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Numerical Solution of the Second Kind Singular Volterra Integral Equations By Modified Navot-Simpson's Quadrature

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Abstract

In this paper, we present a method for solving the second kind Singular Abel integral equation based on the discrete Gronwall inequality, Euler Maclaurin summation formula, variable transforms and modified Simpson's integration rule and Navot's quadrature. An algorithm for solving non linear weakly singular Volterra integral equations is proposed. Then, the convergence of algorithm is derived. Some numerical results show the efficiency and accuracy of the mentioned method.

Keywords**:** ***Non linear weakly singular Volterra integral equation, The second kind singular Abel integral equation, Navot's quadrature, Simpson's rule, Zeta-Riemann function.***

1 Introduction

Many mathematical and physical models in science and engineering lead to analyze the following non linear weakly singular Volterra Abel integral equation of the second kind [5]:

$$\bar{u}(s) = \bar{y}(s) + \int_a^s \bar{k}(s,t,\bar{u}(t))dt, \qquad (1)$$

where $\bar{k}(s,t,\bar{u}(t)) = (s-t)^{\alpha}\hat{k}(s,t,\bar{u}(t))$ and $-1<\alpha<0$, $a \le t \le s \le b$. In equation (1), $\bar{u}$, $\bar{k}$, $\bar{y}$ are initial and non smooth functions. We will transform these functions respectively to u , k , y which are smooth functions in order to compute by classic methods. We assume that $\hat{k}(s,t,\bar{u}(t))$ satisfies in Lipchitz condition, i.e. for fixed s,t with $a \le t \le s \le b$, there is a positive constant L independent of s and t such that,

$$\forall u,v \in \mathbb{R}, \qquad \left|\hat{k}(s,t,u)-\hat{k}(s,t,v)\right| \le L\,|u-v|. \tag{2}$$

There exist very powerful direct methods for solving (1). It is not necessary that the kernel be continuous in order to apply a numerical method. Such methods include product integration methods, convolution quadrature methods [9] and collocation methods on graded meshes [10,11]. But, the method of the present paper is an alternative way of solving (1). By smoothing kernels, we are able to solve such equations by foregoing classic methods. For this purpose, under the change of variable [1,2,7]

$$\gamma(t)=(t-a)^{q}+a, \tag{3}$$

the equation (1) becomes

$$\bar{u}(\gamma(s))=\bar{y}(\gamma(s))+\int_a^s(\gamma(s)-\gamma(t))^{\alpha}\,\hat{k}(\gamma(s),\gamma(t),\bar{u}(\gamma(t)))\gamma'(t)dt, \tag{4}$$

$a \le t \le s \le \gamma^{-1}(b)$,

where, q is a positive integer. If we put

$$u(s)=\bar{u}(\gamma(s)), \quad y(s)=\bar{y}(\gamma(s)), \tag{5}$$

then, we have from equation (4),

$$u(s)=y(s)+\int_a^s(\gamma(s)-\gamma(t))^{\alpha}\,\hat{k}(\gamma(s),\gamma(t),u(t))\gamma'(t)dt. \tag{6}$$

Now, equation (6) can be expressed by,

$$u(s)=y(s)+\int_a^s k^{*}(s,t,u(t))dt, \tag{7}$$

where,

$$k^{*}(s,t,u(t))=(s-t)^{\alpha}k(s,t,u(t)), \tag{8}$$

and,

$$k(s,t,u(t))=\begin{cases}\left(\dfrac{\gamma(s)-\gamma(t)}{s-t}\right)^{\alpha}\hat{k}(\gamma(s),\gamma(t),u(t))\gamma'(t), & s\neq t\\[2mm] (\gamma'(s))^{\alpha}\,\hat{k}(\gamma(s),\gamma(t),u(t))\gamma'(t). & s=t\end{cases} \tag{9}$$

Now, $k(s,t,u(t))$ is a smooth function that can be applied in computational algorithm. In fact, with a suitable choice of the parameter q we can ensure that the solution $u(t)$ is sufficiently smooth [1,5].

2 The numerical method

In the sequel, at first we describe the quadrature formula of the integration with singularity at the end point using Euler Maclaurin summation formula given by Navot in [5,6]. Let

$$I(G)=\int_a^b G(x)dx=\int_a^b (b-x)^{\alpha} g(x)dx, \qquad (10)$$

where, $-1<\alpha<0$ and $G(x)=(b-x)^{\alpha} g(x)$. In (10), $g(x)$ is a smooth function on the interval $[a,b]$. We take the step length $h=\frac{b-a}{N}$ and mesh points $x_i=a+ih, i=0,...,N$, where N is a sufficiently large number.

Since the kernel and the deriving term of the integral equation (7) are expressed by weakly singular integrals, we must use a numerical method which is able to compute these integrals with weakly singularity at the endpoints. For this purpose, Navot's quadrature rule is used. This special quadrature is applied for functions having a singularity of any type on or near the endpoints of the integration interval. This method assigns equal weights to the points $x=a$ and $x=b$ and arbitrary weights to the points in the integration interval with equal distance between the adjacent points. If $a=0$ and $b=1$ then $h=\frac{1}{N}$. In this case, The trapezopidal rule is defined as :

$$Tf=\frac{1}{N}\sum_{j=1}^{N-1} f\left(\frac{j}{N}\right)+\frac{1}{2N}[f(0)+f(1)], \qquad (11)$$

and the midpoint rule is defined as :

$$Mf=\frac{1}{N}\sum_{j=1}^{N} f\left(\frac{2j-1}{2N}\right). \qquad (12)$$

One can improve these relations for the interval $[a,b]$ that can be found in [13]. From (11) and (12), we consider the Simpson's integration rule as a linear combination of the mid-point rule and the trapezoidal rule as follows [3,4] :

$$Sf=\frac{2}{3}Mf+\frac{1}{3}Tf \qquad (13)$$

Now, we can prove the following lemma to construct the quadrature rule.

Lemma 2.1 *Let* $g(x)\in C^{2l+1}[a,b]$ $(l\in Z^{+})$, $G(x)=(b-x)^{\alpha} g(x)$, $h=\frac{b-a}{N}$, N *is even and* $x_i=a+ih$, $i=0,1,...,N$ *.Then the Navot-Simpson's quadrature rule* $S_N(G)$ *has an asymptotic expansion as follows:*

$$S_N(G) = \frac{h}{3}G(x_0) + \frac{4}{3}h\sum_{i=1}^{\frac{N}{2}} G(x_{2i-1}) + \frac{2}{3}h\sum_{i=1}^{\left\lfloor\frac{N-1}{2}\right\rfloor} G(x_{2i})$$
$$-g(b)\left(\frac{2}{3}\zeta(-\alpha,\frac{1}{2}) + \frac{1}{3}\zeta(-\alpha)\right)h^{1+\alpha} \tag{14}$$
$$= \int_a^b (b-x)^\alpha g(x)dx + \sum_{j=1}^{l} P_j G^{(2j-1)}(a)h^{2j}$$
$$+\sum_{j=1}^{2l}(-1)^j \frac{g^{(j)}(b)h^{j+\alpha+1}}{j!}\left(\frac{1}{3}+\frac{2}{3}(2^{-\alpha-j}-1)\right)\zeta(-\alpha-j) + O(h^{2l+1}),$$

where, $\alpha > -1,\ \zeta(-\alpha,\frac{1}{2}) = (2^{-\alpha}-1)\zeta(-\alpha),\ \zeta(x)$ *is the Riemann-Zeta function and* $P_j\ (j = 1,2,...,l)$ *are all constants independent of h.*

Proof. In [1,4,5] the modified trapezoidal rule $T_{h'}(G)$ has been introduced by using Navot's quadrature as follows:

$$T_{h'}(G) = \frac{h'}{2}G(x'_0) + h'\sum_{j=1}^{M-1} G(x'_j) - \zeta(-\alpha)g(b)h'^{1+\alpha}$$
$$= \int_a^b (b-x)^\alpha g(x)dx + \sum_{j=1}^{l}\frac{B_{2j}}{(2j)!}G^{(2j-1)}(a)h'^{2j}$$
$$+\sum_{j=1}^{2l}(-1)^j\zeta(-\alpha-j)\frac{g^{(j)}(b)h'^{j+\alpha+1}}{j!} + O(h'^{2l+1}),$$

where, $\zeta(x)$ is the Riemann–Zeta function and B_{2j} (j = 1, . . . , l) are the Bernoulli numbers and $x'_j = a + jh',\ j = 0,1,...,M-1,\ h' = \frac{b-a}{M}$. We can write the similar formula for the modified rectangular mid-point rule. For this purpose, we consider the following formula [3,12]:

$$M_{h'}(G) = h'\sum_{j=1}^{M} G(x'_j - \frac{h'}{2}) - (2^{-\alpha}-1)\zeta(-\alpha)g(b)h'^{1+\alpha}.$$

Finally, since the number of the points when we combine the modified Trapezoidal and mid-point rules is N=2M which is even, hence $\left\lfloor\frac{N-1}{2}\right\rfloor = \frac{N}{2}-1$ and $\left\lfloor\frac{N}{2}\right\rfloor = \frac{N}{2}$ then if $h = \frac{b-a}{N}$ then, $x_{2j-1} = x'_j - h,\ j = 1,2,...,M$, and $x_{2j} = x'_j,\ j = 1,2,...,M-1,$ hence we can compute $S_N(G)$ by using the relation (13) in the interval [a,b] as follows:

$$S_N(G)=\frac{2}{3}M_{h'}(G)+\frac{1}{3}T_{h'}(G)=\frac{2}{3}(h'\sum_{j=1}^{M}G(x'_j-\frac{h'}{2})-(2^{-\alpha}-1)\zeta(-\alpha)g(b)h'^{1+\alpha})$$
$$+\frac{1}{3}(\frac{h'}{2}G(x'_0)+h'\sum_{j=1}^{M-1}G(x'_j)-\zeta(-\alpha)g(b)h'^{1+\alpha})\ =\frac{h'}{6}G(x'_0)+\frac{2}{3}h'\sum_{j=1}^{M}G(x'_j-\frac{h'}{2})$$
$$+\frac{1}{3}h'\sum_{j=1}^{M-1}G(x'_j)-(\frac{2}{3}(2^{-\alpha}-1)\zeta(-\alpha)+\frac{1}{3}\zeta(-\alpha))g(b)h'^{1+\alpha}\ =\frac{h}{3}G(x_0)+\frac{4}{3}h\sum_{j=1}^{\frac{N}{2}}G(x_{2j-1})$$
$$+\frac{2}{3}h\sum_{j=1}^{\frac{N}{2}-1}G(x_{2j})-(\frac{2}{3}\zeta(-\alpha,\frac{1}{2})+\frac{1}{3}\zeta(-\alpha))g(b)h^{1+\alpha}.$$

Also, if $P_j,\ j=1,2,...,l$, are the constant values independent of *h*, we can prove the following relation similarly.

$$S_N(G)=\int_a^b(b-x)^{\alpha}g(x)dx\ +\sum_{j=1}^{l}P_jG^{(2j-1)}(a)h^{2j}$$
$$+\sum_{j=1}^{2l}(-1)^j\ \frac{g^{(j)}(b)h^{j+\alpha+1}}{j\,!}(\frac{1}{3}+\frac{2}{3}(2^{-\alpha-j}-1))\zeta(-\alpha-j)+O(h^{2l+1}).$$

This completes the proof. □

By lemma 2.1, it is easy to derive that $E_N(G)=O(h^{2l+1+\alpha})$.

Now, we set $s=x_i$ in equation (7), therefore

$$u(x_i)=y(x_i)+\int_{x_0}^{x_i}(x_i-t)^{\alpha}k(x_i,t,u(t))dt. \qquad (15)$$

Using the Navot-Simpson's quadrature formula (14), we obtain the following discrete equations:

$$u_0=y(x_0)$$
$$u_1=y(x_1)+\frac{h}{3}(x_1-x_0)^{\alpha}k(x_1,x_0,u_0)$$
$$u_i=y(x_i)+\frac{h}{3}(x_i-x_0)^{\alpha}k(x_i,x_0,u_0)$$
$$+\frac{4h}{3}\sum_{j=1}^{\left\lfloor\frac{i}{2}\right\rfloor}(x_i-x_{2j-1})^{\alpha}k(x_i,x_{2j-1},u_{2j-1}) \qquad (16)$$
$$+\frac{2h}{3}\sum_{j=1}^{\left\lfloor\frac{i-1}{2}\right\rfloor}(x_i-x_{2j})^{\alpha}k(x_i,x_{2j},u_{2j})$$
$$-h^{1+\alpha}k(x_i,x_i,u_i)(\frac{2}{3}\zeta(-\alpha,\frac{1}{2})+\frac{1}{3}\zeta(-\alpha)),\qquad i=2,...,N.$$

The equations (16), as a non linear diagonal system of equations in $u_j,\ j=1,2,...,\frac{N}{2}$, for partitions of odd and even mesh points, can be computed by
the following iterative relation:

$$u_i^{m+1}=y(x_i)+\frac{h}{3}(x_i-x_0)^{\alpha}k(x_i,x_0,u_0)$$
$$+\frac{4h}{3}\sum_{j=1}^{\lfloor\frac{i}{2}\rfloor}(x_i-x_{2j-1})^{\alpha}k(x_i,x_{2j-1},u_{2j-1}) \quad (17)$$
$$+\frac{2h}{3}\sum_{j=1}^{\lfloor\frac{i-1}{2}\rfloor}(x_i-x_{2j})^{\alpha}k(x_i,x_{2j},u_{2j})$$
$$-h^{1+\alpha}(\frac{2}{3}\zeta(-\alpha,\frac{1}{2})+\frac{1}{3}\zeta(-\alpha))k(x_i,x_i,u_i^m).$$

Algorithm1: (The Navot-Simpson's quadrature rule)

Step1 : Take $\varepsilon>0$ sufficiently small and $\tilde{u}_0=y(x_0)$ and set $i:=1.$
Step2 : Set $u_i^0=\tilde{u}_{i-1}$ and $m:=0$, compute $u_i^{m+1}\ ,i\le N$ by relation (17).
Step3 : if $\left|u_i^{m+1}-u_i^m\right|\le\varepsilon$,then set $\tilde{u}_i=u_i^{m+1}$ and $i:=i+1$, go to Step 2; otherwise set $m:=m+1$ and go to Step 2.
In the above algorithm, $u_i,\tilde{u}_i$ are the approximations of the solution of equations (16) and (17) respectively.

3 Convergence and error estimate

In order to obtain an error estimate of the method, we need the following discrete Gronwall inequality.
Lemma 3.1 [1]. If a non negative sequence $\{y_n,n=0,...,N\}$ satisfies $y_0=0$, $y_n\le A+Bh\sum_{j=0}^{n-1}y_j, 1\le n\le N, h=\frac{1}{N}$, then $\max_{0\le i\le N}y_i\le Ae^B$,
where, A and B are positive constants independent of h.

Now, using quadrature rule (14), equation (16) can be written using Navot-Simpson's quadrature rule as follows:

$$
\begin{aligned}
u(x_i) &= y(x_i) + \frac{h}{3}(x_i - x_0)^{\alpha} k(x_i, x_0, u_0) \\
&+ \frac{4h}{3}\sum_{j=1}^{\left\lfloor \frac{i}{2} \right\rfloor}(x_i - x_{2j-1})^{\alpha} k(x_i, x_{2j-1}, \tilde{u}_{2j-1}) \\
&+ \frac{2h}{3}\sum_{j=1}^{\left\lfloor \frac{i-1}{2} \right\rfloor}(x_i - x_{2j})^{\alpha} k(x_i, x_{2j}, \tilde{u}_{2j}) \\
&- h^{1+\alpha} k(x_i, x_i, \tilde{u}_i)(\frac{2}{3}\zeta(-\alpha, \frac{1}{2}) + \frac{1}{3}\zeta(-\alpha)).
\end{aligned} \tag{18}
$$

By categorizing weights in summation and considering i, j as total mesh points for partitions of odd and even mesh points, we have:

$$
\begin{aligned}
u(x_i) &= y(x_i) + hw_{i0}(x_i - x_0)^{\alpha} k(x_i, x_0, u_0) \\
&+ h\sum_{j=1}^{N-1} w_{ij}(x_i - x_j)^{\alpha} k(x_i, x_j, \tilde{u}_j) \\
&+ hw_{ii} k(x_i, x_i, \tilde{u}_i) \\
&+ E_{i,t}((x_i - t)^{\alpha} k(x_i, t, u(t)),
\end{aligned} \tag{19}
$$

where, $w_{ii} = h^{\alpha}(\frac{2}{3}\zeta(-\alpha, \frac{1}{2}) + \frac{1}{3}\zeta(-\alpha))$, $w_{i0} = \frac{1}{3}$ and

$$
w_{ij} = \begin{cases} \frac{4}{3} & 1 \le j \le i \le \frac{N}{2} \qquad i, j\,(odd) \\ \frac{2}{3} & 1 \le j \le i \le \left\lfloor \frac{N-1}{2} \right\rfloor \qquad i, j\,(even). \end{cases}
$$

By using lemma 2.1, the reminder has the following estimation:

$$
\left| E_{i,t}((x_i - t)^{\alpha} k(x_i, t, u(t)) \right| = O(h^{2l+1+\alpha}) \tag{20}
$$

Setting $e_i = u(x_i) - u_i$, subtracting (16) from (18) and using (19), we obtain:

$e_0 = 0$

$$
\begin{aligned}
e_i &= h\sum_{j=1}^{N-1} w_{ij}(x_i - x_j)^{\alpha}[k(x_i, x_j, u(x_j)) - k(x_i, x_j, u_j)] \\
&+ hw_{ii}[k(x_i, x_i, u(x_i)) - k(x_i, x_i, u_i)] + E_{i,t}((x_i - t)^{\alpha} k(x_i, t, u(t))),
\end{aligned} \tag{21}
$$

$i = 1, 2, ..., N$.

The following lemma is the generalization of discrete Gronwall inequality in lemma 3.1 which has been proposed in [6].

Lemma 3.2 [6]. Suppose sequence $\{e_i\}_{i=1}^{n}$ satisfies

$$
\begin{cases} e_0 = 0, \\ |e_i| \le \sum_{j=1}^{i-1} B_{ij}|e_j| + A, \end{cases} \tag{22}
$$

$1 \le i \le N$

where, $B_{ij} = 2Lh(x_i - x_j)^{\alpha}$, $-1 < \alpha < 0$ and h is sufficiently small to have $Lhw_{ii} \le \frac{1}{2}$. Then, $|e_i| \le HA$ where,

$$H = \sum_{k=0}^{\infty} \frac{R^k}{(k!)^s}, R = 2L(b-a)^s \Gamma(s) e^{\frac{1}{12s}} (\frac{e}{s})^s, s = 1+\alpha.$$

By using the lemma 3.2, we can prove the following theorem.

Theorem 3.1 Assume that h is sufficiently small, then the solutions of the non linear discrete equations (16) exist and are unique and the simple iteration of solution in Navot-Simpson's quadrature rule (17) is geometrically convergent.

Proof. If $\{u_i\}$ and $\{v_i\}$ are solutions of (18), then the differences $\{z_i = u_i - v_i\}$ satisfy in the following relation:

$$\begin{aligned} z_i = u_i - v_i \le & \frac{h}{3}(x_i - x_0)^{\alpha}[k(x_i, x_0, u_0) - k(x_i, x_0, v_0)] \\ & + \frac{4h}{3}\sum_{j=1}^{\frac{N}{2}} (x_i - x_j)^{\alpha}[k(x_i, x_j, u_j) - k(x_i, x_j, v_j)] \\ & + \frac{2h}{3}\sum_{j=1}^{\lfloor \frac{N-1}{2} \rfloor} (x_i - x_j)^{\alpha}[k(x_i, x_j, u_j) - k(x_i, x_j, v_j)] \\ & - h^{1+\alpha}(\frac{2}{3}\zeta(-\alpha, \frac{1}{2}) + \frac{1}{3}\zeta(-\alpha))[k(x_i, x_i, u(x_i)) - k(x_i, x_i, v(x_i))]. \end{aligned} \tag{23}$$

By taking magnitude on two sides of relation (23) and using the Lipchitz condition with $u_0 = v_0$ we conclude:

$$|z_i| \le \frac{4h}{3}\sum_{j=1}^{\frac{N}{2}} (x_i - x_j)^{\alpha} L |u_j - v_j| + \frac{2h}{3}\sum_{j=1}^{\lfloor \frac{N-1}{2} \rfloor} (x_i - x_j)^{\alpha} L |u_j - v_j|. \tag{24}$$

Then, since $z_i = u_i - v_i$ we have:

$$|z_i| \le \frac{4h}{3}\sum_{j=1}^{\frac{N}{2}} (x_i - x_j)^{\alpha} L |z_j| + \frac{2h}{3}\sum_{j=1}^{\lfloor \frac{N-1}{2} \rfloor} (x_i - x_j)^{\alpha} L |z_j|. \tag{25}$$

By factoring,

$$|z_i| \le (\frac{4h}{3}\sum_{j=1}^{\frac{N}{2}} (x_i - x_j)^{\alpha} + \frac{2h}{3}\sum_{j=1}^{\lfloor \frac{N-1}{2} \rfloor} (x_i - x_j)^{\alpha}) L |z_j|. \tag{26}$$

Then,

$$\left|z_i\right| \le \sum_{j=1}^{\frac{N}{2}} B_{ij}\left|z_j\right| + \sum_{j=1}^{\left\lfloor\frac{N-1}{2}\right\rfloor} B_{ij}\left|z_j\right|. \tag{27}$$

So,

$$\left|z_i\right| \le \sum_{j=1}^{\frac{N}{2}} \overline{B}_{ij}\left|z_j\right| + \sum_{j=1}^{\left\lfloor\frac{N-1}{2}\right\rfloor} \overline{B}_{ij}\left|z_j\right|, \tag{28}$$

where, $B_{ij} = 2Lh(x_i - x_j)^{\alpha}$ and

$$\overline{B}_{ij} = \begin{cases} \frac{2}{3}Lh(x_i - x_j)^{\alpha} & i, j\,(odd) \\ \frac{1}{3}Lh(x_i - x_j)^{\alpha} & i, j\,(even). \end{cases} \tag{29}$$

Assuming $\overline{L} = (1+c)L$, we have:

$$\overline{B}_{ij} \le \begin{cases} \frac{2}{3}\overline{L}h(x_i - x_j)^{\alpha} & i, j\,(odd) \\ \frac{1}{3}\overline{L}h(x_i - x_j)^{\alpha} & i, j\,(even). \end{cases} \tag{30}$$

By using discrete Gronwall inequality from lemma 3.2 with $A = 0$, we obtain:

$$z_j = 0, j = 0, \ldots, N.$$

Therefore, we have proved that the simple iteration exists and is unique. For proof of convergence, we obtain from iterative relation (17):

$$\left|u_i^{n+1} - u_i^n\right| \le \left(\frac{2}{3}\zeta(-\alpha, \frac{1}{2}) + \frac{1}{3}\zeta(-\alpha)\right)h^{1+\alpha}\left|k(x_i, x_i, u_i^n) - k(x_i, x_i, u_i^{n-1})\right| \tag{31}$$

$$\le Lhw_{ii}\left|u_i^n - u_i^{n-1}\right| \le \frac{1}{2}\left|u_i^n - u_i^{n-1}\right|.$$

So, we have:

$$\left|u_i^{n+1} - u_i^n\right| \le \frac{1}{2}\left|u_i^n - u_i^{n-1}\right| \le \frac{1}{4}\left|u_i^{n-1} - u_i^{n-2}\right| \le \frac{1}{8}\left|u_i^{n-2} - u_i^{n-3}\right| \le \frac{1}{16}\left|u_i^{n-3} - u_i^{n-4}\right| \le \frac{1}{32}\left|u_i^{n-4} - u_i^{n-5}\right| \le$$

$$\ldots \le \frac{1}{2^k}\left|u_i^{n-(k-1)} - u_i^{n-k}\right| = \frac{1}{2^k}\left|u_i^{n-k+1} - u_i^{n-k}\right| \le \ldots \le \frac{1}{2^{n-1}}\left|u_i^2 - u_i^1\right| \le \frac{1}{2^n}\left|u_i^1 - u_i^0\right|. \tag{32}$$

Then, we conclude that

$$\left|u_i^{n+1} - u_i^n\right| \le \frac{1}{2^n}\left|u_i^1 - u_i^0\right|. \tag{33}$$

According to lemma 3.2, we assume that h is so small that $Lhw_{ii} \leq \frac{1}{2}$. Therefore, we have proved that the simple iterative relation (18) is geometrically convergent. So, its limit is the unique solution of (17). □

Theorem 3.2 If $g(x) \in C^{2l+1}[a,b]$, $l \in Z^{+}$, then there is a positive constant c independent of h such that the errors $e_j = u(x_j) - u_j$, $j = 0,...,N$, have the following estimate

$$\max_{0 \leq i \leq N} |e_i| \leq ch^{2l+1+\alpha} \tag{34}$$

Proof. From relation (21), we easily derive that

$$e_0 = 0 \tag{35}$$

$$|e_i| \leq A + \sum_{j=1}^{i-1} B_{ij} \, |e_j|,$$

where, according to Lemma 3.2,

$$A = \max_{0 \leq i \leq N} \max_{a \leq t \leq b} 2 \left| E_{i,t}((x_i - t)^{\alpha} k(x_i, t, u(t))) \right|,$$

and

$$B_{ij} = 2Lh(x_i - x_j)^{\alpha}.$$

By substituting A, B_{ij} in relation (35) and by means of estimates (20), (30) and lemma 3.2, the theorem is proved. □

4 Numerical Examples

In this section, we apply the algorithm 1 to solve the following examples. The programs have been provided with Maple 11. We can observe that the Navot-Simpson's algorithm which is used in the examples, has two following advantages:

1-The accuracy of the solution in Navot-Simpson's algorithm is almost the same as Navot-trapezoidal algorithm by extrapolation which has been used in [5] but, the cost of operations in Navot-Simpson's algorithm is less than Navot–trapezoidal algorithm by extrapolation in [5].

2-The error estimate in Navot-Simpson's rule from the theorem 3.2 is $O(h^{2l+1+\alpha})$, $l \geq 1$, while the most error estimate in Navot–trapezoidal rule by extrapolation is $O(h^{3+\alpha})$ [1]. This implies that the order of convergence of Navot-Simpson's quadrature is faster.

Example 5.1 We consider the Linear Volterra integral equation with algebraic singularity presented in [5]:

$$u(s) = \frac{1}{2}\pi s + \sqrt{s} + \int_0^s -\frac{u(t)}{\sqrt{s-t}}dt, \qquad 0 \le t \le s \le 1.$$

The exact solution is $u(s) = \sqrt{s}$. For transformation $\gamma(t) = t^2$ and comparison of Navot-Trapezoidal quadrature by extrapolation [5] and Navot-Simpson's quadrature, the errors of approximations at $s = 1.0$ are shown in table 1.

Table 1: Comparison of approximation solutions and errors of Navot-Trapezoidal rule by extrapolation and Navot-Simpson's rule

N	Trapezoidal by extrapolation	Error	Navot-Simpson	Error
10	1.070622162	7.0622162E-2	1.082015673	8.2015673E-2
20	1.026319894	2.6319894E-2	1.068952010	6.8952010E-2
40	1.008649377	8.649377E-3	1.035496745	3.5496745E-2
80	1.003790518	3.790518E-3	1.001790922	1.790922E-3

Example 5.2 We consider the singular integral equation in [8]:

$$u(s) = 1 - \frac{\sqrt{3}}{\pi}\int_0^s \frac{t^{\frac{1}{3}}u^4(t)}{(s-t)^{\frac{2}{3}}}dt, \qquad 0 \le t \le s \le 1,$$

with exact solution that likes behave $u(t) = t^{\frac{2}{3}}$ at the singularity $t = 0$. For transformation $q = 2, a = 0,$ the errors of the approximations for modified Trapezoidal rule and modified Simpson's rule at $s = 0$ are shown in table 2.

Table 2: Approximations and errors for Navot-Trapezoidal and Navot-Simpson's rules.

N	Trapezoidal by extrapolation	Error	Navot-Simpson	Error
10	0.9387219650	6.1278035E-2	0.9528281752	4.71171824E-2
20	0.993963696	6.036304E-3	0.9791345916	2.0865408E-2
40	1.00189888	1.89888E-3	0.9912994100	8.70059E-3
80	0.999499586	5.00414E-4	0.9964753672	3.524632E-3

4 Conclusion

Many of mechanical and physical problems are converted to a type of second kind singular Abel integral equation. In this work, for solving these kinds of integral equations, we presented a numerical method to approximate the solution by means of Navot's quadrature and Simpson's rule. We apply the integral equation which has a singularity at one of the endpoints. One can improve this technique to use the Navot's quadrature and modify it for the case that there are singularity at both of the endpoints of the integration interval.

5 Open Problem

We can develop the idea for integral equation of the first kind and singular integral equation with Cauchy kernel and integro-differential equations.

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