A Method for Solving Fuzzy Fredholm Integral Equations of The Second Kind

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Abstract

In this paper, a numerical method for solving fuzzy Fredholm integral equations of the second kind is introduced. We apply the trapezoidal rule to compute the Riemann integrals. This kind of integral equations convert to a linear system. Then, by solving the linear system , unknowns are determined. Finally, an algorithm is presented to solve the fuzzy integral equation by using the trapezoidal rule. This algorithm is implemented on some numerical examples by using software MATLAB.

Keywords: *Fuzzy Fredholm Integral Equation, Fuzzy derivative, Fuzzy integral, Fuzzy number, Trapezoidal rule.*

1 Introduction

The topic of fuzzy integral equations (FIE) has been developed in recent years. In the first step often including applicable definitions of the fuzzy integrals was followed by introducing FIE and stablishing sufficient conditions for the existence of unique solutions to these equations. Finally, numerical algorithms for calculation approximates to these solutions were designed. Prior to discussing fuzzy integral equations and their associated numerical algorithms, it is necessary to present an appropriate brief introduction to preliminary topics such as fuzzy numbers and fuzzy calculus. The concept of fuzzy sets which was originally introduced by Zadeh [17, 18] led to the definition of the fuzzy number and implementation in fuzzy control [2] and approximate reasoning problems[17, 18]. The basic arithmetic structure for fuzzy numbers was later developed by Mizumoto and Tanaka [12, 13], Nahmias [14], Dubios and Prade [3, 4, 5] and Ralescu [16] all of which observed the fuzzy number as a location of $\alpha-$ levels $0 \leq \alpha \leq 1$ [2].The concept of integration of fuzzy functions was first introduced by Dubois and Prade[5]. Alternative approaches were later suggested by Goetschel and Voxman [8], Kaleva [9], Matloka[11], Nanda [15], and others. While Goetschel and Voxman [8] and later Matloka[11], preferred a Riemann integral type approach, Kaleva[9] chose to define the integral of fuzzy function, using the Lebesgue type concept for integration. One of the first applications of fuzzy integration was given by Wu and Ma [18] who investigated the Fuzzy Fredholm integral equation of the second kind (FF-2). In this work, we concentrate on numerical procedures for solving FIE, whenever these equations posses unique fuzzy solutions. In section 2 we briefly present the basic notations of fuzzy numbers, fuzzy continuous function, fuzzy derivative fuzzy integral, and a trapezoidal rule for integration recalled. Fuzzy Frdholm integral equations is introduced, a numerical solution will present for these kind of integral equation in section 3. Finally, an algorithm for numerical solution is given and illustrated with examples by applying MATLAB software.

2 Preliminaries

The set of all fuzzy numbers is represented by E^1. The parametric definition of fuzzy numbers is defined in [1] as follows:

Definition 2.1 *An arbitrary fuzzy number with an ordered pair of functions* $(\underline{v}(r), \overline{v}(r)),\quad 0 \leq r \leq 1$, *which satisfy in the following requirements.*

1. $\underline{v}(r)$ *is a bounded left continuous non decreasing function in* r *over* $[0, 1]$.
2. $\overline{v}(r)$ *is a bounded left continuous non increasing function in* r *over* $[0, 1]$.
3. $\underline{v}(r) \leq \overline{v}(r),\ 0 \leq r \leq 1$.

For arbitrary $u = (\underline{u}(r), \overline{u}(r)), v = (\underline{v}(r), \overline{v}(r))$ *and* $k \in R$, *we define addition and multiplication by* k *as,*

$$(\underline{u+v})(r) = (\underline{u}(r) + \underline{v}(r)),$$

$$(\overline{u+v})(r) = (\overline{u}(r) + \overline{v}(r)),$$

$$(\underline{ku})(r) = k\underline{u}(r),\ (\overline{ku})(r) = k\overline{u}(r),\quad k \geq 0$$

$$(\underline{ku})(r) = k\overline{u}(r),\ (\overline{ku})(r) = k\underline{u}(r).\quad k < 0$$

Definition 2.2 *The* $n \times n$ *linear system of equations* $AX = Y$ *where the coefficient matrix* $A = (a_{ij}),\ 1 \leq i, j \leq n$ *is a crisp* $n \times n$ *matrix and* $Y = (y_1, \ldots, y_n)^t,\ y_i \in E^1,\ 1 \leq i \leq n$, *is called a fuzzy system of linear equations (FSLE).*

Definition 2.3 *Let $f : [a, b] \longrightarrow E^1$. For each partition $p = \{t_0, t_1, ..., t_n\}$ of $[a, b]$ and for arbitrary $\xi_i : t_{i-1} \leq \xi_i \leq t_i, 1 \leq i \leq n$ if*

$$R_p = \sum f(\xi_i)(t_i - t_{i-1}). \tag{1}$$

Then, the definite integral of f(t) over $[a, b]$ is defined as follows:

$\int_a^b f(t)dt = \lim R_p \qquad \max |t_i - t_{i-1}| \longrightarrow 0, 1 \leq i \leq n,$

provided that this limit exists in the metric D [1, 3].

Definition 2.4 *Let $f : [a, b] \rightarrow E^1$ be continuous in the metric D, then its definite integral over $[a, b]$ exists [**?**]. Furthermore,*

$$\underline{(\int_a^b f(t;r)dt)} = \int_a^b \underline{f}(t,r)dt,$$

$$\overline{(\int_a^b f(t;r)dt)} = \int_a^b \overline{f}(t,r)dt. \tag{2}$$

2.1 The Numerical Method for Integration

To calculate the Riemann integrals in (2) of $\underline{f}(t; r)$ and $\overline{f}(t; r)$ we can apply the trapezoidal rule. In this case the interval $[a, b]$ is partitioned by equally spaced points $a = t_0 < t_1 < ... < t_{n-1} < t_n = b$ where $t_i = a + ih$, $t_i - t_{i-1} = \frac{b-a}{n} = h$,$1 \leq i \leq n$:

Let:

$$\underline{s}_n(r) = h[\underline{f}(a; r) + \underline{f}(b; r) + \sum_{i=1}^{n-1} \underline{f}(t_i; r)],$$

$$\overline{s}_n(r) = h[\overline{f}(a; r) + \overline{f}(b; r) + \sum_{i=1}^{n-1} \overline{f}(t_i; r)].$$

Then, for an arbitrary fixed r we have[7]:

$$\lim_{n \longrightarrow \infty} \underline{s}_n(r) = \underline{F}(r) = \int_a^b \underline{f}(t; r)dt.$$

$$\lim_{n \longrightarrow \infty} \overline{s}_n(r) = \overline{F}(r) = \int_a^b \overline{f}(t; r)dt. \tag{3}$$

Theorem 2.1 *If f is continuous in metric D, $\underline{s}_n(r)$, $\overline{s}_n(r)$ uniformly converge to , $\underline{F}(r)$, $\overline{F}(r)$, respectively, [7].*

3 Fuzzy Fredholm Integral Equation

In this section, the fuzzy integral equations of the second kind are introduced. The Fredholm integral equation of the second kind is [7]

$$f(s) = y(s) + \lambda \int_a^b k(s,t)f(t)dt, \tag{4}$$

Where $\lambda > 0$, k(s,t) is an arbitrary kernel function over the square $a \leq s, t \leq b$ and f, y are fuzzy functions on [a,b]. If f(t) is a crisp function then the solutions of Eqs. (3.3) is crisp .However, if y is a fuzzy function then this equation may only possess fuzzy solutions. Sufficient conditions for the existence of a unique solution to the fuzzy Fredholm integral equations of the second kind, i.e. Eqs. (3.3)where y(t)is fuzzy function, have been given in [7].
In order to design a numerical scheme for solving Eq. (3.3) we first replace it by the system,

$$\underline{f}(s,r) = \underline{y}(s,r) + \lambda \int_a^b \underline{U}(t,r)dt,$$

$$\overline{f}(s,r) = \overline{y}(s,r) + \lambda \int_a^b \overline{U}(t,r)dt,$$

where,

$$\underline{U}(t,r) = \begin{cases} k(s,t)\underline{f}(t,r) & k(s,t) \geq 0, \\ k(s,t)\overline{f}(t,r) & k(s,t) < 0. \end{cases} \tag{5}$$

and

$$\overline{U}(t,r) = \begin{cases} k(s,t)\overline{f}(t,r) & k(s,t) \geq 0, \\ k(s,t)\underline{f}(t,r) & k(s,t) < 0. \end{cases} \tag{6}$$

Without loss of generality, we suppose that $k(s,t) \geq 0$, thus:

$$\underline{U}(t,r) = k(s,t)\underline{f}(t,r),$$

$$\overline{U}(t,r) = k(s,t)\overline{f}(t,r)$$

The other case is similar. Let $\{h_i(s)\}_{i=1}^{\infty}$ be a sequence of independent and complete functions. We consider

$$\underline{f}(s,r) \simeq G_n(s,r) = \sum_{i=1}^{n} \underline{a}_i(r)h_i(s),$$

$$\overline{f}(s,r) \simeq F_n(s,r) = \sum_{i=1}^{n} \overline{a}_i(r)h_i(s).$$

$$\underline{r_n}(s,r) = \underline{y}(s,r) - \sum_{i=1}^{n} b_i(r)l_i(s), \tag{7}$$

$$\overline{r_n}(s,r) = \overline{y}(s,r) - \sum_{i=1}^{n} c_i(r)l_i(s), \tag{8}$$

where,

$$l_i(s) = h_i(s) - k_i(s), \quad 1 \leq i \leq n$$

$$k_i(s) = \lambda \int_a^b k(s,t)h_i(t)dt. \tag{9}$$

$$c_k(r) = \begin{cases} \overline{a}_k(r) & l_k(s) \geq 0, \\ \underline{a}_k(r) & l_k(s) < 0. \end{cases} \tag{10}$$

and

$$b_k(r) = \begin{cases} \underline{a}_k(r) & l_k(s) \geq 0, \\ \overline{a}_k(r) & l_k(s) < 0. \end{cases} \tag{11}$$

Now, by applying the least square method, (3.6) and (3.7) can be transformed to the following system [6]:

SA=Y, $\quad L = [l_{i,j}], \quad l_{i,j} = \int_a^b l_i(s)l_j(s)ds \quad i,j = 1,...,n\,, \quad \det(L) \neq 0,$

$$S = \begin{bmatrix} L & 0 \\ 0 & L \end{bmatrix}, \quad A = \begin{bmatrix} b(r) \\ c(r) \end{bmatrix}, \quad Y = \begin{bmatrix} \underline{y}(r) \\ \overline{y}(r) \end{bmatrix},$$

where,

$$b(r) = \begin{bmatrix} b_1(r) \\ b_2(r) \\ \vdots \\ b_n(r) \end{bmatrix}, \; c(r) = \begin{bmatrix} c_1(r) \\ c_2(r) \\ \vdots \\ c_n(r) \end{bmatrix}, \; \underline{y}(r) = \begin{bmatrix} \underline{y}_1(r) \\ \underline{y}_2(r) \\ \vdots \\ \underline{y}_n(r) \end{bmatrix}, \; \overline{y}(r) = \begin{bmatrix} \overline{y}_1(r) \\ \overline{y}_2(r) \\ \vdots \\ \overline{y}_n(r) \end{bmatrix}.$$

Such that:

compute $l_i(s)$ by using the selection (3.8)

$$\underline{y}_i(r) = \int_a^b \underline{c}(s,r)l_i(s)ds,$$

$$\overline{y}_i(r) = \int_a^b \overline{c}(s,r)l_i(s)ds. \tag{12}$$

where

$$\underline{c}(s,r) = \begin{cases} \underline{y}(s,r) & l_i(s) \geq 0, \\ \overline{y}(s,r) & l_k(s) < 0. \end{cases} \tag{13}$$

and

$$\overline{c}(s,r) = \begin{cases} \overline{y}(s,r) & l_i(s) \geq 0, \\ \underline{y}(s,r) & l_i(s) < 0. \end{cases} \tag{14}$$

The following algorithm evaluate the fuzzy integral equation(3.3):

3.1 Algorithm of The Numerical Procedure

1. Read $a, b, \lambda, n, \underline{y}(s,r), \overline{y}(s,r), k(s,t), \{h_i(s)\}_{i=1}^n$

2. For $i = 1$ to n, compute $l_i(s)$,$\underline{y}_i(r)$,$\overline{y}_i(r)$ by using the selections (3.8) and (3.11)

where compute $\underline{c}(s,r)$, $\overline{c}(s,r)$ by using the relation (3.12) and (3.13)

2-1. For $j = 1$ to n compute $l_{i,j}$

3. Denote $L = [l_{i,j}], \quad i,j = 1,...,n,$

$\underline{y}(r) = [\underline{y}_i(r)], \quad i = 1,...,n,$

$\overline{y}(r) = [\overline{y}_i(r)], \quad i = 1,...,n,$

4. Solve the following linear system:

$$SA = Y \rightarrow \begin{bmatrix} L & 0 \\ 0 & L \end{bmatrix} \begin{bmatrix} b(r) \\ c(r) \end{bmatrix} = \begin{bmatrix} \underline{y}(r) \\ \overline{y}(r) \end{bmatrix}. \tag{15}$$

$$A = \begin{bmatrix} b(r) \\ c(r) \end{bmatrix}, \quad Y = \begin{bmatrix} \underline{y}(r) \\ \overline{y}(r) \end{bmatrix}.$$

$$b(r) = \begin{bmatrix} b_1(r) \\ b_2(r) \\ . \\ . \\ . \\ b_n(r) \end{bmatrix}, \; c(r) = \begin{bmatrix} c_1(r) \\ c_2(r) \\ . \\ . \\ . \\ c_n(r) \end{bmatrix}, \; \underline{y}(r) = \begin{bmatrix} \underline{y}_1(r) \\ \underline{y}_2(r) \\ . \\ . \\ . \\ \underline{y}_n(r) \end{bmatrix}, \; \overline{y}(r) = \begin{bmatrix} \overline{y}_1(r) \\ \overline{y}_2(r) \\ . \\ . \\ . \\ \overline{y}_n(r) \end{bmatrix}.$$

5. Estimate $\underline{f}(s,r)$, $\overline{f}(s,r)$ by computing $\sum_{i=1}^{n} \underline{a}_i(r)h_i(s)$, $\sum_{i=1}^{n} \overline{a}_i(r)h_i(s)$.

6. Write $\underline{f}(s,r)$, $\overline{f}(s,r)$ and then stop.

We use the algorithm trapezoidal rule which evaluate the integral

4 Numerical Example

Example1. Consider the following fuzzy Fredholm equation [7]

$$\underline{y}(t,r) = sin(\frac{t}{2})[\frac{13}{15}(r^2+r) + \frac{2}{15}(4-r^3-r)],$$

$$\overline{y}(t,r) = sin(\frac{t}{2})[\frac{2}{15}(r^2+r) + \frac{13}{15}(4-r^3-r)]$$

and kernel

$$k(s,t) = 0.1\sin(s)\sin(\frac{t}{2}), \quad 0 \leq s, \; t \leq \pi$$

and $a = 0$, $b = 2\pi$. The exact solution is given by

$$\underline{F}(t,r) = (r^2+r)sin(\frac{t}{2})$$

$$\overline{F}(t,r) = (4-r^3-r)sin(\frac{t}{2}).$$

By using the mentioned algorithm and MATLAB package, we obtain the following results :

$k_1(s) = \int_0^{2\pi} 0.1 \sin(s) \sin(\frac{t}{2}) dt = 0.4 \sin(s)$

$k_2(s) = \int_0^{2\pi} 0.1t \sin(s) \sin(\frac{t}{2}) dt = 0.4\pi \sin(s)$

$l_1(s) = h_1(s) - k_1(s) = 1 - 0.4 \sin(s)$

$l_2(s) = h_2(s) - k_2(s) = s - 0.4\pi \sin(s)$

$\underline{y}_1(r) = \int_0^{2\pi} \underline{c}(s,r) l_1(s) ds = 4(\frac{13}{15}(r^2 + r) + \frac{2}{15}(4 - r^3 - r))$

$\overline{y}_1(r) = \int_0^{2\pi} \overline{c}(s,r) l_1(s) ds = 4(\frac{2}{15}(r^2 + r) + \frac{13}{15}(4 - r^3 - r))$

$\underline{y}_2(r) = \int_0^{2\pi} \underline{c}(s,r) l_2(s) ds = -0.027306229r^3 + 12.53906438r^2 + 12.51175815r + 0.109224918$

$\overline{y}_2(r) = \int_0^{2\pi} \overline{c}(s,r) l_2(s) ds = -12.53906438r^3 + 0.027306229r^2 - 12.51175815r + 50.15625752$

$l_{11}(s) = \int_0^{2\pi} l_1(s) l_1(s) ds = 6.785840132$

$l_{22}(s) = \int_0^{2\pi} l_2(s) l_2(s) ds = 103.4357758$

$l_{21}(s) = \int_0^{2\pi} l_2(s) l_1(s) ds = 23.83161963$

$l_{12}(s) = \int_0^{2\pi} l_1(s) l_2(s) ds = 23.83161963$

The approximate and exact solutions are compared at $t = \pi$ in table 1. In this example, the kernel $k(s,t)$ is nonnegative for $0 \leq s \leq \pi$ and negative for $\pi < s < 2\pi$.

r	$\underline{f}$	$\overline{f}$	$\underline{F}$	$\overline{F}$
0.0	0.5332	3.4663	0.5333	3.4667
0.1	0.6150	3.3935	0.6152	3.3938
0.2	0.7137	3.3180	0.7136	3.3184
0.3	0.8279	3.2350	0.8278	3.2353
0.4	0.0954	3.1390	0.0957	3.1392
0.5	1.0997	3.0248	1.1000	3.0250
0.6	1.2562	2.8774	1.2565	2.8775
0.7	1.4255	2.7212	1.4256	2.7214
0.8	1.6063	2.5214	1.6064	2.5216
0.9	1.7982	2.2828	1.7981	2.2829
1.0	1.9998	1.9998	2.0000	2.0000

$Table 1.$

Example2. Consider the following fuzzy Fredholm equation

$$\underline{f}(t,r) = rt + \frac{3}{26} - \frac{3r}{26} - \frac{1}{13}t^2 - \frac{1}{13}t^2 r,$$

$$\overline{f}(t,r) = 2t - rt + \frac{3}{26}r + \frac{1}{13}t^2 r - \frac{3}{26} - \frac{3}{13}t^2$$

and kernel

$$k(s,t) = \frac{s^2 + t^2 - 2}{13}, \quad 0 \leq s, \ t \leq 2$$

and $a = 0$, $b = 2$. The exact solution in this case is given by

$$\underline{F}(t,r) = rt$$

$$\overline{F}(t,r) = (2-r)t.$$

By using the mentioned algorithm and MATLAB package, we obtain the following results:
The approximate and exact solution are compared at $t = 1$ in table 2.

r	$\underline{f}$	$\overline{f}$	$\underline{F}$	$\overline{F}$
0.0	0.0001	1.9998	0.0000	2.0000
0.1	0.1001	1.9002	0.1000	1.9000
0.2	0.2002	1.8002	0.2000	1.8000
0.3	0.2999	1.6999	0.3000	1.7000
0.4	0.4001	1.6001	0.4000	1.6000
0.5	0.4998	1.4998	0.5000	1.5000
0.6	0.6001	1.3999	0.6000	1.4000
0.7	0.6998	1.3003	0.7000	1.3000
0.8	0.7997	1.1999	0.8000	1.2000
0.9	0.9001	1.1001	0.9000	1.1000
1.0	0.9998	0.9998	1.0000	1.0000

$Table(2).$

5 Conclusion

In this work, we present a numerical method for solving the fuzzy Fredholm integral equation of second kind. The integrals are computed by using the trapezoidal rule. Finally, The stability and precise of the method is illustrated by comparing the results with the exact solutions.

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