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Majorization Problems for Certain Analytic Functions

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Abstract

A subclass $\mathcal{A}(\alpha, \beta, j)$ of certain analytic functions in the open unit disk $\mathbb{U}$ is introduced. For the class $\mathcal{A}(\alpha, \beta, j)$, a majorization problem for $f(z)$ belonging to $\mathcal{A}(\alpha, \beta, j)$ is considered. Furthermore, we give the open problem for the coefficients $|c_n|$ of $f(z)$ belonging to $\mathcal{A}(\alpha, \beta, 1)$.

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1 Introduction

Let $\mathcal{A}(\alpha, \beta, j)$ be the class of functions $h(z)$ of the form

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (c_n \in \mathbb{C}) \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and satisfy

$$\mathrm{Re}\{h(z) + \alpha z^j h^{(j)}(z)\} > \beta \quad (z \in \mathbb{U}) \tag{1.2}$$

for some $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) \geqq 0$ and $0 \leqq \beta < 1$, where $j \in \mathbb{N} = \{1, 2, 3, \cdots\}$.

For $j = 1$, we can show the example in a function $h(z) \in \mathcal{A}(\alpha, \beta, 1)$ making use of the manner due to Owa, Hayami and Kuroki [7].

Example 1 For $h(z)$ in the class $\mathcal{A}(\alpha, \beta, 1)$, we define the function $F(z)$ by

$$F(z)=\frac{h(z)+\alpha zh'(z)-\beta}{1-\beta}. \tag{1.3}$$

Then, $F(z)$ is the Carathéodory function, since $F(0)=1$ and $\mathrm{Re}F(z)>0$. Hence, we can write

$$F(z)=\frac{h(z)+\alpha zh'(z)-\beta}{1-\beta}=\int_{|x|=1}\frac{1+xz}{1-xz}d\mu(x), \tag{1.4}$$

where $\mu(x)$ is the probability measure on $X=\{x\in\mathbb{C}:|x|=1\}$ (cf. [4]). Since (1.4) is equivalent to

$$\alpha\left(\frac{1}{\alpha}h(z)+zh'(z)\right)=\beta+(1-\beta)\int_{|x|=1}\left(1+\sum_{n=1}^{\infty}2x^nz^n\right)d\mu(x), \tag{1.5}$$

we have that

$$z^{\frac{1}{\alpha}-1}\left(\frac{1}{\alpha}h(z)+zh'(z)\right)=\frac{1}{\alpha}z^{\frac{1}{\alpha}-1}\left\{1+(1-\beta)\int_{|x|=1}\left(\sum_{n=1}^{\infty}2x^nz^n\right)d\mu(x)\right\}. \tag{1.6}$$

Integrating both sides of (1.6), we know that

$$\begin{aligned}&\int_0^z\zeta^{\frac{1}{\alpha}-1}\left(\frac{1}{\alpha}h(\zeta)+\zeta h'(\zeta)\right)d\zeta\\&\quad=\frac{1}{\alpha}\int_{|x|=1}\left\{\int_0^z\left(\zeta^{\frac{1}{\alpha}-1}+2(1-\beta)\left(\sum_{n=1}^{\infty}x^n\zeta^{n+\frac{1}{\alpha}-1}\right)\right)d\zeta\right\}d\mu(x),\end{aligned} \tag{1.7}$$

that is, that

$$z^{\frac{1}{\alpha}}h(z)=z^{\frac{1}{\alpha}}+2(1-\beta)\int_{|x|=1}\left(\sum_{n=1}^{\infty}\frac{x^n}{1+\alpha n}z^{n+\frac{1}{\alpha}}\right)d\mu(x). \tag{1.8}$$

This implies that $h(z)\in\mathcal{A}(\alpha,\beta,1)$ if and only if

$$h(z)=1+\sum_{n=1}^{\infty}\left(\frac{2(1-\beta)}{1+\alpha n}\int_{|x|=1}x^nd\mu(x)\right)z^n. \tag{1.9}$$

Let $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$. Then $f(z)$ is said to be subordinate to $g(z)$ if there exists an analytic function $w(z)$ in $\mathbb{U}$ satisfying $w(0) = 0$, $|w(z)| \leqq |z|$ $(z \in \mathbb{U})$ and $f(z) = g(w(z))$. We denote this subordination by

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \qquad \text{(cf. Nehari[6, p. 226])}. \tag{1.10}$$

If $g(z)$ is univalent in $\mathbb{U}$, then this subordination $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$ (cf. Duren [2] or Goodman [3]).

Further, $f(z)$ is said to be quasi-subordinate to $g(z)$ if there exists an analytic function $w(z)$ such that $\dfrac{f(z)}{w(z)}$ is analytic in $\mathbb{U}$,

$$\frac{f(z)}{w(z)} \prec g(z) \quad (z \in \mathbb{U}), \tag{1.11}$$

and $|w(z)| \leqq 1$ $(z \in \mathbb{U})$. We also denote this quasi-subordination by

$$f(z) \underset{q}{\prec} g(z) \quad (z \in \mathbb{U}). \tag{1.12}$$

Note that the quasi-subordination (1.12) is equivalent to

$$f(z) = w(z)g(\phi(z)), \tag{1.13}$$

where $|w(z)| \leqq 1$ $(z \in \mathbb{U})$ and $|\phi(z)| \leqq |z|$ $(z \in \mathbb{U})$ (see Robertson [8]).

In the quasi-subordination (1.12), if $w(z) \equiv 1$, then (1.12) becomes the subordination (1.10).

For analytic functions $f(z)$ and $g(z)$ in $\mathbb{U}$, we say that $f(z)$ is majorized by $g(z)$ if there exists an analytic function $w(z)$ in $\mathbb{U}$ satisfying $|w(z)| \leqq 1$ and $f(z) = w(z)g(z)$ $(z \in \mathbb{U})$. We denote this majorization by

$$f(z) \ll g(z) \quad (z \in \mathbb{U}) \qquad \text{(see MacGregor[5])}. \tag{1.14}$$

If we take $\phi(z) = z$ in (1.13), then the quasi-subordination (1.12) becomes the majorization (1.14).

Altintas and Owa [1] have considered some problems for the majorizations of $f(z)$.

2 A majorization problem

To consider our problems, we need the following lemmas.

Lemma 1 *If $h(z)$ is in the class $\mathcal{A}(\alpha,\beta,j)$ with $c_n=|c_n|e^{i(n\theta+\pi)}$ $(n=1,2,3,\cdots)$, then*

$$\sum_{n=1}^{\infty}|c_n|+\mathrm{Re}(\alpha)\sum_{n=j}^{\infty}\frac{n!}{(n-j)!}|c_n|\leqq 1-\beta. \tag{2.1}$$

Proof For $h(z)\in\mathcal{A}(\alpha,\beta,j)$, we note that

$$1+\mathrm{Re}\left\{\sum_{n=1}^{\infty}c_nz^n+\alpha\sum_{n=j}^{\infty}\frac{n!}{(n-j)!}c_nz^n\right\}>\beta \quad (z\in\mathbb{U}). \tag{2.2}$$

Since $c_n=|c_n|e^{i(n\theta+\pi)}$, we consider z such that $z=|z|e^{-i\theta}$ $(z\in\mathbb{U})$. Then we can write

$$c_nz^n=|c_n||z|^ne^{i\pi}=-|c_n||z|^n. \tag{2.3}$$

This implies that

$$1-\mathrm{Re}\left\{\sum_{n=1}^{\infty}|c_n||z|^n+\alpha\sum_{n=j}^{\infty}\frac{n!}{(n-j)!}|c_n||z|^n\right\}>\beta. \tag{2.4}$$

Letting $|z|\to 1^-$, we see from (2.4) that

$$\sum_{n=1}^{\infty}|c_n|+\mathrm{Re}\,(\alpha)\sum_{n=j}^{\infty}\frac{n!}{(n-j)!}|c_n|\leqq 1-\beta, \tag{2.5}$$

which completes the proof of our lemma.

Taking $j=1$ in Lemma 1, we have the following result which is the improvement of the lemma by Altintas and Owa [1].

Corollary 1 *If $h(z)$ is in the class $\mathcal{A}(\alpha,\beta,1)$ with $c_n=|c_n|e^{i(n\theta+\pi)}$ $(n=1,2,3,\cdots)$, then*

$$\sum_{n=1}^{\infty}(1+n\mathrm{Re}(\alpha))\,|c_n|\leqq 1-\beta.$$

With the help of Lemma 1, we have

Lemma 2 *If $h(z)$ is in the class $\mathcal{A}(\alpha,\beta,j)$ with $c_n = |c_n|e^{i(n\theta+\pi)}$ $(n = 1,2,3,\cdots)$, then*

$$1-\frac{1-\beta}{1+A_j j!\mathrm{Re}(\alpha)}|z| \leqq \mathrm{Re}\,(h(z)) \leqq |h(z)| \leqq 1+\frac{1-\beta}{1+A_j j!\mathrm{Re}(\alpha)}|z| \quad (z\in\mathbb{U}) \tag{2.6}$$

where $A_j = \begin{cases} 0 & (n<j) \\ 1 & (n\geqq j) \end{cases}$.

Proof Since $h(z)\in\mathcal{A}(\alpha,\beta,j)$, we have

$$|h(z)| \leqq 1+|z|\sum_{n=1}^{\infty}|c_n|. \tag{2.7}$$

Noting that

$$\frac{n!}{(n-j)!} \geqq j! \quad (n=j,j+1,j+2,\cdots), \tag{2.8}$$

we see that by Lemma 1

$$\sum_{n=1}^{\infty}|c_n| + j!\mathrm{Re}(\alpha)\sum_{n=j}^{\infty}|c_n| \leqq 1-\beta, \tag{2.9}$$

which is equivalent to

$$\sum_{n=1}^{\infty}|c_n| \leqq \frac{1-\beta}{1+A_j j!\mathrm{Re}(\alpha)}, \tag{2.10}$$

where $A_j = \begin{cases} 0 & (n<j) \\ 1 & (n\geqq j) \end{cases}$.

Therefore, with the help of (2.7) and (2.10), we obtain

$$|h(z)| \leqq 1+\frac{1-\beta}{1+A_j j!\mathrm{Re}(\alpha)}|z| \quad (z\in\mathbb{U}). \tag{2.11}$$

On the other hand, by means of (2.10), we see that

$$\mathrm{Re}\,(h(z)) = 1+\mathrm{Re}\left(\sum_{n=1}^{\infty}c_n z^n\right) \geqq 1-\left|\sum_{n=1}^{\infty}c_n z^n\right| \tag{2.12}$$

$$\geqq 1 - |z|\sum_{n=1}^{\infty}|c_n|$$
$$\geqq 1 - \frac{1-\beta}{1+A_j j!\mathrm{Re}(\alpha)}|z|.$$

Therefore, the proof of the lemma is completed.

If we take $j = 1$ in Lemma 2, we have the following corollary which is the improvement of the result due to Altintas and Owa [1].

Corollary 2 *If $h(z)$ is in the class $\mathcal{A}(\alpha, \beta, 1)$ with $c_n = |c_n|e^{i(n\theta+\pi)}$ $(n = 1, 2, 3, \cdots)$, then*

$$1 - \frac{1-\beta}{1+\mathrm{Re}(\alpha)}|z| \leqq \mathrm{Re}\,(h(z)) \leqq |h(z)| \leqq 1 + \frac{1-\beta}{1+\mathrm{Re}(\alpha)}|z| \quad (z \in \mathbb{U}).$$

Now, we derive

Theorem 1 *Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$ $(a_1 \neq 0)$ be analytic in $\mathbb{U}$. If $f(z) \ll g(z)$ and $\frac{zg'(z)}{g(z)} = 1 + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{A}(\alpha, \beta, j)$ with $c_n = |c_n|e^{i(n\theta+\pi)}$ $(n = 1, 2, 3, \cdots)$, then*

$$|f'(z)| \leqq |g'(z)| \quad (|z| \leqq r(\alpha, \beta, j)), \tag{2.13}$$

where $r(\alpha, \beta, j)$ is the root of the following equation

$$(1-\beta)r^3 - (1 + A_j j!\mathrm{Re}(\alpha))\, r^2 + (\beta - 2A_j j!\mathrm{Re}(\alpha) - 3)\, r + 1 + A_j j!\mathrm{Re}(\alpha) = 0 \tag{2.14}$$

contained in the interval $(0, 1)$.

Proof For $g(z)$ such that $\frac{zg'(z)}{g(z)} \in \mathcal{A}(\alpha, \beta, j)$,we have from Lemma 2 that

$$\left|\frac{zg'(z)}{g(z)}\right| \geqq 1 - \frac{1-\beta}{1+A_j j!\mathrm{Re}(\alpha)}r \quad (|z| = r), \tag{2.15}$$

or

$$|g(z)| \leqq \frac{(1 + A_j j!\mathrm{Re}(\alpha))\, r}{1 + A_j j!\mathrm{Re}(\alpha) - (1-\beta)r} |g'(z)| \quad (|z| = r). \tag{2.16}$$

Since $f(z) \ll g(z)$, there exists an analytic function $w(z)$ such that $f(z) = w(z)g(z)$ and $|w(z)| \leqq 1$ $(z \in \mathbb{U})$. Thus we have

$$f'(z) = w(z)g'(z) + w'(z)g(z). \tag{2.17}$$

Noting that $w(z)$ satisfies

$$|w'(z)| \leqq \frac{1 - |w(z)|^2}{1 - |z|^2} \quad (z \in U) \quad (cf\,.\,[6,\ p.\,168]), \tag{2.18}$$

we see that

$$|f'(z)| \leqq \frac{H(X)|g'(z)|}{(1 - r^2)\,(1 + A_j j!\mathrm{Re}(\alpha) - (1-\beta)r)}, \tag{2.19}$$

where $X = |w(z)|$ and $H(X)$ is defined by

$$H(X) = -\,(1 + A_j j!\mathrm{Re}(\alpha))\, rX^2 + (1 - r^2)\,(1 + A_j j!\mathrm{Re}(\alpha) - (1-\beta)r)\, X$$
$$+\,(1 + A_j j!\mathrm{Re}(\alpha))\, r \quad (0 \leqq X \leqq 1).$$

Then we see that $H(X)$ takes its maximum value at $X = 1$ with the condition (2.14). Also, if $0 \leqq a \leqq r(\alpha, \beta, j)$ for $r(\alpha, \beta, j)$ $(0 < r(\alpha, \beta, j) < 1)$ to be the root of the equation (2.14), then the function

$$\psi(X) = -\,(1 + A_j j!\mathrm{Re}(\alpha))\, aX^2$$
$$+ (1 - a^2)\,(1 + A_j j!\mathrm{Re}(\alpha) - (1-\beta)a)\, X + (1 + A_j j!\mathrm{Re}(\alpha))\, a \tag{2.20}$$

increases in the interval $0 \leqq X \leqq 1$ so that $\psi(X)$ does not exceed

$$\psi(1) = (1 - a^2)(1 + A_j j!\mathrm{Re}(\alpha) - (1-\beta)a).$$

Therefore, from this fact, the inequality (2.19) gives the inequality (2.13).

Letting $j = 1$ in Theorem 1, we obtain the following corollary which is the improvement of the theorem by Altintas and Owa [1].

Corollary 3 *Let* $f(z) = \sum_{n=1}^{\infty} a_n z^n$ $(a_1 \neq 0)$ *be analytic in* $\mathbb{U}$. *If* $f(z) \ll g(z)$ *and* $\frac{zg'(z)}{g(z)} = 1 + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{A}(\alpha, \beta, 1)$ *with* $c_n = |c_n| e^{i(n\theta+\pi)}$ $(n = 1, 2, 3, \cdots)$, *then*

$$|f'(z)| \leqq |g'(z)| \qquad (|z| \leqq r(\alpha, \beta, 1)),$$

where $r(\alpha, \beta, 1)$ *is the root of the equation*

$$(1-\beta) r^3 - (1 + \mathrm{Re}(\alpha))\, r^2 + (\beta - 2\mathrm{Re}(\alpha) - 3)\, r + 1 + \mathrm{Re}(\alpha) = 0$$

contained in the interval $(0, 1)$.

3 Open problem for the coefficients

In Example 1, we give the function $h(z) \in \mathcal{A}(\alpha, \beta, 1)$ as

$$h(z) = 1 + \sum_{n=1}^{\infty} \left(\frac{2(1-\beta)}{1+\alpha n} \int_{|x|=1} x^n d\mu(x) \right) z^n .$$

Since

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n ,$$

we see that

$$c_n = \frac{2(1-\beta)}{1+\alpha n} \int_{|x|=1} x^n d\mu(x) \qquad (n = 1, 2, 3, \cdots). \tag{3.1}$$

Further in Corollary 1, we consider $h(z) \in \mathcal{A}(\alpha, \beta, 1)$ with

$$c_n = |c_n| e^{i(n\theta+\pi)} = -\,|c_n| e^{in\theta} \qquad (n = 1, 2, 3, \cdots). \tag{3.2}$$

Therefore, we need to find the probability measure $\mu(x)$ which satisfies

$$-\,|c_n| e^{in\theta} = \frac{2(1-\beta)}{1+\alpha n} \int_{|x|=1} x^n d\mu(x),$$

that is,

$$\int_{|x|=1} x^n d\mu(x) = -\frac{(1+\alpha n)|c_n|}{2(1-\beta)} e^{in\theta} \qquad (n = 1, 2, 3, \cdots). \tag{3.3}$$

In this paper, we don't find such a probability measure $\mu(x)$. How can we find the probability measure $\mu(x)$ which satisfies (3.3) for each $n = 1, 2, 3, \cdots$?

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