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Optimal choice of multipole coefficients of the modified Green's function in elasticity (case of circular boundaries)

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Abstract

In this paper, we are interested to determine the terms of optimal choice of multipole coefficients and to obtain an estimation of the norm of the modified Green's function in elasticity for the case of circular boundaries.

Keywords: *Multipole coefficients, Gren's function, integral equations, linear elasticity.*

1 Introduction

The problem of scattering of waves (water, acoustic, elastic and electromagnetic waves) in a domain containing an inhomogeinety (cavity, inclusion or others) is very often formulated in terms of a boundary value problem. The solution to such a problem can be sought using different methods (finite difference, finite element and so on). In the case of infinite domain, the boundary integral equation method seems to be more appropriate for solving this type of problem. This method reduces the solving of the problem to an integral equation on the internal boundary of the domain. However, a problem of uniqueness of the solution of the boundary integral equation appears. This anomaly is related to the method of the resolution used rather than to the physical nature of the problem. Some methods, to overcome this anomally, were proposed (see [1] for a detailled discussion of the proposed solutions)

Indeed, when using the method of integral representations, the two problems; exterior problem (which has a unique solution) and the interior one (which has no unique solution for a certain specter of values of the frequency of waves) are represented by two integral equations with adjoint kernels, and therefore they will have the same number of solutions [2] which presents a contardiction. To recover the uniqueness of the solution of the interior problem, Jones [3] and Ursell [4] developed a technique in acoustics. In 1986, Bencheikh [1] has developed this technique (called a modified Green's function technique) in the case of elastic waves, by adding to the fundamental solution a set of functions called multipole, physically talk, this technique is based on injection of points or small circles, absorbent inside the domain, to transform the phenomenon of stationary waves (interior problem) to a phenomenon of diverging progressive waves (exterior problem). This modification involves the complex coefficients called multipoles coefficients and which should satisfy a large condition (1.7).

In the case of acoustic waves, a method of determination of an optimal choice of those coefficients was elaborated by Roach and Kleinman [5]. This method is based on the minimization of the norm of the modified integral operator. In [6] Argyropoulos, Kiriaki and Roach determined another optimal choice for these coefficients by minimizing the norm of the kernel of the modified integral operator for the case of three dimensional elastic waves. The minimization of the norm of this integral operator or of the norm of its kernel is related to the convergence of the iterative method used for the resolution of that modified integral equation, namely the method of successive approximations. We have established in [7] and [8] an optimal choice of the multipole coefficients in the case of two dimensional elastic waves, by using two different optimality criteria, the first one is the minimization of the norm of the integral operator and the second is the minimization of its kernel.

The aim of this paper is to test the results found in [8] for simple geometric forms, circles or a slightly distorted circles. Because in the general case, it is difficult to get the explicit calculation of multipole coefficients and the norm of the Green's function. But, it is possible in the case of circular boundaries, as we will see later in this paper.

2 Preliminary notes

2.1 Formulation of the problem

Consider a domain $D \subset IR^2$ which is homogeneous, elastic and isotropic, unbounded externally and bounded internally by ∂D, and seek a function $U \in L^2(D)$ satisfying :

i) The equation in D

$$\frac{1}{k^2}grad(divU(p)) - \frac{1}{K^2}rot(rotU(p)) + U(p) = 0, \qquad p \in D \qquad (1.1)$$

ii) The boundary conditions on ∂D

$TU(p) = f(p)$(Neumann condition) or $U(p) = g(p)$ (Dirichlet condition), for $p \in \partial D$

iii) The radiation conditions (1)

with $k^2 = \frac{\rho\,\omega^2}{\lambda+2\mu}$, $K^2 = \frac{\rho\,\omega^2}{\mu}$ and ρ is the density. λ, μ are the Lamé constants and ω^2 is the frequency of the waves. T is the traction operator which acts on the function $u\,(p)$ in the point p, and f, g are two given functions.

This last boundary problem is represented by the following modified integral equation:

$$\frac{1}{2}U(p) - (\overline{K_1^*}U)(p) = -(S_1 f)(p) \qquad p \in \partial D \qquad (1.2)$$

With K_1 is the modified integral operator defined by:

$$(K_1U)(p) = \int_{\partial D} T_p G_1(p,q).U(q).ds_q \quad p \in \partial D \qquad (1.3)$$

and S_1 is the single layer potential defined by:

$$(S_1U)(P) = \int_{\partial D} G_1(P,q).U(q).ds_q \qquad P \in D \qquad (1.4)$$

where $G_1(p,q)$ is the modified Green's function defined by:

$$G_1(p,q) = G_0(p,q) + \frac{i}{4\mu K^2}\sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}\sum_{\ell=1}^{2}\left[\begin{array}{c} a_m^{\sigma\ell}.F_m^{\sigma\ell}(p)\otimes F_m^{\sigma\ell}(q) \\ +(-1)^{\sigma+\ell}.b_m.F_m^{\sigma\ell}(p)\otimes F_m^{(3-\sigma)(3-\ell)}(q) \end{array}\right] \qquad (1.5)$$

where

$$F_m^{\sigma 1}(p) = grad(H_m^1(k.r_p).E_m^\sigma(\theta_p))$$
$$(1.6)$$
$$F_m^{\sigma 2}(p) = rot(H_m^1(K.r_p).E_m^\sigma(\theta_p)).e_3$$

and $G_0(p,q)$ is the initial Green's function.

(r_p, θ_p) are the polar coordinates of the point p, $H_m^1(.)$ is the Hankel's function of order m and type 1,

$E_m^\sigma(\theta) = \sqrt{\varepsilon_m}.\begin{cases} \cos(m\theta_p) & \sigma = 1 \\ \sin(m\theta_p) & \sigma = 2 \end{cases}$ with $\varepsilon_m = \begin{cases} 1 & m = 0 \\ 2 & m > 0 \end{cases}$, $\otimes$ design the tensorial product and e_3 is the unit vector in the direction of z.

$a_m^{\sigma\ell}$ and b_m are successively, the simple and cross multipole coefficients (complex coefficients)which must satisfy the following largest condition [1]:

$$\overline{b_m}(a_m^{\sigma 1} + \frac{1}{2}) + b_m(\overline{a_m^{\sigma 2}} + \frac{1}{2}) = 0 \qquad (\forall m = 0 : \infty, \quad \forall \sigma, \quad \ell = 1 : 2) \qquad (1.7)$$

and

$$|a_m^{\sigma\ell} + \frac{1}{2}|^2 + |b_m|^2 - \frac{1}{4} < 0$$

2.2 General case

Taking as optimal criterion, a minimization of the norm of the kernel of the integral modified operator K_1, (i.e., a minimization of the norm of the modified Green's function G_1). This criterion is motivated by the fact that we want to ensure the convergence of the successive approximations method which will be used for the resolution of the modified integral equation. The expressions of optimal choice of simple and cross multipole coefficients for a domain D of any boundary are given by Theorem 2.1 (for more details see [8],[9] and [10]).

Theorem 2.1 *If the kernel of the modified integral operator K_1, namely the function of Green G_1 is defined by (1.5) then the quantity*

$$\int_{r_p=A} \|G_1\|^2_{L_2(\partial D} \ . \ ds_p \qquad \forall\ A \geq \max(r_q), \qquad q \in \partial D \qquad (2.1)$$

is minimized if the simple and cross multipole coefficients are selected as follows :

$$a_m^{\sigma l} = \frac{\begin{array}{c}\left(\overline{B}_m^{\sigma\ell}\right)\left[M_{m,\ 1}^{\sigma\ell}\right] + \left(\overline{\beta}_m^{\sigma\ell}\right)\left[M_{m,\ 2}^{\sigma\ell}\right] \\ - \left(A_m^{(3-\sigma)(3-\ell)}\right)\left[N_{m,\ 1}^{\sigma\ell}\right] - \left(\alpha_m^{(3-\sigma)(3-\ell)}\right)\left[N_{m,\ 2}^{\sigma\ell}\right]\end{array}}{\triangle_{m,\ \partial D}^{\sigma}}$$

and

$$(-1)^{\sigma+l}\ .b_m = \frac{\begin{array}{c}\left(B_m^{\sigma\ell}\right)\left[N_{m,\ 1}^{\sigma\ell}\right] + \left(\beta_m^{\sigma\ell}\right)\left[N_{m,\ 2}^{\sigma\ell}\right] \\ - \left(A_m^{\sigma\ell}\right)\left[M_{m,\ 1}^{\sigma\ell}\right] - \left(\alpha_m^{\sigma\ell}\right)\left[M_{m,\ 2}^{\sigma\ell}\right]\end{array}}{\triangle_{m,\ \partial D}^{\sigma}} \qquad (2.2)$$

with:

$$M_{m,\ 1}^{\sigma\ell} = \left(\triangle_{m,\ A}^{\sigma}\right)\ .\ \left[B_m^{(3-\sigma)(3-\ell)}.g_m^{(3-\sigma)(3-\ell)} - A_m^{(3-\sigma)(3-\ell)}.h_m^{\sigma\ell}\right]$$

$$M_{m,\ 2}^{\sigma\ell} = \left(\triangle_{m,\ \partial D}^{\sigma}\right)\ .\ \left[\beta_m^{(3-\sigma)(3-\ell)}.g_m^{(3-\sigma)(3-\ell)} - \alpha_m^{(3-\sigma)(3-\ell)}.h_m^{\sigma\ell}\right]$$

$$N_{m,\ 1}^{\sigma\ell} = \left(\triangle_{m,\ A}^{\sigma}\right)\ .\ \left[B_m^{(3-\sigma)(3-\ell)}.h_m^{(3-\sigma)(3-\ell)} - A_m^{(3-\sigma)(3-\ell)}.g_m^{\sigma\ell}\right]$$

$$N_{m,\ 2}^{\sigma\ell} = \left(\triangle_{m,\ \partial D}^{\sigma}\right)\ .\ \left[\beta_m^{(3-\sigma)(3-\ell)}.h_m^{(3-\sigma)(3-\ell)} - \alpha_m^{(3-\sigma)(3-\ell)}.g_m^{\sigma\ell}\right]$$

$$g_m^{\sigma\ell} = -\langle\overline{\beta}_m^{\sigma\ell}.\widehat{F}_m^{(3-\sigma)(3-\ell)} + \alpha_m^{\sigma\ell}\ .\ \widehat{F}_m^{\sigma\ell}\ ,\ F_m^{\sigma\ell}\rangle_{\partial D}$$

$$h_m^{\sigma\ell} = -\ \left\langle\overline{\beta}_m^{\sigma\ell}\ .\ \widehat{F}_m^{(3-\sigma)(3-\ell)} + \alpha_m^{\sigma\ell}\ .\ \widehat{F}_m^{\sigma\ell},\ F_m^{(3-\sigma)(3-\ell)}\right\rangle_{\partial D}$$

$$\alpha_m^{\sigma\ell} = \left\|F_m^{\sigma\ell}\right\|_A^2 \quad , \qquad \beta_m^{\sigma\ell} = \left\langle F_m^{\sigma\ell}\ ,\ F_m^{(3-\sigma)(3-\ell)}\right\rangle_A$$

$$A_m^{\sigma\ell} = \left\|F_m^{\sigma\ell}\right\|_{\partial D}^2 \quad , \qquad B_m^{\sigma\ell} = \left\langle F_m^{\sigma\ell}\ ,\ F_m^{(3-\sigma)(3-\ell)}\right\rangle_{\partial D}$$

$$\triangle_{m,\ A}^{\sigma} = \left(\alpha_m^{\sigma 1}\ .\ A_m^{\sigma 1}\ .\ \alpha_m^{(3-\sigma)2}\ .\ A_m^{(3-\sigma)2} - \beta_m^{\sigma 1}\ .\ B_m^{\sigma 1}\ .\ \overline{\beta}_m^{\sigma 1}\ .\ \overline{B}_m^{\sigma 1}\right)_A$$

$$\triangle_{m,\ \partial D}^{\sigma} = \left(\alpha_m^{\sigma 1}\ .\ A_m^{\sigma 1}\ .\ \alpha_m^{(3-\sigma)2}\ .\ A_m^{(3-\sigma)2} - \beta_m^{\sigma 1}\ .\ B_m^{\sigma 1}\ .\ \overline{\beta}_m^{\sigma 1}\ .\ \overline{B}_m^{\sigma 1}\right)_{\partial D}$$

where $\langle,\rangle_{\partial D}$ *is the inner product calculated on the boundary* ∂D*, and* $\langle,\rangle_A$ *is the inner product calculated on a large circle of radius* A *which contains the boundary* ∂D

Proof.

Step 1: We have :

$$G_1(p,q) = G_0(p,q) + \frac{i}{4\mu K^2}\sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}\sum_{\ell=1}^{2}\left[\begin{array}{c} a_m^{\sigma\ell}.F_m^{\sigma\ell}(p)\otimes F_m^{\sigma\ell}(q) \\ +(-1)^{\sigma+\ell}.b_m.F_m^{\sigma\ell}(p)\otimes F_m^{(3-\sigma)(3-\ell)}(q)\end{array}\right]$$

$$= \frac{i}{4\mu K^2}\sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}\sum_{\ell=1}^{2}\left[\begin{array}{c} F_m^{\sigma\ell}(P)\otimes\widehat{F}_m^{\sigma\ell}(Q) + a_m^{\sigma\ell}.F_m^{\sigma\ell}(P)\otimes F_m^{\sigma\ell}(Q) \\ +(-1)^{\sigma+l}.b_m.F_m^{\sigma\ell}(P)\otimes F_m^{(3-\sigma)(3-\ell)}(Q)\end{array}\right] \tag{2.3}$$

We put:

$$f_m^{\sigma\ell}(Q) = \left[\widehat{F}_m^{\sigma\ell}(Q) + a_m^{\sigma\ell}.F_m^{\sigma\ell}(Q) + (-1)^{\sigma+l}.b_m.F_m^{(3-\sigma)(3-\ell)}(Q)\right] \tag{2.4}$$

So the modified Green's function is written in the form :

$$G_1(P,Q) \ = \sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}\sum_{\ell=1}^{2}[F_m^{\sigma\ell}(P)\otimes f_m^{\sigma\ell}(Q)] \tag{2.5}$$

hence :

$$\int_{r_p=A}\|G_1\|^2_{L_2(\partial D}\ .\ ds_p \qquad = \int_{r_p=A}\int_{\partial D} G_1(P,q)\ :\ \overline{G}_1(q,P)\ .\ ds_p.ds_q$$

$$\sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}\sum_{\ell=1}^{2}\sum_{n=0}^{\infty}\sum_{\nu=1}^{2}\sum_{k=1}^{2}\int_{r_p=A}F_m^{\sigma\ell}(P).\overline{F}_n^{\nu k}(P).ds_p\ .\quad \int_{\partial D} f_m^{\sigma\ell}(q).\overline{f}_n^{\nu k}(q).ds_q$$

Using the relations of the inner products of the functions $\left\{F_m^{\sigma\ell}\right\}_{m=0:\infty}^{\sigma,l=1:2}$ on the circle of radius A [**7**] and [**11**], we obtain :

$$\int_{r_p=A}\|G_1\|^2_{L_2(\partial D}\ .\ ds_p$$

$$= \sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}\Big[\begin{array}{c} \|F_m^{\sigma1}\|_A^2\ .\langle f_m^{\sigma1}\ ,\ f_m^{\sigma1}\rangle_{\partial D}\ +\ \langle F_m^{\sigma1}\ ,\ F_m^{\sigma2}\rangle_A\ .\langle f_m^{\sigma1}\ ,\ f_m^{(3-\sigma)2}\rangle_{\partial D} \\ +\ \langle F_m^{\sigma2}\ ,\ F_m^{\sigma1}\rangle_A\ .\langle f_m^{(3-\sigma)2}\ ,\ f_m^{\sigma1}\rangle_{\partial D} + \|F_m^{\sigma2}\|_A^2\ .\langle f_m^{(3-\sigma)2}\ ,\ f_m^{(3-\sigma)2}\rangle_{\partial D}\end{array}\Big] \tag{2.6}$$

calculating the expressions $\langle f_m^{\sigma1}\ ,\ f_m^{\sigma1}\rangle_{\partial D}$, $\left\langle f_m^{\sigma1}\ ,\ f_m^{(3-\sigma)2}\right\rangle_{\partial D}$, $\left\langle f_m^{(3-\sigma)2}\ ,\ f_m^{\sigma1}\right\rangle_{\partial D}$ and $\langle f_m^{(3-\sigma)2}\ ,\ f_m^{(3-\sigma)2}\rangle_{\partial D}$, and substituting in (2.6)

we get:

$$\int_{r_p=A}\|G_1\|^2_{L_2(\partial D}\ .\ ds_p$$

$$= \sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}\alpha_m^{\sigma1}\ .\left(\begin{array}{c} \|\widehat{F}_m^{\sigma1}\|^2_{\partial D}\ +\overline{a}_m^{\sigma1}\langle\widehat{F}_m^{\sigma1}\ ,\ F_m^{\sigma1}\rangle_{\partial D} - (-1)^{\sigma}\overline{b}_m\langle\widehat{F}_m^{\sigma1}\ ,\ F_m^{(3-\sigma)2}\rangle_{\partial D} \\ +\ a_m^{\sigma1}\left\langle F_m^{\sigma1}\ ,\ \widehat{F}_m^{\sigma1}\right\rangle_{\partial D} + a_m^{\sigma1}.\ \overline{a}_m^{\sigma1}.\ A_m^{\sigma1} - (-1)^{\sigma}\ a_m^{\sigma1}\ \overline{b}_m\ B_m^{\sigma1} \\ -(-1)^{\sigma}\ b_m\left\langle F_m^{(3-\sigma)2}\ ,\ \widehat{F}_m^{\sigma1}\right\rangle_{\partial D} - (-1)^{\sigma}\ \overline{a}_m^{\sigma1}\ b_m\ \overline{B}_m^{\sigma1} + b_m\ \overline{b}_m\ A_m^{(3-\sigma)2}\end{array}\right)$$

$$+\beta_m^{\sigma1}\ .\left(\begin{array}{c} \left\langle\widehat{F}_m^{\sigma1}\ ,\ \widehat{F}_m^{(3-\sigma)2}\right\rangle_{\partial D} + \overline{a}_m^{(3-\sigma)2}\left\langle\widehat{F}_m^{\sigma1}\ ,\ F_m^{(3-\sigma)2}\right\rangle_{\partial D} - (-1)^{\sigma}\ \overline{b}_m\left\langle\widehat{F}_m^{\sigma1}\ ,\ F_m^{\sigma1}\right\rangle_{\partial D} \\ +\ a_m^{\sigma1}\left\langle F_m^{\sigma1}\ ,\ \widehat{F}_m^{(3-\sigma)2}\right\rangle_{\partial D} + a_m^{\sigma1}.\ \overline{a}_m^{(3-\sigma)2}.\ B_m^{\sigma1} - (-1)^{\sigma}\ a_m^{\sigma1}\ \overline{b}_m\ A_m^{\sigma1} \\ -(-1)^{\sigma}\ b_m\left\langle F_m^{(3-\sigma)2}\ ,\ \widehat{F}_m^{(3-\sigma)2}\right\rangle_{\partial D} - (-1)^{\sigma}\ \overline{a}_m^{(3-\sigma)2}\ b_m\ A_m^{(3-\sigma)2} + b_m\ \overline{b}_m\ \overline{B}_m^{\sigma1}\end{array}\right)$$

$$+\overline{\beta}_m^{\sigma1}\ .\left(\begin{array}{c} \left\langle\widehat{F}_m^{(3-\sigma)2}\ ,\ \widehat{F}_m^{\sigma1}\right\rangle_{\partial D} + a_m^{(3-\sigma)2}\left\langle F_m^{(3-\sigma)2}\ ,\ \widehat{F}_m^{\sigma1}\right\rangle_{\partial D} - (-1)^{\sigma}\ \overline{b}_m\left\langle\widehat{F}_m^{(3-\sigma)2}\ ,\ F_m^{(3-\sigma)2}\right\rangle_{\partial D} \\ +\ \overline{a}_m^{\sigma1}\left\langle\widehat{F}_m^{(3-\sigma)2}\ ,\ F_m^{\sigma1}\right\rangle_{\partial D} + a_m^{(3-\sigma)2}.\ \overline{a}_m^{\sigma1}.\ \overline{B}_m^{\sigma1} - (-1)^{\sigma}\ a_m^{(3-\sigma)2}\ \overline{b}_m\ A_m^{(3-\sigma)2} \\ -(-1)^{\sigma}\ b_m\left\langle F_m^{\sigma1}\ ,\ \widehat{F}_m^{\sigma1}\right\rangle_{\partial D} - (-1)^{\sigma}\ \overline{a}_m^{\sigma1}\ b_m\ A_m^{\sigma1} + b_m\ \overline{b}_m\ B_m^{\sigma1}\end{array}\right)$$

$$+\alpha_m^{(3-\sigma)2}\cdot\left(\begin{array}{c}\left\|\widehat{F}_m^{(3-\sigma)2}\right\|_{\partial D}^2 \quad +\overline{a}_m^{(3-\sigma)2}\left\langle\widehat{F}_m^{(3-\sigma)2}\ ,\ F_m^{(3-\sigma)2}\right\rangle_{\partial D}\\ -(-1)^\sigma\,\overline{b}_m\left\langle\widehat{F}_m^{(3-\sigma)2}\ ,\ F_m^{\sigma 1}\right\rangle_{\partial D}+\ a_m^{(3-\sigma)2}\left\langle F_m^{(3-\sigma)2}\ ,\ \widehat{F}_m^{(3-\sigma)2}\right\rangle_{\partial D}\\ +a_m^{(3-\sigma)2}.\ \overline{a}_m^{(3-\sigma)2}.\ A_m^{(3-\sigma)2}-(-1)^\sigma\,a_m^{(3-\sigma)2}\,\overline{b}_m\,\overline{B}_m^{\sigma 1}\\ -(-1)^\sigma\,b_m\left\langle F_m^{\sigma 1}\ ,\ \widehat{F}_m^{(3-\sigma)2}\right\rangle_{\partial D}-(-1)^\sigma\,\overline{a}_m^{(3-\sigma)2}\,b_m\,B_m^{\sigma 1}+b_m\,\overline{b}_m\,A_m^{\sigma 1}\end{array}\right) \tag{2.7}$$

This is a standard problem of minimization in which the necessary condition for the existence of the minimum is the cancelation of the gradient.

So, if we cancel the gradient with respect to the coefficients $\overline{a}_m^{\sigma 1}$, $\overline{a}_m^{(3-\sigma)2}$ and b_m we obtain the following relations :

$$\left(\alpha_m^{\sigma 1}.\ A_m^{\sigma 1}\right)\ .\ a_m^{\sigma 1}+\left(\overline{\beta}_m^{\sigma 1}.\ \overline{B}_m^{\sigma 1}\right)\ .\ a_m^{(3-\sigma)2}-(-1)^\sigma\left(\alpha_m^{\sigma 1}\overline{B}_m^{\sigma 1}+A_m^{\sigma 1}\overline{\beta}_m^{\sigma 1}\right)b_m=\ g_m^{\sigma 1}$$

$$\begin{array}{c}\left(\beta_m^{\sigma 1}.\ B_m^{\sigma 1}\right)\ .\ a_m^{\sigma 1}+\left(\alpha_m^{(3-\sigma)2}.\ A_m^{(3-\sigma)2}\right)\ .\ a_m^{(3-\sigma)2}\\ -(-1)^\sigma\left(\alpha_m^{(3-\sigma)2}B_m^{\sigma 1}+A_m^{(3-\sigma)2}\beta_m^{\sigma 1}\right)b_m\ =\ g_m^{(3-\sigma)2}\end{array}$$

$$\begin{array}{c}-(-1)^\sigma\left(\begin{array}{c}B_m^{\sigma 1}\alpha_m^{\sigma 1}\\ +A_m^{\sigma 1}\beta_m^{\sigma 1}\end{array}\right)\ .\ a_m^{\sigma 1}-(-1)^\sigma\left(\begin{array}{c}A_m^{(3-\sigma)2}\ \overline{\beta}_m^{\sigma 1}\\ +\overline{B}_m^{\sigma 1}\alpha_m^{(3-\sigma)2}\end{array}\right)\ .\ a_m^{(3-\sigma)2}\\ +\left(\begin{array}{c}A_m^{(3-\sigma)2}\alpha_m^{\sigma 1}+\overline{B}_m^{\sigma 1}\beta_m^{\sigma 1}\\ +B_m^{\sigma 1}\ \overline{\beta}_m^{\sigma 1}+A_m^{\sigma 1}\alpha_m^{(3-\sigma)2}\end{array}\right)b_m=-(-1)^\sigma\ \left(h_m^{\sigma 1}+h_m^{(3-\sigma)2}\right)\end{array}$$

According to the Schwartz inequality and the linear independence of the functions $\left\{F_m^{\sigma\ell}\right\}_{m\ =\ 0\ :\ \infty}^{\sigma,l\ =\ 1\ :\ 2}$, the latter system admits the following non-zero determinant

$$\triangle_m^\sigma=\left[\begin{array}{c}A_m^{\sigma 1}\ .\ \alpha_m^{(3-\sigma)2}\\ +\alpha_m^{\sigma 1}\ .\ A_m^{(3-\sigma)2}\\ -\left(\begin{array}{c}B_m^{\sigma 1}\ .\ \overline{\beta}_m^{\sigma\ell}\\ +\beta_m^{\sigma 1}\ \ \overline{B}_m^{\sigma\ell}\end{array}\right)\end{array}\right].\left(\begin{array}{c}A_m^{\sigma 1}\ .\ A_m^{(3-\sigma)2}\\ -\left|B_m^{\sigma 1}\right|^2\end{array}\right).\left(\begin{array}{c}\alpha_m^{\sigma 1}\ .\ \alpha_m^{(3-\sigma)2}\\ -\left|\beta_m^{\sigma 1}\right|^2\end{array}\right)$$

Therefore, the solutions of the above system will be given by (2.2).

Step 2 : Now, we will show that the choice of multipole coefficients as defined in (2.2) verify the minimum of the quantity (2.1). For this, we suppose that the quantity (2.1) is a function of six variables $x_m^{\sigma 1}$, $y_m^{\sigma 1}, x_m$, $x_m^{(3-\sigma)2}$, $y_m^{(3-\sigma)2}$ and y_m where :

$$a_m^{\sigma 1}\ =x_m^{\sigma 1}+\ i\ y_m^{\sigma 1},\ \ a_m^{(3-\sigma)2}\ =x_m^{(3-\sigma)2}+\ i\ y_m^{(3-\sigma)2}$$

and

$$b_m \ = x_m + \ i\ y_m.$$

It is clear that the optimal choice defined by (2.2) annul the gradient of the quantity (2.1). Moreover, if the Hessien of (2.1) is a semi-defined positive matrix, then our choice will verify the minimum of the quantity (2.1). But, since the Hessien is a symmetric matrix, then it is sufficient that the main determinants were strictly positives.

After the long calculations, we obtain:

$$|H[\int_{r_p=A} \|G_1\|^2_{L_2(\partial D)}\ .\ ds_p]| = \left(16\ \alpha_m^{\sigma 1}\ .\ A_m^{\sigma 1}\ .\ \alpha_m^{(3-\sigma)2}\ .\ A_m^{(3-\sigma)2}\right).[\triangle_m^\sigma]$$

$$+16(Re^4(\beta_m^{\sigma 1}\ .\ B_m^{\sigma 1}) + Im^4(\beta_m^{\sigma 1}\ .\ B_m^{\sigma 1})) > 0$$

$$|H_{11}\ \left[\int_{r_p=A} \|G_1\|^2_{L_2(\partial D)}\ .\ ds_p\right]| = \left(8\ \ \alpha_m^{(3-\sigma)2}\ .\ A_m^{(3-\sigma)2}\right).[\triangle_m^\sigma] > 0$$

$$|H_{22}\ \left[\int_{r_p=A} \|G_1\|^2_{L_2(\partial D)}\ .\ ds_p\right]| = \left(8\ \ \alpha_m^{(3-\sigma)2}\ .\ A_m^{(3-\sigma)2}\right).[\triangle_m^\sigma] > 0$$

and

$$|H_{33}\ \left[\int_{r_p=A} \|G_1\|^2_{L_2(\partial D)}\ .\ ds_p\right]| = \left(8\ \alpha_m^{\sigma 1}\ .\ A_m^{\sigma 1}\right).[\triangle_m^\sigma] > 0.$$

Hence the optimal choice as defined in (2.2) really verifies the minimum of the quantity (2.1) and this completes the proof. □

Next, we shall test the expressions (2.2) for relatively simple geometric forms (the boundary is a circle or a slightly distorted circle).

3 Mains results

3.1 Case of the circle

Consider in the following a domain D with circular boundary of radius a, and calculate the simple and cross multipole coefficients $a_m^{\sigma\ell}$and b_m, and the value of the Green's function.

Theorem 3.1 *(***Calculation of the simple and cross multipole coefficients** $a_m^{\sigma\ell}$ **and** b_m*)*

If the boundary of the domain D is a circle of radius a, then the multipole coefficients will be given by the following relatively simple expressions :

$$a_m^{11} = \frac{(\overline{c}_m.\widehat{c}_m - \widehat{a}_m^1.a_m^2)}{\Delta_{m,\ a}} \qquad (3.1)$$

$$a_m^{21} = \frac{(\overline{c}_m.\widehat{c}_m - \widehat{a}_m^1.a_m^2)}{\Delta_{m,\ a}} \qquad (3.2)$$

$$a_m^{22} = \frac{(c_m.\widehat{d}_m - a_m^1.\widehat{a}_m^2)}{\Delta_{m,\ a}} \qquad (3.3)$$

$$a_m^{12} = \frac{\left(c_m.\widehat{d}_m - a_m^1.\widehat{a}_m^2\right)}{\Delta_{m,\ a}} \qquad (3.4)$$

$$b_m = \frac{\left(a^2{}_m.\widehat{d}_m - \widehat{a}_m^2.\overline{c}_m\right)}{\Delta_m} = \frac{(\widehat{c}_m.a_m^1 - \ c_m.\widehat{a}_m^1)}{\Delta_{m,\ a}} \qquad (3.5)$$

where (see [7]):

$$a_m^1 = \ \left\langle F_m^{\sigma 1},\ F_m^{\sigma 1}\right\rangle_a = 2\pi a k^2\left[\left|H_m'(ka)\right|^2 + \frac{m^2}{(ka)^2}\left|H_m(ka)\right|^2\right],$$

$$a_m^2 = \left\langle F_m^{\sigma 2},\ F_m^{\sigma 2}\right\rangle_a = 2\pi a K^2\left[\left|H_m'(Ka)\right|^2 + \frac{m^2}{(Ka)^2}\left|H_m(Ka)\right|^2\right],$$

$$c_m = (-1)^{\sigma+1}\left\langle F_m^{\sigma 1},\ F_m^{(3-\sigma)2}\right\rangle_a = 2\pi a k K\left[\begin{array}{c} \frac{m}{Ka}H_m'(ka).\overline{H}_m(Ka) \\ +\frac{m}{ka}H_m(ka).\overline{H}_m'(Ka)\end{array}\right],$$

$$\widehat{a}_m^1 = \left\langle \widehat{F}_m^{\sigma 1},\ F_m^{\sigma 1}\right\rangle_a = 2\pi a k^2\left[J_m'(ka).\overline{H}_m'(ka) + \frac{m^2}{(ka)^2}J_m(ka).\overline{H}_m(ka)\right],$$

$$\widehat{a}_m^2 = \left\langle \widehat{F}_m^{\sigma 2},\ F_m^{\sigma 2}\right\rangle_a = 2\pi a K^2\left[J_m'(Ka).\overline{H}_m'(Ka) + \frac{m^2}{(Ka)^2}J_m(Ka).\overline{H}_m(Ka)\right],$$

$$\widehat{c}_m = (-1)^{\sigma+1}\left\langle \widehat{F}_m^{\sigma 1},\ F_m^{(3-\sigma)2}\right\rangle_a = 2\pi a k K\left[\begin{array}{c} \frac{m}{Ka}J_m'(ka).\overline{H}_m(Ka) \\ +\frac{m}{ka}J_m(ka).\overline{H}_m'(Ka)\end{array}\right],$$

$$\widehat{d}_m = (-1)^{2-\sigma} \left\langle \widehat{F}_m^{\sigma 2},\ F_m^{(3-\sigma)1} \right\rangle_a = 2\pi akK \begin{bmatrix} \frac{m}{ka} J_m'(Ka).\overline{H}_m(ka) \\ + \frac{m}{Ka} J_m(Ka).\overline{H}_m'(Ka) \end{bmatrix}$$

and $J_m(.)$ *is the Bessel's function of order* m *and type* 1.

Proof.

To calculate the simple and cross miltipole coefficients $a_m^{\sigma\ell}$ and b_m, we should first calculate $g_m^{\sigma 1}$, $g_m^{(3-\sigma)2}$, $h_m^{\sigma 1}$, and $h_m^{(3-\sigma)2}$

We have :

$$g_m^{\sigma\ell} = -\ \overline{\beta}_m^{\sigma\ell} \ \left\langle \ \widehat{F}_m^{(3-\sigma)(3-\ell)},\ F_m^{\sigma\ell} \right\rangle_a - \alpha_m^{\sigma\ell} \ \left\langle \widehat{F}_m^{\sigma\ell},\ F_m^{\sigma\ell} \right\rangle_a$$

$$= -\ \left\langle \ F_m^{(3-\sigma)(3-\ell)},\ F_m^{\sigma\ell} \right\rangle_a \ \left\langle \ \widehat{F}_m^{(3-\sigma)(3-\ell)},\ F_m^{\sigma\ell} \right\rangle_a - \left\| F_m^{\sigma\ell} \right\|_a^2 \ \left\langle \widehat{F}_m^{\sigma\ell},\ F_m^{\sigma\ell} \right\rangle_a$$

so :

$$g_m^{\sigma 1} = -\ (-\ (-1)^{\sigma}\ \overline{c}_m\ ).\ \left(-\ (-1)^{\sigma}\ \widehat{d}_m\right) - \left(a_m^1\right)\ .\ \left(\widehat{a}_m^1\right)$$

and

$$g_m^{(3-\sigma)2} = -\ (-\ (-1)^{\sigma}\ c_m).\ (-\ (-1)^{\sigma}\ \widehat{c}_m) - \left(a_m^2\right)\ .\ \left(\widehat{a}_m^2\right)$$

On the other hand :

$$h_m^{\sigma\ell} = -\ \overline{\beta}_m^{\sigma\ell} \ \left\langle \ \widehat{F}_m^{(3-\sigma)(3-\ell)},\ F_m^{(3-\sigma)(3-\ell)} \right\rangle_a - \alpha_m^{\sigma\ell} \ \left\langle \widehat{F}_m^{\sigma\ell},\ F_m^{(3-\sigma)(3-\ell)} \right\rangle_a$$

$$= -\ \left\langle \ F_m^{(3-\sigma)(3-\ell)},\ F_m^{\sigma\ell} \right\rangle_a \ \left\langle \ \widehat{F}_m^{(3-\sigma)(3-\ell)},\ F_m^{(3-\sigma)(3-\ell)} \right\rangle_a - \left\| F_m^{\sigma\ell} \right\|_a^2 \ \left\langle \widehat{F}_m^{\sigma\ell},\ F_m^{(3-\sigma)(3-\ell)} \right\rangle_a$$

So,

$$h_m^{\sigma 1} = -\ (-\ (-1)^{\sigma}\ \overline{c}_m\ ).\ \left(\widehat{a}_m^2\right) - \left(a_m^1\right)\ .\ (-\ (-1)^{\sigma}\ \widehat{c}_m)$$

and

$$h_m^{(3-\sigma)2} = -\ (-\ (-1)^{\sigma}\ c_m).\ \left(\widehat{a}_m^1\right) - \left(a_m^2\right)\ .\ \left(-\ (-1)^{\sigma}\ \widehat{d}_m\right).$$

Then, the expressions of the coefficients $a_m^{\sigma\ell}$ and b_m will be given by (3.1) –(3.5).

Note that $a_m^{11} = a_m^{21}$ and $a_m^{12} = a_m^{22}$. Also, we can show (see [7]) that the two expressions found for b_m which appear different, are equal. □

Lemma 3.2 *(***Calculation of the modified Green's function***)*
If the boundary of the domain D is a circle of radius a, then $\|K_1\| = 0$

Proof.
We have :

$$G_1(p,q) = G_0(p,q) + \frac{i}{4\mu K^2} \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} \sum_{\ell=1}^{2} \left[\begin{array}{c} a_m^{\sigma\ell}.F_m^{\sigma\ell}(p) \otimes F_m^{\sigma\ell}(q) \\ +(-1)^{\sigma+\ell}.b_m.F_m^{\sigma\ell}(p) \otimes F_m^{(3-\sigma)(3-\ell)}(q) \end{array} \right].$$

By replacing the expressions of multipole coefficients obtained above, we obtain:

$$G_1(p,q) = G_0(p,q) + \frac{i}{4\mu K^2} \sum_{m=0}^{\infty} \left[\begin{array}{c} \frac{\left(\bar{c}_m.\widehat{c}_m - \widehat{a}_m^1.a_m^2\right)}{\Delta_{m,\ a}}.F_m^{11}(p) \otimes F_m^{11}(q) \\ + \frac{\left(a^2{}_m.\widehat{d}_m - \widehat{a}_m^2.\bar{c}_m\right)}{\Delta_m} F_m^{11}(p) \otimes F_m^{22}(q) \\ + \frac{\left(c_m.\widehat{d}_m - a_m^1.\widehat{a}_m^2\right)}{\Delta_{m,\ a}}.F_m^{12}(p) \otimes F_m^{12}(q) \\ - \frac{\left(a^2{}_m.\widehat{d}_m - \widehat{a}_m^2.\bar{c}_m\right)}{\Delta_m} F_m^{12}(p) \otimes F_m^{21}(q) \\ + \frac{\left(\bar{c}_m.\widehat{c}_m - \widehat{a}_m^1.a_m^2\right)}{\Delta_{m,\ a}}.F_m^{21}(p) \otimes F_m^{21}(q) \\ - \frac{\left(a^2{}_m.\widehat{d}_m - \widehat{a}_m^2.\bar{c}_m\right)}{\Delta_m} F_m^{21}(p) \otimes F_m^{12}(q) \\ + \frac{\left(c_m.\widehat{d}_m - a_m^1.\widehat{a}_m^2\right)}{\Delta_{m,\ a}}.F_m^{22}(p) \otimes F_m^{22}(q) \\ + \frac{\left(a^2{}_m.\widehat{d}_m - \widehat{a}_m^2.\bar{c}_m\right)}{\Delta_m} F_m^{22}(p) \otimes F_m^{11}(q) \end{array} \right]. \quad (3.6)$$

But according to the development of the modified Green's function G_1established in [**7**], the expression of the modified Green's function (3.6) implies that the norm of the modified integral operator K_1 is equal to zero, which completes the proof of Lemma 1. □

In this case, we say that if the border of our domain is a circle of radius a, then the optimal choice of the simple and cross multipole coefficients defined in (3.1) – (3.5), lead us to the exact Green's function for the Dirichlet problem. In other words, we have:

$$G_1^D\,(P,\ Q) = G_{ex}^D\,(P,\ Q)$$

This result, avoid us the obligation to verify the large condition (1.7), because:

$$G_1^D(P,Q) = G_{ex}^D(P,Q) \quad \Longrightarrow \|K_1^D\| = 0.$$

Thus, in this case we don't need to verify the large condition (1.7) from the moment that we have no integral equation for solving. In other words, the solution of our boundary problem is obtained directly, as follows :

1- Dirichlet boundary condition $U(p) = g(p)$:

- integral equation on ∂D :

$$W(p) = 2g(p), \qquad p \in \partial D$$

- Representation in D :

$$U(P) = D_1 [W(P)] = 2\ D_1 [g(P)], \qquad P \in D$$

or :

- integral equation on ∂D :

$$W(p) = 2\ D_{1n} [g(p)], \qquad p \in \partial D$$

- Representation in D :

$$U(P) = D_1 [g(P)] - S_1 [W(P)] = D_1 [g(P)] - 2\ S_1 [D_{1n} [g(P)]], \qquad P \in D$$

2- Neumann boundary condition $TU(p) = f(p)$:

- integral equation on ∂D :

$$W(p) = -\ 2\ S_1 [f(p)], \qquad p \in \partial D$$

- Representation in D :

$$U(P) = D_1 [W(P)]\ -\ S_1 [f(P)] = -\ 2\ D_1 [S_1 [f(P)]] -\ S_1 [f(P)], \qquad P \in D$$

or :

- integral equation on ∂D :

$$W(p) = -\ 2\ f(p), \qquad p \in \partial D$$

- Representation in D :

$$U(P) = S_1 [W(P)] = -\ 2\ S_1 [f(P)], \qquad P \in D$$

with D_1 is the double layer potential defined by :

$$(D_1 W)(P) = \int_{\partial D} T_q G_1(P, q).W(q).ds_q, \qquad P \in D$$

and

$$D_{1n} [g(P)] = T\ \ [D_1 [g(P)]], \qquad P \in D$$

3.2 Case of the slightly distorted circle

After having treated the case of a circular border, we propose to consider the case of a boundary in the shape of a slightly distorted circle. The Parametric equation of this slightly distorted circle is defined in polar coordinates as follows:

$$r = a + \varepsilon\varphi(\theta), \qquad 0 \leq \theta \leq 2\pi \qquad (3.7)$$

where a is the radius of the circle not distorted and φ and $\frac{\partial\varphi}{\partial\theta}$ are two bounded functions . we note by p_ε, q_ε the points of ∂D and which are defined by :

$$Op_\varepsilon = (a + \varepsilon\varphi(\theta_p))\ \widehat{r}_p \qquad , \qquad Oq_\varepsilon = (a + \varepsilon\varphi(\theta_q))\ \widehat{r}_q \qquad (3.8)$$

where

$$\widehat{r} = (\cos(\theta) \quad , \quad \sin(\theta)) \qquad (3.9)$$

The points of the circle of radius a are defined by :

$$Op_0 = a\ \widehat{r}_p \qquad , \qquad Oq_0 = a\widehat{r}_q \qquad (3.10)$$

It is easy to see that :

$$|p_\varepsilon q_\varepsilon| = |p_0 q_0| + O(\varepsilon)\ (3.11).$$

Moreover, since φ is continuously derivable then we can show that

$$\frac{|\varphi(\theta_p) - \varphi(\theta_q)|}{|p_0 q_0|}$$

is bounded. Therefore,

$$\frac{1}{|p_\varepsilon q_\varepsilon|} = \frac{1}{|p_0 q_0|} + O(\varepsilon) \qquad (3.12)$$

on the slightly distorted circle ∂D we can easily show that :

$$\widehat{n} = \frac{\widehat{r} - \varepsilon\frac{\partial\varphi}{\partial\theta}\widehat{\theta}}{\sqrt{1 + \varepsilon^2\left(\frac{\partial\varphi}{\partial\theta}\right)^2}} = \widehat{r} + O(\varepsilon) \qquad (3.13)$$

and that the element ds written in the form :

$$ds = r\sqrt{1 + \varepsilon^2\left(\frac{\partial\varphi}{\partial\theta}\right)^2}\, d\theta = a\ d\theta\ + O(\varepsilon) \qquad (3.14)$$

Lemma 3.3 *If the border ∂D is defined by* (3.7) *then we have,*

$$F_m^{\sigma\ell}(p_\varepsilon) = F_m^{\sigma\ell}(p_0) + O(\varepsilon) \qquad (3.15)$$

$$A_m^{\sigma\ell} = \left\|F_m^{\sigma\ell}\right\|_{\partial D}^2 = \left\|F_m^{\sigma\ell}\right\|_a^2 + O(\varepsilon) = A_m^{\sigma\ell}(0) + O(\varepsilon) \qquad (3.16)$$

$$B_m^{\sigma\ell} = \left\langle F_m^{\sigma\ell}\ ,\ F_m^{(3-\sigma)(3-\ell)}\right\rangle_{\partial D} = \left\langle F_m^{\sigma\ell}\ ,\ F_m^{(3-\sigma)(3-\ell)}\right\rangle_a + O(\varepsilon) = B_m^{\sigma\ell}(0) + O(\varepsilon) \qquad (3.17)$$

$$\triangle_{m,\ \partial D}^{\sigma} = \begin{pmatrix} \alpha_m^{\sigma 1}\ .\ A_m^{\sigma 1}\ .\ \alpha_m^{(3-\sigma)2}\ .\ A_m^{(3-\sigma)2} \\ -\beta_m^{\sigma 1}\ .\ B_m^{\sigma 1}\ .\ \overline{\beta}_m^{\sigma 1}\ .\ \overline{B}_m^{\sigma 1} \end{pmatrix}_{\partial D} = \triangle_{m,\ a}^{\sigma} + O(\varepsilon) \qquad (3.18)$$

$$g_m^{\sigma\ell} = -\ \left\langle \overline{\beta}_m^{\sigma\ell}\ .\ \widehat{F}_m^{(3-\sigma)(3-\ell)} + \alpha_m^{\sigma\ell}\ .\ \widehat{F}_m^{\sigma\ell}\ ,\ F_m^{\sigma\ell}\right\rangle_{\partial D} = g_m^{\sigma\ell}(0) + O(\varepsilon) \qquad (3.19)$$

$$h_m^{\sigma\ell} = -\ \left\langle \overline{\beta}_m^{\sigma\ell}\ .\ \widehat{F}_m^{(3-\sigma)(3-\ell)} + \alpha_m^{\sigma\ell}\ .\ \widehat{F}_m^{\sigma\ell},\ F_m^{(3-\sigma)(3-\ell)}\right\rangle_{\partial D} = h_m^{\sigma\ell}(0) + O(\varepsilon) \qquad (3.20)$$

Proof.

From the definition of $F_m^{\sigma\ell}$ (1.6) and taking into account the fact that the hankel's function $H_m^1(x)$ is analytical with only one pole in $x = 0$

then we have :

$$H_m^1(a+\varepsilon) = H_m^1(a) + O(\varepsilon) \qquad (3.21)$$

Then followed (3.15). In the same way and using (3.14) and parametric repesentation of ∂D defined by (3.7), we obtain (3.16)

using (3.15) and (3.16) we obtain directly (3.17), (3.18), (3.19) and (3.20)
□

Theorem 3.4 *(***Calculation of the simple and cross multipole coefficients** $a_m^{\sigma\ell}$ **and** b_m*)*

If the boundary of the domain D is a slightly distorted circle, then the multipole coefficients will be given by the following relatively simple expressions:

$$a_m^{11} = a_m^{11}(0) + O(\varepsilon) \qquad (3.22)$$

$$a_m^{21} = a_m^{21}(0) + O(\varepsilon) \qquad (3.23)$$

$$a_m^{22} = a_m^{22}(0) + O(\varepsilon) \qquad (3.24)$$

$$a_m^{12} = a_m^{12}(0) + O(\varepsilon) \qquad (3.25)$$

$$b_m = b_m(0) + O(\varepsilon) \qquad (3.26)$$

where $a_m^{11}(0), a_m^{21}(0), a_m^{22}(0)$, $a_m^{12}(0)$ *and* $b_m(0)$ *are the multipole coefficients calculated in the case of circle* (3.1) *to* (3.5)

Proof.

To calculate the simple and cross miltipole coefficients $a_m^{\sigma\ell}$ and b_m for the case of the slightly distorted circle, we should first calculate $g_m^{\sigma 1}$, $g_m^{(3-\sigma)2}$, $h_m^{\sigma 1}$, and $h_m^{(3-\sigma)2}$

We have :

$$g_m^{\sigma\ell} = -\,\overline{\beta}_m^{\sigma\ell}\left\langle \widehat{F}_m^{(3-\sigma)(3-\ell)},\ F_m^{\sigma\ell}\right\rangle_{\partial D} - \alpha_m^{\sigma\ell}\left\langle \widehat{F}_m^{\sigma\ell},\ F_m^{\sigma\ell}\right\rangle_{\partial D}$$

$$= -\left\langle F_m^{(3-\sigma)(3-\ell)},\ F_m^{\sigma\ell}\right\rangle_{\partial D}\left\langle \widehat{F}_m^{(3-\sigma)(3-\ell)},\ F_m^{\sigma\ell}\right\rangle_{\partial D} - \left\|F_m^{\sigma\ell}\right\|_{\partial D}^2\left\langle \widehat{F}_m^{\sigma\ell},\ F_m^{\sigma\ell}\right\rangle_{\partial D}$$

using Lemma 3.3 we obtain :

$$g_m^{\sigma 1} = \left((-1)^{\sigma}\ \bar{c}_m(0)\right).\left(-\ (-1)^{\sigma}\ \widehat{d}_m(0)\right) - \left(a_m^1(0)\right).\left(\widehat{a}_m^1(0)\right) + O(\varepsilon) = g_m^{\sigma 1}(0) + O(\varepsilon)$$

and

$$g_m^{(3-\sigma)2} = \left((-1)^{\sigma}\ c_m(0)\right).\left(-\ (-1)^{\sigma}\ \widehat{c}_m(0)\right) - \left(a_m^2(0)\right).\left(\widehat{a}_m^2(0)\right) + O(\varepsilon) = g_m^{(3-\sigma)2}(0) + O(\varepsilon)$$

On the other hand :

$$h_m^{\sigma\ell} = -\,\overline{\beta}_m^{\sigma\ell}\left\langle \widehat{F}_m^{(3-\sigma)(3-\ell)},\ F_m^{(3-\sigma)(3-\ell)}\right\rangle_{\partial D} - \alpha_m^{\sigma\ell}\left\langle \widehat{F}_m^{\sigma\ell},\ F_m^{(3-\sigma)(3-\ell)}\right\rangle_{\partial D}$$

$$= -\left\langle F_m^{(3-\sigma)(3-\ell)},\ F_m^{\sigma\ell}\right\rangle_{\partial D}\left\langle \widehat{F}_m^{(3-\sigma)(3-\ell)},\ F_m^{(3-\sigma)(3-\ell)}\right\rangle_{\partial D} - \left\|F_m^{\sigma\ell}\right\|_{\partial D}^2\left\langle \widehat{F}_m^{\sigma\ell},\ F_m^{(3-\sigma)(3-\ell)}\right\rangle_{\partial D}$$

Using Lemma 3.3, we obtain :

$$h_m^{\sigma 1} = \left((-1)^{\sigma}\ \bar{c}_m(0)\ \right).\left(\widehat{a}_m^2(0)\right) - \left(a_m^1(0)\right).\left(-\ (-1)^{\sigma}\ \widehat{c}_m(0)\right) + O(\varepsilon) = h_m^{\sigma 1}(0) + O(\varepsilon)$$

and

$$h_m^{(3-\sigma)2} = \left((-1)^{\sigma}\ c_m(0)\right).\left(\widehat{a}_m^1(0)\right) - \left(a_m^2(0)\right).\left(-\ (-1)^{\sigma}\ \widehat{d}_m(0)\right) + O(\varepsilon) = h_m^{(3-\sigma)2}(0) + O(\varepsilon)$$

so the expressions of the coefficients $a_m^{\sigma\ell}$ and b_m will be given by (3.22) to (3.26)

Lemma 3.5 *(***Calculation of the modified Green's function***)*
If the boundary of the domain D is a slightly distorted circle, then $\|K_1\| = O(\varepsilon)$

Proof.
We have:

$$G_1(p,q) = G_0(p,q) + \frac{i}{4\mu K^2} \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} \sum_{\ell=1}^{2} \begin{bmatrix} a_m^{\sigma\ell}.F_m^{\sigma\ell}(p) \otimes F_m^{\sigma\ell}(q) \\ +(-1)^{\sigma+\ell}.b_m F_m^{\sigma\ell}(p) \otimes F_m^{(3-\sigma)(3-\ell)}(q) \end{bmatrix}.$$

Replacing the expressions of multipole coefficients obtained above, we obtain :

$$G_1(p,q) = G_0(p,q)\,(0) + O(\varepsilon)$$

$$+\frac{i}{4\mu K^2} \sum_{m=0}^{\infty} \begin{bmatrix} \frac{\left(\bar{c}_m.\widehat{c}_m - \widehat{a}_m^1.a_m^2\right)(0)+O(\varepsilon)}{\Delta_{m,\ a}+O(\varepsilon)}.F_m^{11}(p) \otimes F_m^{11}(q) \\ +\left(\frac{\left(a^2{}_m.\widehat{d}_m - \widehat{a}_m^2.\bar{c}_m\right)}{\Delta_m} + O(\varepsilon)\right) F_m^{11}(p) \otimes F_m^{22}(q) \\ +\frac{\left(c_m.\widehat{d}_m - a_m^1.\widehat{a}_m^2\right)(0)+O(\varepsilon)}{\Delta_{m,\ a}+O(\varepsilon)}.F_m^{12}(p) \otimes F_m^{12}(q) \\ -\left(\frac{\left(a^2{}_m.\widehat{d}_m - \widehat{a}_m^2.\bar{c}_m\right)}{\Delta_m} + O(\varepsilon)\right) F_m^{12}(p) \otimes F_m^{21}(q) \\ +\frac{\left(\bar{c}_m.\widehat{c}_m - \widehat{a}_m^1.a_m^2\right)(0)+O(\varepsilon)}{\Delta_{m,\ a}+O(\varepsilon)}.F_m^{21}(p) \otimes F_m^{21}(q) \\ -\left(\frac{\left(a^2{}_m.\widehat{d}_m - \widehat{a}_m^2.\bar{c}_m\right)}{\Delta_m} + O(\varepsilon)\right) F_m^{21}(p) \otimes F_m^{12}(q) \\ +\frac{\left(c_m.\widehat{d}_m - a_m^1.\widehat{a}_m^2\right)(0)+O(\varepsilon)}{\Delta_{m,\ a}+O(\varepsilon)}.F_m^{22}(p) \otimes F_m^{22}(q) \\ +\left(\frac{\left(a^2{}_m.\widehat{d}_m - \widehat{a}_m^2.\bar{c}_m\right)}{\Delta_m} + O(\varepsilon)\right) F_m^{22}(p) \otimes F_m^{11}(q) \end{bmatrix}. \qquad (3.27)$$

Or:

$$G_1(p,q) = G_1(p,q)\,(0) + O(\varepsilon). \qquad (3.28)$$

But from the development of the modified Green's function G_1 established in [7], the result (3.28) implies that the norm of the modified integral operator K_1 is of order ε. In other words,

$$G_1(p,\ q) = G_1(p,q)\,(0) + O(\varepsilon) \quad \Longrightarrow \quad \|K_1\| = O(\varepsilon).$$

The interest of this result, is that, in the case of a slightly deformed circular border, the method of successive approximations to be used for solving our integral equation, will have a large radius of convergence (spectrum of values of the frequency waves ω^2), as the form :

$$0 < \omega^2 < \frac{1}{\|K_1\|} \Longrightarrow 0 < \omega^2 < \frac{1}{O(\varepsilon)}$$

4 Conclusion

The work presented in this paper, aims at testing the expressions of the optimal choice of the multipole coeffcients found for the general case in [**8**]. For this, we consider the case where the boundary ∂D is a circle of radius $'a'$. in this case, a relatively simple form for the simple and cross multipole coeffcients, has been achieved. Moreover, our optimal choice led us to the exact Green's function, where the norm of the modified integral operator $\|K_1\|$ becomes void, which will lead to the direct solution of the boundary problem without resorting to solving an integral equation . It is noted that in this case, the large condition (1.7) imposed on the multipole coefficients need not to be satisfied from the moment that we have no integral equation to solve. (we recall here that the origin of the large condition (1.7), which is a sufficient condition but not necessary, is the inversibility of the operator $I - \overline{K_1^*}$ (1.2)).

In the second part of this work, we consider the case where the boundary ∂D is a slightly distorted circle, where we show that the expression of the multipole coefficients is relative to that found in the circular border, and moreover the norm of our modified integral operator will be very small or of order of ε, that enlarges us the radius of convergence of the numerical method to be used for the resolution of our integral equation defined on ∂D.

5 Open problems

The modified Green's function techniques which use the multipole coefficients has many open problems which deserve to be treated. In this way we can mentioned the following:

1- Checking the large condition (1.7) for the general case where the border take any form.

2- Consider the cases of other simple geometric forms, such as square, rectangle, triangle, ellipse, ...

3- Treat the same subject by changing the criterion of optimality and consider for example the minimization of the condition number of the integral operator associated with our boundary problem, (in the case of three dimensions, see [12] for acoustic waves and [13] for elastic waves).

4- Establish the numerical applications for the results obtained in this paper (some numerical applications given in [14] and [15]).

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