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On the Spectrum of Non-selfadjoint Differential Operators

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Abstract

For Sturm–Liouville type operators generated by the Sturm–Liouville differential expression

$$\tau \equiv \frac{d}{dx}(-p\frac{d}{dx}) + q(x)$$

on $[0,\infty)$***, the associated boundary value problems are non-selfadjoint if*** p ***or*** q ***is a complex-valued or the boundary conditions are non-real. Some important links between the spectral properties of a selfadjoint Sturm–Liouville operator and the analyticity properties of corresponding Titchmarsh–Weyl functions*** $m(\lambda)$ ***have been investigated.The purpose of this paper is that to extend some of those results to non-selfadjoint problems with*** $p \equiv 1$ ***and*** q ***a continuous complex valued function under condition*** $\lim_{x\to\infty} \Im q(x) = L$***.***

Keywords: *Sturm–Liouville problem; spectral theory ; Titmarsh–Weyl–sims theory.*

1 Introduction

Consider the ordinary second order differential expression

$$L[y] = -\frac{d}{dx}(p\frac{dy}{dx}) + qy \quad x \in [0,\infty)$$

, where $L[y]$ is regular at 0 and singular at ∞ and p, q are real-valued functions satisfying the following conditions.

i) p is positive, locally absolutely continuous on $[0,\infty)$.

ii) q is continuous on $[0, x]$ for all $x > 0$.
Let $L[y]$ be limit point at infinity in the sense of [4], and $\{\phi(x,\lambda), \theta(x,\lambda)\}$ a fundamental set of solutions of the equation

$$L[y] = \lambda y, \quad \lambda = \mu + i\nu \in \mathbf{C} \tag{1}$$

satisfying

$$\begin{cases} \phi(0,\lambda) = -\sin\alpha, & \phi'(0,\lambda) = p(0)^{-1}\cos\alpha \\ \theta(0,\lambda) = \cos\alpha, & \theta'(0,\lambda) = p(0)^{-1}\sin\alpha\,, \end{cases}$$

where $\alpha \in [0, \pi)$ and $[\phi, \theta](0) = 1$ such that $[f, g](x) = p(x)W[f, \bar{g}]$. We define a selfadjoint operator T_α on the Hilbert space $\mathcal{H} = L^2[0, \infty)$ by $T_\alpha f = L[f]$ for all $f \in \mathcal{D}_\alpha$, where

$$\mathcal{D}_\alpha = \{f \in \mathcal{D} : f(0)\cos\alpha + f'(0)\sin\alpha = 0\} \tag{2}$$

and $\mathcal{D}$ is the domain of the maximal operator associated with $L[f]$. Corresponding to T_α there is a Herglots function $m_\alpha(\lambda)$ which is regular on the half-planes $\Im\lambda > 0$, $\Im\lambda < 0$ and is such that the solution
$\psi(x,\lambda) = \theta(x,\lambda) + m_\alpha(\lambda)\phi(x,\lambda)$ is square integrable.
Under condition $(1 + x)q(x) \in L^2[0, \infty)$ G.Freiling and V.Yurko [9] obtained some results about the non-selfadjoint second order differential operators on half line with a discontinuity in an interior point.They established properties of the spectrum and investigate the inverse problem of recovering the operator from spectrum. E.B. Davies [8] has given a method to analyze the spectrum of non-selfadjoint differential operators emphasizing the differences from the selfadjoint theory. A numerical method for determining the Titchmarsh–Weyl $m(\lambda)$ function for the singular $L[y]$ equation on $[a, \infty)$, where a is finite in [10] is presented and the computational techniques have been applied to the problem of finding best constant in the Hardy-Littlewood inequality. in [11] the authors have extended the pioneering work of Sims on second order linear differential equations with a complex coefficient, they did generalization of features not visible in the special case of Sims's paper, an m function constructed and the relationship between its properties and the spectrum of underlying m-accretive differential operators analysed. it is known that the spectral properties of T_α are closely correlated with the boundary properties of the analytic function $m_\alpha(\lambda)$ on the real axis. [1]

2 Preliminary results

Theorem 2.1 (Chaudhuri–Everitt) *Let $L[f]$ be limit point case at infinity then:*

i) The complex number λ' belongs to the resolvent set $\rho(T_\alpha)$ of T_α if and only if $m_\alpha(\lambda)$ is regular at λ'. The resolvent operator at such points for all $f \in \mathcal{H}$is given by

$$\Phi(x,\lambda';f) = \psi(x,\lambda')\int_0^x \phi(t,\lambda')f(t)dt + \phi(x,\lambda')\int_x^\infty \psi(t,\lambda')f(t)dt$$

ii) The complex number μ' belongs to the point spectrum $\sigma_p(T_\alpha)$ of T_α if and only if $m_\alpha(\lambda)$ has a simple pole at μ'; in this case $\phi(x,\mu'), \theta(x,\mu') + r\phi_\lambda(x,\mu') \in L^2[0,\infty)$ and the resolvent operator at such points is given by

$$\begin{aligned}\Phi(x,\mu';f) &= \theta(x,\mu')\int_0^x \phi(t,\mu')f(t)dt + r\phi(x,\mu')\int_0^x \phi_\lambda(t,\mu')f(t)dt \\ &+ \phi(x,\mu')\int_x^\infty \{\theta(t,\mu') + r\phi_\lambda(x,\mu')\}f(t)dt\end{aligned}$$

for all $f \in L^2[0,\infty) \ominus \{\phi(x,\mu')\}$, where r is the residue of $m_\alpha(\lambda)$ at μ' and $\phi_\lambda(x,\mu') = \frac{\partial\phi(x,\lambda)}{\partial\lambda}|_{\lambda=\mu'}$ and $\{\phi(x,\mu')\}$ is the eigenspace at μ'
iii) The complex number μ' belongs to the continuous spectrum $\sigma_c(T_\alpha)$ of T_α if and only if $m_\alpha(\lambda)$ is not regular at μ' and $\lim_{\nu\to 0}\nu m_\alpha(\mu' + i\nu) = 0$.

We shall extend part i) of this theorem to the non-selfadjoint case, also part ii) under certain conditions and by giving a counter example we show that part iii) can not be extended.

3 Main results

We now consider the corresponding non-selfadjoint differential operator T_α under the condition $\alpha \in C$, $p \equiv 1$ and $q = q_1 + iq_2$ is a continuous function such that $\lim_{x\to\infty} q_2(x) = L < \infty$ on the interval $[0,\infty)$. In this case T_α is defined by $T_\alpha f = \tau f$ for all $f \in \mathcal{D}_\alpha$, where
$\tau f = -f'' + q(x)f$ and $\mathcal{D}_\alpha$ is the set of all functions f in $\mathcal{H}$ satisfying the following conditions
i) f and f' are locally absolutely continuous on the interval $[0,\infty)$.
ii) $f(0)\cos\alpha + f'(0)\sin\alpha = 0$.
If $\Im\lambda = \nu \neq L$ then there always exists an L^2-solution $\psi(x,\lambda)$ of the equation (1)and a meromorphic function $m_\alpha(\lambda)$ satisfying

$$\psi(x,\lambda) = \theta(x,\lambda) + m_\alpha(\lambda)\phi(x,\lambda)\,, \tag{3}$$

where λ is a regular point of $m_\alpha(\lambda)$ [2]. Let f and g be two functions for which the expression

$$\tau f = -\frac{d^2 f}{dx^2} + q(x)f \tag{4}$$

makes sense. If $[f,g](x) = W[f,\bar{g}](x)$, and if $q(x)$ is real, then we have

$$\tau f\bar{g} - f\tau\bar{g} = \frac{d}{dx}(f\bar{g}' - f'\bar{g})(x) = \frac{d}{dx}[f,g](x)\,, \tag{5}$$

which is called the *Lagrange's identity.* Integrating both sides of (5) on the finite interval $[0,x]$ we obtain *Green's formula*

$$\int_0^x (\tau f\bar{g} - f\overline{\tau g})dx = (f\bar{g}' - f'\bar{g})|_0^x = [f,g]_0^x$$

However if the function q in the expression (4)is a complex valued function then we have

$$\tau f\bar{g} - f\overline{\tau g} = \frac{d}{dx}(f\bar{g}' - f'\bar{g})(x) + f\bar{g}(q-\bar{q}) = \frac{d}{dx}[f,g] + 2iq_2 f\bar{g} \tag{6}$$

Integrating both side of(6)on the finite interval $[0,x]$, imply

$$\int_0^x (\tau f\bar{g} - f\overline{\tau g})dx = [f,g]_0^x + 2i\int_0^x q_2 f\bar{g}dx$$

hence

$$[f,g](x) = 2i\int_0^x (\nu - q_2)f\bar{g}dx + [f,g](0). \tag{7}$$

Lemma 3.1 *Let $f \in L^2[0,\infty)$, and let $\nu \neq L$. Suppose that λ is a regular value of $m_\alpha(\lambda)$ and define the function $\Phi(x,\lambda;f)$ on $[0,\infty)$ by*

$$\Phi(x,\lambda;f) = \psi(x,\lambda)\int_0^x \phi(t,\lambda)f(t)dt + \phi(x,\lambda)\int_x^\infty \psi(t,\lambda)f(t)dt\,,$$

where ϕ and ψ are solutions of the equation $L[f] = \lambda f$ satisfying condition ii) and (3) respectively for some $\alpha \in C$. Then $\Phi \in L^2[0,\infty)$ and there exists $K > 0$ such that $\|\Phi\| \leq K\|f\|$ for all $f \in L^2[0,\infty)$.

Proof: First we note that Φ is well defined, since f and ψ are $L^2[0,\infty)$ and ϕ and f are square integrable on $[0,x]$ for all $x > 0$. Let $\nu > L$. Then there exists a real number $r > 0$ so that

$$\nu - q_2(x) > \frac{1}{2}(\nu - L) > 0 \tag{8}$$

for all $x \in [r,\infty)$, so proceeding as in [5], §5, there are square integrable solutions ψ_0 and ψ_1 and meromorphic functions m_0 and m_1 satisfying $\psi_0(x,\lambda) = \tilde{\theta}(x,\lambda) + m_0(\lambda)\tilde{\phi}(x,\lambda)$, $\psi_1(x,\lambda) = \tilde{\theta}(x,\lambda) + m_1(\lambda)\tilde{\phi}(x,\lambda)$, where $\psi_0(x,\lambda) \in L^2[0,\infty)$ satisfies the boundary condition $f(0)\cos\alpha + f'(0)\sin\alpha = 0$, and $\psi_1(x,\lambda) \in L^2[r,\infty)$. The fundamental set $\{\tilde{\theta},\tilde{\phi}\}$ is defined in the usual way in terms of the boundary condition $\tilde{\alpha} = 0$ at $x = r$ i.e. $\tilde{\phi}(r,\lambda) =$

0, $\tilde{\theta}(r,\lambda) = 1$, $\tilde{\phi}'(r,\lambda) = -1$, $\tilde{\theta}'(r,\lambda) = 0$. Hence, since we are in case I, there are non-zero scalars $k_1(\lambda)$ and $k_2(\lambda)$ depending on λ such that

$$\psi_0(x,\lambda') = k_1(\lambda)\phi(x,\lambda') \quad , \quad \psi_1(x,\lambda') = k_2(\lambda)\psi(x,\lambda').$$

Now define the function f_b on the interval $[0,\infty)$ by

$$f_b(x) = \begin{cases} f(x) & \text{if } x \le b \\ 0 & \text{if } x > b \end{cases}$$

for some $b > r$, and let

$$\begin{aligned} \Phi_b = \Phi(x,\lambda;f_b) &= \frac{1}{W[\psi_0,\psi_1]}\left\{\psi_1(x,\lambda)\int_0^x \psi_0(t,\lambda)f_b(t)dt\right. \\ &+ \left.\psi_0(x,\lambda)\int_x^b \psi_1(t,\lambda)f_b(t)dt\right\} \end{aligned} \tag{9}$$

$$\begin{aligned} \int_0^b \bar{\Phi}\tau\Phi - \Phi\overline{\tau\Phi} &= \int_0^b \bar{\Phi}_b(-\Phi_b{}'' + q\Phi_b) - \Phi_b(-\bar{\Phi_b}'' + \bar{q}\bar{\Phi}_b) \\ &= \int_0^b (\Phi_b\bar{\Phi_b}' - \bar{\Phi}_b\Phi_b{}')' + \int_0^b 2iq_2|\Phi_b|^2 \\ &= W[\Phi_b,\bar{\Phi}_b]_0^b + 2i\int_0^b q_2|\Phi_b|^2. \end{aligned} \tag{10}$$

On the other hand Φ_b satisfies the non-homogenous differential equation $\tau\Phi_b - \lambda\Phi_b = f$ on $[0,b]$ so

$$\begin{aligned} \int_0^b \bar{\Phi}_b\tau\Phi_b - \Phi_b\overline{\tau\Phi_b} &= \int_0^b \bar{\Phi}_b(\lambda\Phi_b + f_b) - \Phi_b(\bar{\lambda}\bar{\Phi}_b + \bar{f}_b) \\ &= \int_0^b 2i\nu|\Phi_b|^2 + \int_0^b 2i\Im(\bar{\Phi}_b f) \end{aligned} \tag{11}$$

By (10) and (11) we have

$$2i\int_0^b (\nu - q_2)|\Phi_b|^2 = W[\Phi_b,\bar{\Phi}_b]_0^b - 2i\int_0^b \Im(\bar{\Phi}_b f) \tag{12}$$

However, from (10) we can write

$$W[\Phi_b,\bar{\Phi}_b](b) = \frac{1}{|W[\psi_0,\psi_1](b)|^2}W[\psi_1,\bar{\psi_1}](b)|\int_0^b \psi_0(t,\lambda)f_b(t)dt|^2$$

$$W[\Phi_b,\bar{\Phi}_b](0) = \frac{1}{|W[\psi_0,\psi_1](0)|^2}W[\psi_0,\bar{\psi_0}](0)|\int_0^b \psi_1(t,\lambda)f_b(t)dt|^2$$

Also by (7) , $W[\psi_1, \bar{\psi_1}](r) = 2i\Im m_1$ and $W[\psi_0, \bar{\psi_0}](r) = 2i\Im m_0$, we have

$$W[\psi_1, \bar{\psi_1}](b) = 2i[\int_r^b (\nu - q_2)|\psi_1|^2 dx + \Im m_1]$$

$$W[\psi_0, \bar{\psi_0}](0) = 2i[\int_0^r (\nu - q_2)|\psi_0|^2 dx - \Im m_0]$$

Using these results in (12), we obtain $\int_0^b (\nu - q_2)|\Phi_b|^2 =$

$$\frac{1}{|W^2[\psi_0\psi_1]|^2}(|\int_0^b \psi_0(t,\lambda) f_b(t) dt|^2[\int_r^b (\nu - q_2)|\psi_1|^2 dt + \Im m_1]$$

$$+|\int_0^b \psi_1(t,\lambda) f_b(t) dt|^2[\int_0^r (\nu - q_2)|\psi_0|^2 dt - \Im m_0]) + \int_0^b \Im(\Phi_b \bar{f_b})$$

Since from inequality (5) of [5] we have $\int_r^b (\nu - q_2)|\psi_1|^2 < -\Im m_1$ and $\int_0^r (\nu - q_2)|\psi_0|^2 < \Im m_0$ thus by the Cauchy-Schwartz inequality

$$\int_0^b (\nu - q_2)|\Phi_b|^2 \leq \int_0^b \Im(\Phi_b \bar{f}) \leq \int_0^b |\Phi_b \bar{f}| \leq (\int_0^b |\Phi_b|^2 \int_0^b |f|^2)^{\frac{1}{2}} \tag{13}$$

i.e.

$$\int_0^b (\nu - q_2)|\Phi_b|^2 \leq (\int_0^b |\Phi_b|^2 \int_0^b |f|^2)^{\frac{1}{2}}$$

or

$$\int_r^b (\nu - q_2)|\Phi_b|^2 \leq (\int_0^b |\Phi_b|^2 \int_0^b |f|^2)^{\frac{1}{2}} - \int_0^r (\nu - q_2)|\Phi_b|^2$$

By the continuity of $q(x)$ on $[0, r]$, there exists a positive constant K' such that $|\nu - q_2| < K'$. Hence, using also (8) we have

$$\frac{1}{2}(\nu - L)\int_r^b |\Phi_b|^2 \leq (\int_0^b |\Phi_b|^2 \int_0^b |f|^2)^{\frac{1}{2}} + K'(\int_0^r |\Phi_b|^2 \int_0^b |\Phi_b|^2)^{\frac{1}{2}}$$

On the other hand since $\nu > L$, we can write

$$\frac{1}{2}(\nu - L)\int_0^r |\Phi_b|^2 \leq \frac{1}{2}(\nu - L)(\int_0^r |\Phi_b|^2)^{\frac{1}{2}}(\int_0^b |\Phi_b|^2)^{\frac{1}{2}}$$

Hence, we may add the last two inequalities to obtain

$$(\int_0^b |\Phi_b|^2)^{\frac{1}{2}} \leq \frac{2}{\nu - L}(\int_0^b |f|^2)^{\frac{1}{2}} + (\frac{2K'}{\nu - L} + 1)(\int_0^r |\Phi_b|^2)^{\frac{1}{2}}$$

But from (9), there exists a constant K'' such that

$$(\int_0^r |\Phi_b|^2)^{\frac{1}{2}} \leq K''(\int_0^r |f|^2)^{\frac{1}{2}} \leq K''(\int_0^b |f|^2)^{\frac{1}{2}}$$

since the functions $\psi_0(x,\lambda)$, $\psi_1(x,\lambda)$ and $f(x)$ are in $L^2[0,\infty)$, so using the above inequality in the previous one gives

$$(\int_0^b |\Phi_b|^2)^{\frac{1}{2}} \leq K(\int_0^b |f|^2)^{\frac{1}{2}}\,, \tag{14}$$

where $K = \frac{(K'K''+1)}{\nu-L} + K''$ is not dependent on f. On the other hand

$$W[\psi_1,\psi_0](x) = k_1(\lambda)k_2(\lambda)W[\psi,\phi](x) = k_1(\lambda)k_2(\lambda)$$

so we have

$$\Phi_b = \frac{1}{k_1(\lambda)k_2(\lambda)}[k_2(\lambda)\psi(x,\lambda)\int_0^x k_1(\lambda)\phi(t,\lambda)f_b(t)dt +$$

$$k_1(\lambda)\phi(x,\lambda)\int_x^b k_2(\lambda)\psi(t,\lambda)f_b(t)dt]$$

or

$$\Phi_b = \Phi(x,\lambda;f_b) = \psi(x,\lambda)\int_0^x \phi(t,\lambda)f_b(t)dt + \phi(x,\lambda)\int_x^b \psi(t,\lambda)f_b(t)dt$$

Now let $b \to \infty$. Then $\Phi_b \to \Phi$ and by Fatou's theorem we conclude from (142.12) that $\int_0^\infty |\Phi|^2 \leq K^2 \int_0^\infty |f|^2$ and $\|\Phi\| \leq K\|f\|$ as required. If $\nu < L$ the proof is similar to the case $\nu > L$.

Theorem 3.2 *Consider the differential equation $\tau f = \lambda f$ generated by the non-selfadjoint differential expression τ on $[0,\infty)$ and let λ' be a complex parameter such that $\Im\lambda' \neq L$. Then λ' is in the resolvent set $\rho(T_\alpha)$ of T_α if and only if the corresponding m-function, $m_\alpha(\lambda)$, is regular at λ' and the resolvent operator $R_{\lambda'}(T_\alpha)$ is given by*

$$\Phi(x,\lambda';f) = R_{\lambda'}(T_\alpha)(f)(x) = \int_0^\infty G(x,t,\lambda')f(t)dt\,, \tag{15}$$

where

$$G(x,t,\lambda') = \begin{cases} \psi(x,\lambda')\phi(t,\lambda') & \text{if } 0 \leq t < x < \infty \\ \psi(t,\lambda')\phi(x,\lambda') & \text{if } 0 \leq x < t < \infty \end{cases}$$

for all $f \in L^2[0,\infty)$

Proof: Suppose that $\lambda' = \mu' + i\nu'$ is a fixed point in $\rho(T_\alpha)$, where $\nu' > L$. Then there exists an L^2-solution of the equation

$$\tau f = \lambda' f \tag{16}$$

and the corresponding m-function $m_\alpha(\lambda)$ in the limit point case is meromorphic in the region $\nu' > L$ by the theorem in [2]. If $m_\alpha(\lambda)$ is regular at λ' then the solution $\psi(x,\lambda')$ is square integrable, whereas if $m_\alpha(\lambda)$ has a singularity (a pole) at λ' we can show the solution $\phi(x,\lambda')$ is square integrable as follows.First suppose $\alpha \neq 0$ and let Θ and χ be two linearly independent solutions of (16) satisfying

$$\Theta(0,\lambda) = -1, \quad \Theta'(0,\lambda) = 0$$

$$\chi(0,\lambda) = 0, \quad \chi'(0,\lambda) = 1$$

Then there exists a meromorphic function $M(\lambda)$ such that

$$\Psi(x,\lambda') = \Theta(x,\lambda') + M(\lambda')\chi(x,\lambda')$$

is an L^2-solution whenever λ' is a point of regularity of $M(\lambda)$. Also there is a constant $k(\lambda)$ such that for all λ, which are points of regularity of $m_\alpha(\lambda)$ and $M(\lambda)$, with $\Im\lambda > L$,

$$\Theta(x,\lambda) + M(\lambda)\chi(x,\lambda) = k(\lambda)[\theta(x,\lambda) + m_\alpha(\lambda)\phi(x,\lambda)] \tag{17}$$

because changing boundary conditions does not affect the existence of square integrable solutions. Applying (17) and its derivative at $x = 0$ we obtain

$$\Theta(0,\lambda) + M(\lambda)\chi(0,\lambda) = k(\lambda)(\theta(0,\lambda) + m_\alpha(\lambda)\phi(0,\lambda))$$

$$\Theta'(0,\lambda) + M(\lambda)\chi'(0,\lambda) = k(\lambda)(\theta'(0,\lambda) + m_\alpha(\lambda)\phi'(0,\lambda))$$

which gives

$$-1 = k(\lambda)(m_\alpha(\lambda)\sin\alpha + \cos\alpha)$$

$$M(\lambda) = k(\lambda)(-m_\alpha(\lambda)\cos\alpha + \sin\alpha)$$

and hence the m-function satisfies

$$m_\alpha(\lambda) = \frac{-\sin\alpha + M(\lambda)\cos\alpha}{\cos\alpha + M(\lambda)\sin\alpha}$$

It follows that $m_\alpha(\lambda)$ has a pole at $\lambda = \lambda'$ iff $\cos\alpha + M(\lambda)\sin\alpha$ has a zero at $\lambda = \lambda'$ but when $\cos\alpha + M(\lambda')\sin\alpha = 0$ we have

$$\cos\alpha(\Theta(0,\lambda') + M(\lambda')\chi(0,\lambda')) + \sin\alpha(\Theta'(0,\lambda') + M(\lambda')\chi'(0,\lambda') = 0$$

so the L^2-solution $\Psi(x,\lambda')$ satisfies the boundary condition α at $x = 0$, and hence $\phi(x,\lambda')$ is a scalar multiple of $\Psi(x,\lambda')$. Therefore $\phi(x,\lambda') \in L^2[0,\infty)$, so that λ' is an eigenvalue and $\lambda' \in \sigma(T_\alpha)$ whenever $m_\alpha(\lambda)$ has a pole at $\lambda = \lambda'$. If $\alpha = 0$ the argument is similar; however, it is now necessary to choose the basis $\{\Theta, \chi\}$ so that

$$\chi(0,\lambda)\cos 0 + \chi'(0,\lambda)\sin 0 \neq 0$$

i.e. so that $\chi(x,\lambda)$ does not satisfy the boundary condition $\alpha = 0$ at $x = 0$.
Now suppose that $m_\alpha(\lambda)$ is regular at λ', where $\nu' > L$. Since $m_\alpha(\lambda)$ is meromorphic [2], λ' is not a pole of $m_\alpha(\lambda)$ and we will show that $\lambda' \in \rho(T_\alpha)$. To achieve this we first note that Φ is a bounded operator defined on $\mathcal{H}$ by Lemma 2.1, so that $\Phi(x,\lambda;f) \in L^2[0,\infty)$ wherever $f \in L^2[0,\infty)$. To complete the proof that $\Phi(x,\lambda';f) \in \mathcal{D}_\alpha$ the domain of T_α we can show that Φ satisfies the boundary condition

$$\Phi(0,\lambda';f)\cos\alpha + \Phi'(0,\lambda';f)\sin\alpha = 0 \tag{18}$$

For since

$$\Phi(0,\lambda';f) = \phi(0,\lambda')\int_0^\infty \psi(t,\lambda')f(t)dt = \sin\alpha\int_0^\infty \psi(t,\lambda')f(t)dt$$

$$\Phi'(0,\lambda';f) = \phi'(0,\lambda')\int_0^\infty \psi(t,\lambda')f(t)dt = -\cos\alpha\int_0^\infty \psi(t,\lambda')f(t)dt$$

(18) follows immediately.
We also prove that $\Phi(x,\lambda';\,\cdot\,)$ is the inverse operator for the operator $T_\alpha - \lambda' I$. Obviously we have

$$\Phi(x,\lambda';(T_\alpha - \lambda' I)f) = \Phi(x,\lambda';(-f'' + qf - \lambda' f)) =$$

$$\psi(x,\lambda')\int_0^x \phi(t,\lambda')(-f'' + qf - \lambda' f)dt + \phi(x,\lambda')\int_x^\infty \psi(t,\lambda')(-f'' + qf - \lambda' f)dt$$

Integrating by parts twice we obtain

$$\Phi(x,\lambda';(-f'' + qf - \lambda' f)) = \psi(x,\lambda')\int_0^x (-\phi'' + q\phi - \lambda'\phi)f +$$

$$\phi(x,\lambda')\int_x^\infty (-\psi'' + q\psi - \lambda'\psi)f + \psi(x,\lambda')[(-f'\phi + f\phi')]_0^x +$$

$$\phi(x,\lambda')[(-f'\psi + f\psi')]_x^\infty$$

The last two terms are zero so

$$\begin{aligned}\Phi(x,\lambda';(T_\alpha - \lambda')f) &= \psi(x,\lambda')(f(x)\phi'(x) - f'(x)\phi(x)\\ &+ f(0)\phi'(0) - f'(0)\phi(0)) + \phi(x,\lambda')\\ &\times (W_\infty[f,\psi] - f(x)\psi'(x) + f'(x)\psi(x))\end{aligned}$$

Since $f \in \mathcal{D}_\alpha$ then $W_0[f,\phi] = 0$, so that

$$\Phi(x,\lambda';(T_\alpha - \lambda' I)f) = f(x)W_x[\psi,\phi] + \phi(x,\lambda')W_\infty[f,\psi] = f(x)$$

since we are in the limit point case and $\mathcal{D}_\alpha = \{f \in \mathcal{D} : W_\infty[f\psi] = 0\}$ ([5] p.267). On the other hand

$$\begin{aligned}
(T_\alpha - \lambda')\Phi &= -\Phi''(x,\lambda';f) + (q-\lambda')\Phi(x,\lambda';f) \\
&= -\psi''(x,\lambda')\int_0^x \phi(t,\lambda')f(t)dt - \phi''(x,\lambda')\int_x^\infty \psi(t,\lambda')f(t)dt \\
&+ f(x)W_x[\phi,\psi] + (q-\lambda')\Phi(x,\lambda';f) \\
&= f(x) + [-\psi''(x,\lambda') + (q-\lambda')\psi(x,\lambda')]\int_0^x \phi(t,\lambda')f(t)dt \\
&+ [-\phi''(x,\lambda') + (q-\lambda')\phi(x,\lambda')]\int_x^\infty \psi(t,\lambda')f(t)dt \\
&= f(x)
\end{aligned}$$

for all $f \in L^2[0,\infty)$. Hence we can write for each λ with the property $\Im\lambda > L$

$$(T_\alpha - \lambda)^{-1} = \Phi(\,\cdot,\ \lambda;\ \cdot\,)$$

The operator $\Phi(\,\cdot,\ \lambda;\ \cdot\,)$ therefore has all of the properties that make it identically equal to the resolvent operator $R_\lambda(T_\alpha)$ for each λ with $\Im\lambda > L$, and we conclude that $\lambda' \in \rho(T_\alpha)$.
If $\Im\lambda' < L$ proof is the same as when $\Im\lambda' > L$. The special case of $\Im\lambda' = L$ is considered in the next theorem.

Theorem 3.3 *Let $\nu' = L$ and $\lambda' \in \rho(T_\alpha)$. Then $m_\alpha(\lambda)$ is regular at λ'.*

Proof: Let $\nu' = L$ and $\lambda' \in \rho(T_\alpha)$, Since $\rho(T_\alpha)$ is an open set in the complex plane, there is a disk $D_\delta(\lambda')$ around λ' such that $D_\delta(\lambda') \subseteq \rho(T_\alpha)$. Therefore, noting that there is no loss of generality if we take $L \neq 1$, by [3] Corollary 4.6.1 we have

$$m_\alpha(\lambda) - m_\alpha(i) = (\lambda - i)\int_0^\infty \psi(x,\lambda)\psi(x,i)dx \tag{19}$$

for $\lambda \in \mathcal{D}_\delta, \Im\lambda \neq L$. From the properties of Φ as a function we see that

$$(\tau-\lambda)(i-\lambda)\Phi(x,\lambda;\psi(t,i)) = (i-\lambda)\psi(x,i)$$

and since

$$(\tau-\lambda)\psi(x,i) = (i-\lambda)\psi(x,i)$$

it follows that $(i-\lambda)\Phi(x,\lambda;\psi(t,i))$ and $\psi(x,i)$ are solutions of the non-homogeneous equation $(\tau-\lambda)f = (i-\lambda)\psi(x,i)$, so their difference is a solution of the homogeneous equation $\tau f = \lambda f$. Hence

$$(i-\lambda)\Phi(x,\lambda;\psi(t,i)) - \psi(x,i) = c_1\phi(x,\lambda) + c_2\psi(x,\lambda)\,, \tag{20}$$

where c_1 and c_2 are constants, which can be determined, since if we set $x = 0$ in (12) and its derivative, and use (11), we obtain $c_1 = 0$ and $c_2 = -1$. We have therefore

$$\psi(x, \lambda) = \psi(x, i) + (\lambda - i)\Phi(x, \lambda; \psi(t, i))$$

Also we can use Theorem 3.1 and write $\Phi(x, \lambda; \psi(t, i)) = R_\lambda(T_\alpha)\psi(t, i)(x)$ for $\lambda \in \mathcal{D}_\delta$, $\Im\lambda \neq L$ Then substituting for $\psi(x, \lambda)$ in (7) gives

$$m_\alpha(\lambda) = m_\alpha(i) + (\lambda - i)\int_0^\infty \psi(x, i)^2 dx + (\lambda - i)^2 (R_\lambda(T_\alpha)\psi(t, i)(x), \bar{\psi}(x, i))$$

But this equation implies, since the function $\lambda \longmapsto (g, R_\lambda(T_\alpha)f)$ is an analytic function from $\rho(T_\alpha)$ to C for given fixed functions f and g in $\mathcal{H}$ [6] p.101., that $m_\alpha(\lambda)$ is analytic in the neighborhood D_δ of λ', so that by analytic continuation m_α is regular on the resolvent set at λ'. We believe that the converse of Theorem 3.2 is also true, but have not been able to prove this. However, the following result provides a partial converse. Then with slightly modification of the above proofs all the above results are valid under the new conditions.

Example 3.4 Consider the simplest case of a boundary value problem

$$-y'' + q(x)y = \lambda y$$

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0\,,$$

where $q(x) = 0,\ \forall x \in [0, \infty)$, and a fundamental set of solutions $\{\phi, \theta\}$ satisfying the boundary conditions

$$\phi(0, \lambda) = \sin\alpha, \quad \phi'(0, \lambda) = -\cos\alpha$$

$$\theta(0, \lambda) = \cos\alpha, \quad \theta'(0, \lambda) = \sin\alpha\,,$$

where $\alpha \in C$. Then

$$\theta(x, \lambda) = \cos\alpha\cos(x\sqrt{\lambda}) + \lambda^{-\frac{1}{2}}\sin\alpha\sin(x\sqrt{\lambda})$$

$$\phi(x, \lambda) = \sin\alpha\cos(x\sqrt{\lambda}) + -\lambda^{-\frac{1}{2}}\cos\alpha\sin(x\sqrt{\lambda})$$

and we obtain the m-function $m_\alpha(\lambda)$ explicitly as

$$m_\alpha(\lambda) = \frac{\sin\alpha - i\sqrt{\lambda}\cos\alpha}{\cos\alpha + i\sqrt{\lambda}\sin\alpha}\,,$$

where $\Im\sqrt{\lambda} > 0$. We now show that m is regular on the whole complex plane except on the set $\{\lambda : \Im\lambda = 0,\ \Re\lambda > 0\}$ and at poles of the m-function, which satisfy the equation $i\cot\alpha = \sqrt{\lambda}$. To show that this is true, note that $m_0(\lambda) = -i\sqrt{\lambda}$ is the m-function for the selfadjoint problem with $q(x) = 0, \alpha = 0$, and is analytic on

$$S = C\backslash\{\lambda : \Im\lambda = 0, \Re\lambda \geq 0\}$$

It then follows from the expression for $m_\alpha(\lambda)$ that for $\alpha \in C\backslash\{0\}$, $m_\alpha(\lambda)$ is regular on S, apart from isolated poles at the zeros of $\cos\alpha + i\sqrt{\lambda}\sin\alpha$. Let $\alpha = \alpha_1 + i\alpha_2$. Using complex trigonometry we have

$$\lambda = -\cot^2\alpha = \frac{\sin 2\alpha_1 - i\sinh 2\alpha_2}{2|\sin\alpha|^2},$$

where $\alpha_2 \neq 0$ and the condition $\Im\sqrt{\lambda} > 0$ is equivalent to

$$n\pi < \alpha_1 < (n + \frac{1}{2})\pi,\ n \in Z.$$

Taking $\alpha_1 = 0, \alpha_2 = 1$ then m-function is

$$m(\lambda) = \frac{i\sinh 1 - i\sqrt{\lambda}\cosh 1}{\cosh 1 - \sqrt{\lambda}\sinh 1}$$

or

$$m(\lambda) = i\frac{1 - \sqrt{\lambda}\coth 1}{\coth 1 - \sqrt{\lambda}}$$

Note that the only pole of $m(\lambda)$ is at $\lambda = \lambda_0 = \coth^2 1$, so that $m(\lambda)$ is regular on S. To investigate whether Theorem 1.1(i) remains true in general in the non-selfadjoint case, we consider the behavior of $\nu m(\mu' + i\nu)$ as $\nu \to 0$ for $\mu' = \lambda_0$. We have:

$$\lim_{\nu\to 0} \nu m(\mu' + i\nu) = \lim_{\nu\to 0} \frac{\nu[1 - \sqrt{\lambda_0 + i\nu}\coth 1]}{\coth 1 - \sqrt{\lambda_0 + i\nu}}$$

Using l'Hôpital rule we have

$$\begin{aligned}\lim_{\nu\to 0} \nu m(\mu' + i\nu) &= \lim_{\nu\to 0} 2i(1 - \coth^2 1 - i\nu\coth 1)\coth 1 \\ &= 2i(1 - \coth^2 1)\coth 1 \neq 0\end{aligned}$$

Since there is no $L^2[0,\infty)$ solution of the equation $-y'' = \lambda y$ for any $\lambda \geq 0$, S lies in the essential spectrum and λ_0 is not an eigenvalue. Hence λ_0 is a point of the continuous spectrum, but Theorem 2.1(i) is not satisfied for $\mu' = \lambda_0$. This shows that Theorem 2.1(i) is not generally true in the non-selfadjoint case.

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