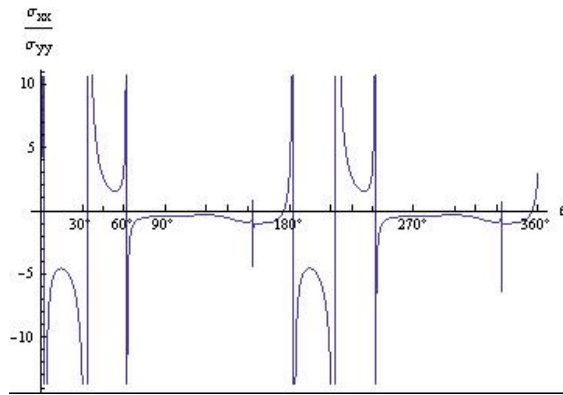


Fig. 15

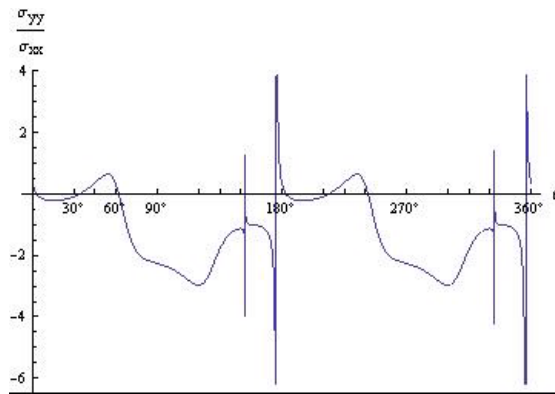


Fig. 16

6 Open problems

Singular integral equations arise in many problems of mathematical models of physical phenomena, specifically in various kinds of mixed boundary value problems of mathematical physics and engineering problems.

The important theory of these integral equations is contained with some of its applications in the work of Muskhelishvili (1953) and Abdou (2003)

The solution of a large class of mixed boundary value problems of a great variety of contact and crack problems in solid mechanics, physics and engineering can be related to the singular integral equations that has a simple Cauchy-type singularity, see Abdou (2003).

$$\mu \phi(x) + \frac{\lambda}{\pi} \int_{-1}^1 \frac{\phi(y) dy}{y-x} + \lambda \int_{-1}^1 k(x,y) \phi(y) dy = f(x) \quad (6.1)$$

where $\phi(x)$ is the unknown function, the kernel $k(x,y)$ is continuous and known function, also $f(x)$ is a known function. The coefficient λ is a constant,

may be complex, and has a physical meaning and the constant μ defines the kind of the integral equation, when $\mu = 0$ we have the integral equation of the first kind, while for $\mu = \text{const} \neq 0$ we have an integral equation of the second kind.

The integral equation with a simply Cauchy kernel (6.1) may in principal, be regularized and hence, it can be solved numerically by conventional methods see Atkinson (1976), Golberg (1990), Delves and Mohamed (1985) and Abdou et.al (2003).

In physical problems the unknown function $\phi(x)$ may be either a potential (e.g. temperature, displacement velocity potential, electrostatic field) or flux type quantity (e.g. heat flux, stress, dislocation, velocity charge density).

The end points ± 1 are points of geometric singularity. At these points $\phi(x)$ is bounded if it is a potential and $\phi(x)$ has an integrable singularity if it is a flux- type quantity of the singular integral equation (6.1).

This equation may be arising from the formulation of elasticity problems for the parallel layers compressed by stamps with arbitrary profile. If the contact between the parallel layers and the stamps is frictionless the corresponding constant $\mu = 0$, i.e. there are no interface cracks, or the cracks are finite, and the related integral equation is of the first kind, while if the contact is perfect adhesion, i.e. the cracks are infinite between the stamp and the layers, the related equation is a singular integral equations of the second kind, where the length of the cracks or size of the stamp are not equals.

If the crack takes a closed form, it leads us to a new type of problems which is called fundamental problems.

Conclusion:

From the above results and discussions the following may be concluded :

- 1) In the theory of two dimensional linear elasticity one of the most useful techniques for the solutions of boundary value problem for a region weakened by a curvilinear hole is to transform the region into a simpler shape.
- 2) The mapping function (1.8) maps the curvilinear hole C in z -plane into the domain outside a unit circle in ζ -plane under the condition $\omega'(\zeta) \neq 0$ or ∞ for $|\zeta| > 1$.
- 3) The physical interest of the mapping (1.8) comes from its different shapes of holes it treats and different directions it takes as shown in Fig's. 1-6.
- 4) The complex variable method (Cauchy method) is considered as one of the best methods for solving the integro-differential equations (1.1), and obtaining the two complex potential functions $\phi(z)$ and $\psi(z)$ directly.

- 5) Stress is an internal force whereas positive values of it mean that stress is in the positive direction, i.e. stress acts as a tension force. On the other side, negative values of stress mean that stress is in the negative direction, i.e. stress acts as a press force.
- 6) The most important issue deduced from (Fig's. 7, 8 and 12, 13) is that $\max \sigma_{xx} = -\min \sigma_{yy}$ and vice versa ($\min \sigma_{xx} = -\max \sigma_{yy}$).
- 7) By following the intendency of $\frac{\sigma_{xx}}{\sigma_{yy}}$ and $\frac{\sigma_{yy}}{\sigma_{xx}}$ at (Fig's. 10, 11 and 15, 16), we find that $\frac{\sigma_{xx}}{\sigma_{yy}} \rightarrow 0$ at the same points where $\frac{\sigma_{yy}}{\sigma_{xx}} \rightarrow \infty$ and vice versa ($\frac{\sigma_{xx}}{\sigma_{yy}} \rightarrow \infty$ at the same points where $\frac{\sigma_{yy}}{\sigma_{xx}} \rightarrow 0$).
- 8) When $\frac{\sigma_{xx}}{\sigma_{yy}} \rightarrow 0$ that means that the perpendicular stress on y -axis is the maximum value and presents the body interior resistance of treatment (like rocks for example). Whereas, the perpendicular stress on x -axis is small according to y -axis. Thereon, it is better to treat the problem at points determined by angles that gives minimum values of $\frac{\sigma_{xx}}{\sigma_{yy}}$.

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