Int. J. Open Problems Compt. Math., Vol. 1, No. 2, September 2008

Analysis of a Bulk Queue with Unreliable Server and Single Vacation

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Abstract

In this paper, the operating characteristics of an $M^X/G/1$ *queueing system with unreliable server and single vacation are analyzed. The server is subjected to fail, while it is on, and the arrival rate depends on the up and down states of the server. The time to failure is exponentially distributed and the repair times follow general distribution. Server is assigned to a secondary job (vacation) when the system becomes empty. The model is studied by the embedded Markov chain technique and level crossing analysis. The probability generating function of the steady state system size at an arbitrary time is obtained. An expression for the expected number of customers in the system, expected length of busy period and idle period is also derived. The application of the proposed model for a manufacturing system is discussed. A cost model for the queueing systems is discussed with a numerical illustration.*

Key words: Batch arrival queue, Single Vacation, System size, Unreliable Server.

1 Introduction

The objective of this paper is to analyze an $M^X/G/1$ queueing system under single vacation policy and unreliable server, that exists in a manufacturing industry, assembly systems, etc., In a Globe Valve manufacturing industry, after turning operation the components arrive from job shop in batches to CNC turning center for facing and turning processes. The operator of CNC turning center starts the processes immediately. After processing these components, if no components arrive, the operator will begin doing other work *(vacation state)* such as arranging the tooling, writing the coding, removing the chips, changing the coolant, etc. When the operator returns from other work and finds one or more components waiting for service in the queue, he immediately begins to serve them until the system becomes empty. On the other hand, if the operator finds an empty system again at the end of the vacation, he remains idle until a component arrives. The service of components may be interrupted when operator encounters unpredicted *breakdowns* such as accident event, blunt tool, troubles in coolant, etc. Arumuganathan and Jeyakumar [1] proposed a cost model for a practical situation in Globe Valve manufacturing industry with reliable server.

When service interruptions occur (*breakdowns*), it is emergently recovered with a random time. As soon as the broken server is repaired, the server immediately returns to provide his services until the system is empty, and the service time is cumulative. The above process can be modeled as an $M^X/G/1$ queueing system with unreliable server and single vacation.

Concerning queueing models with server vacations, an excellent survey of queueing systems with server vacations was found in Doshi [3] and Takagi [10]. Lee and Srinivasan [7] and Kella [6] studied queueing systems with the threshold policy (N Policy) and multiple vacations, including some applications. They respectively dealt with the batch arrival M/G/1 and the single unit arrival M/G/1 queueing systems, examined the system performances, and obtained the optimal policy under a stationary cost function, in which the arrival occurs in bulk and single respectively.

Queueing models with server breakdowns are more realistic representation of the systems. Wang [12] proposed an N Policy M/M/1 queueing system with server breakdowns. He developed an analytic closed – form solutions and provided a sensitivity analysis, but service followed an exponential distribution without any interruptions for a single arrival. Wang [13] and Wang et al. [14] extended Wang's model to the N policy $M/E_k/1$ and $M/H_2/1$ queueing system cases, respectively. They focused on single arrival Erlangian service time queueing model with a reliable server. J.C. Ke [4] studied a modified T vacation policy for an M/G/1 queueing system where an unreliable server may take at most J vacations repeatedly until at least one customer appears in the queue upon returning from a vacation, and the server needs a startup time before starting each of his service periods, also derived various system performances, but arrival occurs one at a time. Wang et al [11] performed a comparative analysis between the exact results and the maximum entropy results, also demonstrated through the maximum entropy results that the maximum entropy principle approach is accurate enough for practical purposes.

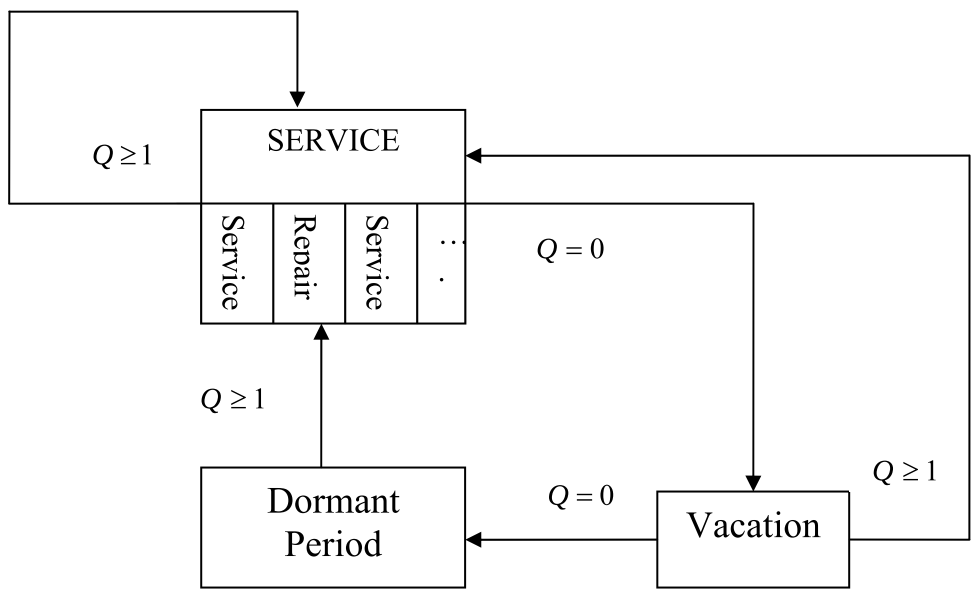
Choudhury [2] successfully modeled a batch arrival queueing system with a single vacation policy which extends the results of Levy and Yechiali [8] and Doshi [3]. J.C. Ke [5] discussed some operating characteristics analysis on the $M^X/G/1$ system with a variant vacation policy and balking, but both considered the service without any breakdown concepts. This paper generalizes the above said models by considering bulk arrival general service queueing system in which the server avails vacation and is subject to fail.

This paper deals with an $M^X/G/1$ queueing system under single vacation policy and unreliable server. Customers arrive according to the compound Poisson process with random arrival size. Arrival rate varies according to the server's status: busy or broken down. When he returns from a vacation, if there are one or more customers waiting, he serves until the system becomes empty; otherwise, he stays in the system waiting for the first one to arrive. The service is interrupted if break down occurs, and the server is immediately repaired. When the repair is completed, the server immediately returns for service. Breakdown times are exponentially distributed and the repair times follow general distribution. The model is studied by the embedded Markov chain technique and level crossing analysis.

The paper is organized in the following scenario:

- In the first part of the paper, system size distribution at busy period initiation epoch, at departure epoch and at an arbitrary epoch are discussed.
- In the second part of the paper, various performance measures such as expected number of customers in the system, expected length of idle period and expected length of busy period are obtained .
- In the final part, numerical illustration is provided with a cost model.

Important contribution is the study of cost model for a practical situation and how the results are useful in optimizing the cost.



Schematic Representation of the Model Q - Queue length

2 Notations

λ	arrival rate when the server is up
λ_0	arrival rate when the server is down
γ	failure rate when the server is busy
X	arrival size random variable
g_k	Pr(X=k)
X(z)	probability generating function (PGF) of X
S	service time random variable
V	vacation time random variable
$G(\cdot)$	cumulative distribution function of the repair time
$S(\cdot)$	cumulative distribution function of the service time
$V(\cdot)$	cumulative distribution function of the vacation time
$\tilde{V}(\theta)$	Laplace-Stieltjes transform (LST) of V
$\tilde{S}(\theta)$	Laplace-Stieltjes transform (LST) of S
$\tilde{G}(\theta)$	Laplace-Stieltjes transform (LST) of G

3 Queue size distribution at busy period initiation epoch

In this section, the steady state queue size distribution at a busy period initiation epoch is developed. α_n $(n \geq 1)$ is defined as the steady state probability that an arbitrary (tagged) customer finds n customers in the system at the busy period initiation epoch. If T_l (l = 0,1,2,….) is the initiation epochs of the busy period and $N(T_l)$ is the number of customers in the system at the time instant T_l, then

$$\alpha_n = \lim_{l \to \infty} \Pr ob[N(T_l) = n], \quad n \geq 1 .$$

Conditioning on the number of customers which arrive during the first vacation after a busy period , the following state equation is obtained.

$$\alpha_n = \sum_{k=1}^{n} a_k\, g_n^{(k)} + a_0\, g_n, \quad n \geq 1, \tag{3.1}$$

$a_k = \Pr ob['k'$batches arrive during a vacation time 'V'] = $\displaystyle\int_0^\infty \frac{e^{-\lambda t}(\lambda t)^k}{k!}\, dV(t)$

The following probability generating functions X(z) and $\alpha(z)$ are defined as follows.

$$X(z) = \sum_{n=1}^{\infty} z^n g_n \qquad \text{and} \qquad \alpha(z) = \sum_{n=1}^{\infty} z^n \alpha_n .$$

Multiplying (3.1) by appropriate powers of z and then taking summation over all possible values of 'n', we get

$$\alpha(z)=\tilde{V}(\lambda-\lambda X(z)) + \tilde{V}(\lambda)[X(z)-1], \qquad \textbf{(3.2)}$$

$\alpha(z)$ represents the PGF of the number of customers in the system at the completion epoch of the idle period.

Let E(α) and E[α(α − 1)] be the first two factorial moments of the distribution of the queue size at busy period initiation epoch, then

$$E(\alpha)=\left[\frac{d\alpha(z)}{dz}\right]_{z=1} = \lambda E(X)E(V)+\tilde{V}(\lambda)E(X) \qquad \textbf{(3.3)}$$

and

$$E[\alpha(\alpha-1)]= \left[\frac{d^2\alpha(z)}{dz^2}\right]_{z=1} = [\lambda E(X)]^2 E(V^2)+E[X(X-1)][\lambda E(V)+\tilde{V}(\lambda)]$$

Thus the variance of the busy period initiation queue size distribution is given by

$$Var(\alpha)=E[\alpha(\alpha-1)]+E(\alpha)-[E(\alpha)]^2$$

$$Var(\alpha)= [\lambda E(X)]^2 E(V^2)+[\lambda E(V)+\tilde{V}(\lambda)]\left\{E(X^2)-E^2(X)[\lambda E(V)+\tilde{V}(\lambda)]\right\}$$

The expression (3.3) represents the expected number of arrivals during an idle period.

4 System size distribution at departure epoch

In this section, the probability generating function of the departure point system size distribution is derived. π_j (j = 0,1,2,3,…..) is defined as the steady-state probability that 'j' customers are left in the system at a departure epoch of customer. The probability π_j is obtained by embedded Markov chain technique.

Here $\{\pi_j;$ j = 0,1,2,3,…….$\}$ forms a Markov chain with transition probability matrix,

$$P = \begin{pmatrix} \alpha_1 r_0 & (\alpha_1 r_1+\alpha_2 r_0) & (\alpha_1 r_2+\alpha_2 r_1+\alpha_3 r_0) & \dots & \dots \\ r_0 & r_1 & r_2 & \dots & \dots \\ 0 & r_0 & r_1 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

Also π_j, j = 0,1,2,3,……. satisfies the following steady-state equation:

$$\pi_j = \pi_0 \sum_{k=1}^{j+1} \alpha_k r_{j-k+1} + \sum_{k=1}^{j+1} \pi_k r_{j-k+1} \;;\; j \geq 0 \quad \text{with} \sum_{j=0}^{\infty} \pi_j = 1 \qquad \textbf{(4.1)}$$

where r_i is the probability of 'i ' customers arrive during the period starting with the initiation of a service of a customer and ending with the completion of its service.

Conditioning on the actual service length of this customer and the number of breakdowns during this service, we get

$$r_i = \int_0^{\infty} \int_0^{\infty} \sum_{j=0}^{\infty} \left(\sum_{m=0}^{i} \left[\sum_{k=0}^{m} \frac{e^{-\lambda t}(\lambda t)^k}{k!} \frac{e^{-\lambda_0 y}(\lambda_0 y)^{m-k}}{(m-k)!} \right] g_i^{(m)} \right) \frac{e^{-\gamma t}(\gamma t)^j}{j!} d\,G^{(j)}(y)\, d\,S(t)$$

(4.2)

The PGF of $\{r_i, i = 0,1,2,3,.......\}$ is obtained as

$$r(z) = \sum_{i=0}^{\infty} z^i r_i$$

$$= \tilde{S}\left[\gamma + \lambda(1 - X(z)) - \gamma \; \tilde{G}(\lambda_0(1 - X(z))) \right] \qquad \textbf{(4.3)}$$

The expected number of arrivals during the service period of a customer is given by

$$E(r) = \sum_{i=1}^{\infty} i\, r_i$$

Using (4.3), we have,

$$E(r) = r^1(1)$$

$$= E(X)\,E(S)\left[\lambda + \gamma\lambda_0 E(G)\right] \qquad \textbf{(4.4)}$$

where E(.) is the expectation operator.

Let $\prod(z)$ be the PGF of $\{\pi_j, \text{j} > 0\}$, then

$$\prod(\text{z}) = \sum_{\text{j}=0}^{\infty} \pi_{\text{j}} \text{z}^{\text{j}}$$

$$= \frac{\pi_0\, \text{r(z)}\left[1 - \alpha(\text{z})\right]}{\text{r(z)} - \text{z}} \qquad \textbf{(4.5)}$$

Now using (4.3) and (3.2) in (4.5) and solving this equation, we get finally

$$\prod(z) = \frac{\pi_0 \left[1 - \tilde{V}(\lambda - \lambda X(z)) - \tilde{V}(\lambda)[X(z) - 1] \right] \tilde{S}\left[\gamma + \lambda(1 - X(z)) - \gamma \; \tilde{G}(\lambda_0(1 - X(z))) \right]}{\tilde{S}\left[\gamma + \lambda(1 - X(z)) - \gamma \; \tilde{G}(\lambda_0(1 - X(z))) \right] - z}$$

(4.6)

Now, since $\sum_{j=0}^{\infty}\pi_j = 1$, by taking the limit of $\prod(z)$ as $z \to 1$ is unity, we get

$$\pi_0 = \frac{1-E(X)\,E(S)\left[\lambda+\gamma\lambda_0 E(G)\right]}{\lambda E(X)\,E(V)+\tilde{V}(\lambda)\,E(X)}$$

$$\pi_0 = \frac{1-\rho}{\lambda E(X)\,E(V)+\tilde{V}(\lambda)\,E(X)}, \text{ where } \rho = E(X)\,E(S)\left[\lambda+\gamma\lambda_0 E(G)\right]$$

and $\rho < 1$ is the condition to be satisfied for the existence of steady state for the model under consideration.

Substituting π_0 in (4.6), the PGF of departure point system size is obtained as

$$\prod(z) = \frac{(1-\rho)\left[1-\tilde{V}(\lambda-\lambda X(z)) - \tilde{V}(\lambda)[X(z)-1]\right]\tilde{S}\left[\gamma+\lambda(1-X(z)) - \gamma\,\tilde{G}(\lambda_0(1-X(z)))\right]}{\left\{\tilde{S}\left[\gamma+\lambda(1-X(z)) - \gamma\,\tilde{G}(\lambda_0(1-X(z)))\right]-z\right\}\left[\lambda E(X)\,E(V)+\tilde{V}(\lambda)\,E(X)\right]}$$

(4.7)

5 Expected System Size at a Departure epoch

The expected number of customers in the system at a departure epoch is obtained as

$$L = \sum_{n=0}^{\infty} n\pi_n = \prod^{1}(1)$$

Using L'Hospital's rule and evaluating $\underset{z\to 1}{Lim}\ \frac{d\prod(z)}{dz}$, we get

$$L = \frac{1}{2}\left[\frac{E(\alpha^2)}{E(\alpha)}-1\right] + \rho$$
$$+\left[\frac{E(S)\,X''(1)(\lambda+\gamma\lambda_0 E(G))+\gamma\lambda_0^{2}E(G^2)E^2(X)E(S)+E(S^2)E^2(X)\{\lambda+\gamma\lambda_0 E(G)\}^2}{2(1-\rho)}\right]$$

(5.1)

where $E(\alpha^2) = [\lambda\, E(X)]^2\, E(V^2) + E(X^2)\lambda\, E(V) + \tilde{V}(\lambda)\,E(X^2)$

6 The Steady State Probability Distribution at an Arbitrary time

Let p_n be the probability distribution of 'n' customers in the system at an arbitrary time epoch. Let q_n *and* q_n^0 be the steady state probabilities that there are 'n' customers in the system at an arbitrary epoch when the server is up and down, respectively. Then $p_n = q_n + q_n^0$ and P(z) = q(z) + $q^0(z)$, $|z| \le 1$, where P(z), q(z) and $q^0(z)$ are the probability generating functions of p_n, q_n *and* q_n^0, respectively. Instead of relating p_n *to* π_n, we relate q_n *and* q_n^0 to π_n. This is achieved by relating the rates of up and down crossings of the process $\{M(t), t \ge 0\}$, where *M(t)* is the number customers in the system at time t. Let $\hat{\lambda}$ be the effective arrival rate in the steady state.

The rate of down crossings to level 'n' is given by (Shanthikumar and Chandra[9]).

$$r_d(n) = \hat{\lambda} \sum_{i=0}^{n} g_{i+1} \pi_{n-i} \quad , \quad n \ge 0 \tag{6.1}$$

The rates of up crossings over level n are

$$r_u(0) = \lambda g_1 q_0 , \qquad n = 0$$

$$r_u(n) = \lambda \sum_{i=0}^{n} g_{i+1} q_{n-i} + \sum_{i=1}^{n} \gamma q_i \left(1 - \sum_{j=0}^{n-i} b_j \right) , \quad n \ge 1 \tag{6.2}$$

where $b_i = \int_0^\infty \sum_{m=0}^{i} \left(e^{-\lambda_0 t}\right) \frac{(\lambda_0 t)^m}{m!} g_i^{(m)} \, dG(t) , i \ge 0$ is the probability that 'i' customers arrive during a down time of the server.

Taking the PGF of (6.1) and (6.2) and equating them, we get

$$\frac{\hat{\lambda} X(z) \prod(z)}{z} = \left(\frac{\lambda X(z) q(z)}{z} + \gamma q(z) \left[\frac{1-b(z)}{1-z} \right] \right) - \gamma q_0 \left[\frac{1-b(z)}{1-z} \right] \tag{6.3}$$

where $b(z) = \tilde{G}(\lambda_0 (1 - X(z)))$ is the PGF of $\{b_i\}$. Solving (6.3) for q(z), we get

$$q(z) = \frac{\hat{\lambda} X(z)(1-z) \prod(z) + q_0 z \gamma [1 - b(z)]}{\lambda (1-z) X(z) + z \gamma [1 - b(z)]} \quad , \ |z| \le 1 \tag{6.4}$$

substituting n = 0 in (6.1) and (6.2), and equating them, we get $q_0 = \frac{\hat{\lambda} \pi_0}{\lambda}$ **(6.5)**

Alternate equation for rate of up crossings over level n is

$$r_u(n) = \lambda \sum_{i=0}^{n} g_{i+1}\, q_{n-i} + \lambda_0 \sum_{i=0}^{n} g_{i+1}\, q_{n-i}^{0} \qquad \textbf{(6.6)}$$

Taking the PGF of (6.1) and (6.6) and equating them, we get

$$\lambda\, q(z) + \lambda_0\, q^0(z) = \hat{\lambda} \prod(z) \qquad \textbf{(6.7)}$$

Therefore $q^0(z) = \left(\dfrac{\hat{\lambda} \prod(z) - \lambda\, q(z)}{\lambda_0} \right)$ **(6.8)**

Using equations (6.4) and (6.8), we get

$$q^0(z) = \frac{\hat{\lambda}\, \gamma\, z[1 - b(z)]\prod(z) - \lambda q_0 z \gamma [1 - b(z)]}{\lambda_0 [\lambda(1 - z)\, X(z) + z\gamma(1 - b(z))]}\ , \ |z| \le 1 \qquad \textbf{(6.9)}$$

The PGF of $\{p_n, n = 0,1,2,3,.......\}$ is defined as $P(z) = \sum_{n=0}^{\infty} z^n p_n$

Using equations (6.4) and (6.9) in P(z) = q(z) + $q^0(z)$, after some algebra we get

$$\begin{aligned} P(z) = {} & \frac{\hat{\lambda}}{\lambda_0} \frac{[\lambda_0 X(z)(1 - z) + z\gamma(1 - b(z))]\prod(z)}{[\lambda(1 - z)\, X(z) + z\gamma(1 - b(z))]} \\ & + \frac{(\lambda_0 - \lambda) q_0 z \gamma [1 - b(z)]}{\lambda_0 [\lambda(1 - z)\, X(z) + z\gamma(1 - b(z))]}, \qquad |z| \le 1 \end{aligned} \qquad \textbf{(6.10)}$$

Using $\lim_{z \to 1} P(z) = 1$ and solving for $\hat{\lambda}$, we get

$$\hat{\lambda} = \frac{\lambda[\lambda + \gamma\lambda_0 E(G)E(X)]}{\lambda[1 + \gamma E(G)E(X)] - (\lambda - \lambda_0)\pi_0 \gamma E(G)E(X)} \qquad \textbf{(6.11)}$$

Thus, (6.10) and (6.11) give the probability generating function P(z) of the number of customers at an arbitrary time.

7 Expected System Size at an Arbitrary Time

The expected number of customers in the system at an arbitrary time epoch is given by the following equation:

$$E(N) = \sum_{n=0}^{\infty} n\, P(N = n) = P^1(1)$$

Evaluating $\lim_{z \to 1} \dfrac{dP(z)}{dz}$ using L'Hospital's rule, we get

$$E(N) = \frac{\hat{\lambda}\left[1+\gamma E(G)E(X)\right]}{\left[\lambda+\gamma\lambda_0 E(G)E(X)\right]} L$$
$$+ \frac{\hat{\lambda}(\lambda-\lambda_0)\gamma\left\{2E(X)E(G)+E(G)X''(1)+E^2(X)\left[\lambda_0 E(G^2)-2E(G)\right]\right\}}{2\left[\lambda+\gamma\lambda_0 E(G)E(X)\right]^2}$$
$$+ \frac{(\lambda_0-\lambda)\gamma q_0 E(G)E(X)}{[\lambda+\gamma\lambda_0 E(G)E(X)]} \tag{7.1}$$
$$+ \frac{(\lambda_0-\lambda)\gamma q_0\left\{\left[\lambda+\gamma\lambda_0 E(G)E(X)\right]V_1 - E(G)E(X)V_2\right\}}{2\left[\lambda+\gamma\lambda_0 E(G)E(X)\right]^2}$$

where L is given by the equation (5.1), $V_1 = \lambda_0 E^2(X)E(G^2)+E(G)X''(1)$ and
$V_2 = 2\left[\lambda+\gamma\lambda_0 E(G)\right]E(X)+\gamma\lambda_0^2 E^2(X)E(G^2)+\gamma\lambda_0 E(G)X''(1)$

8 Expected length of busy period

Let B be the random variable "busy period". We define a random variable U and Sr as follows,

$$U = \begin{cases} 0, & \text{if the server finds no customer in the queue at a service completion epoch} \\ 1, & \text{if the server finds at least one customer in the queue at a service completion epoch} \end{cases}$$

and S_r is the time taken to serve a customer including the repair times if any.

Hence $E(S_r) = E(S)\left[1+\gamma E(G)\right]$

Now,

$$\begin{aligned} E(B) &= E(B/U=0)P(U=0) + E(B/U=1)P(U=1) \\ &= E(S_r)P(U=0) + [E(S_r) + E(B)]\,P(U=1) \\ &= E(S_r)\,\pi_0 + [E(S_r) + E(B)]\,(1-\pi_0) \end{aligned} \tag{8.1}$$

solving for E(B), we get

$$E(B) = \frac{E(S)\left[1+\gamma E(G)\right]E(X)[\lambda E(V)+\tilde{V}(\lambda)]}{1-\rho} \tag{8.2}$$

9 Expected length of idle period

Let I be the random variable "Idle Period". To find the expected length of idle period E(I), a random variable J is defined as, J = 0, if the server finds at least one customer in the queue at the end of a vacation and J = 1, if the server finds no customer at the end of the vacation. Let D be the random variable "Dormant Period" and E(D) be the expected length of dormant period.

Now,

$$
\begin{aligned}
E(I) &= E(I/J=0)P(J=0) + E(I/J=1)P(J=1) \\
&= E(V)\,P(J=0)+[E(V)+E(D)]\,P(J=1) \\
&= E(V) + \frac{1}{\lambda}\tilde{V}(\lambda) \qquad \textbf{(9.1)}
\end{aligned}
$$

10 Special cases

Case 1: In particular, if $\gamma = 0$ i.e. no server breakdowns, then (4.7) becomes

$$\Pi(z) = \frac{(1-\rho)\left[1-\tilde{V}(\lambda-\lambda X(z)) - \tilde{V}(\lambda)[X(z)-1]\right]\tilde{S}[\lambda(1-X(z))]}{\left\{\tilde{S}[\lambda(1-X(z))]-z\right\}\left[\lambda E(X)E(V)+\tilde{V}(\lambda)E(X)\right]}$$

and (8.3) becomes

$$E(B) = \frac{E(S)\,E(X)[\lambda E(V)+\tilde{V}(\lambda)]}{1-\rho}, \text{ where } \rho=\lambda E(S)E(X)$$

which agree with the result obtained by Choudhury [2].

Case 2: If Pr[V = 0] = 1 and $\gamma=0$, our model can be reduced to the ordinary $M^{[x]}/G/1$ queueing system.

Therefore (5.1) becomes $L=\frac{1}{2}\left[\frac{E(X^2)}{E(X)}-1\right]+\rho+\left[\frac{E(S)X''(1)\lambda+E(S^2)E^2(X)\lambda^2}{2(1-\rho)}\right]$,

where $\rho=\lambda E(S)E(X)$. In this case, the result coincides with those of Takagi's system [10].

11 Cost Model

In this section, we find the average cost with the following assumptions:

C_s = Start up cost C_h = Holding cost per customer

C_o = Operating cost per unit time C_r = Reward cost per unit time due to vacation

Since the length of the cycle is the sum of the idle and busy periods from the equations (8.2) and (9.1), the expected length of cycle, $E(T_c)$ is obtained as

$$E(T_c) = E(I) + E(B)$$

Now, the **total average cost (TAC)** per unit time is obtained as

$$\text{Total Cost} = C_s \frac{1}{E(T_c)} + C_h\ E(N) + C_o\ \rho - C_r\ \frac{E(I)}{E(B)}\ ,$$

The significance of the cost model will be discussed with practical example in the next section.

12 Illustrative Example

A numerical example is presented in this section to illustrate how the management of Globe valve manufacturing industry can use the above results to take decision effectively.

In the valve manufacturing industry, the arrival of valves occur in bulk from turning center to CNC turning center follows Poisson process with arrival rate λ when the server is up and the arrival rate λ_0 when the server is down. The operator takes a vacation when he finds no valves available. The operator utilizes this time for doing some other work viz., checking coolant, removing chips, etc. When the operator returns from other work and if any valves are available, he immediately starts his service. Otherwise he waits for a new arrival. During service time if the operator faces any failures i.e., unpredicted interruptions in service, immediately it is recovered with a random time.

The above system can be modeled as $M^X/G/1$ queueing system under single vacation policy and server failures with the following assumptions.

Service time distribution is exponential and batch size distribution of the arrivals is geometric with mean E(X) = 2. Repair times are also exponential.

Start up cost	= Rs. 4.00
Operating cost per unit time	= Rs. 5.00
Holding cost per customer	= Rs. 0.50
Reward cost per unit time due to vacation	= Rs. 2.00
Repair rate	α = 10
Failure rate	γ = 0.3
Arrival rate when the server is down	λ_0 = 0.6
Vacation time	t = 0.2

L, E(N), E(I) and E(B) are mean system size at departure epochs, mean system size at an arbitrary time, expected length of idle period and expected length of busy period respectively.

Numerical results are tabulated in tables 1 to 9. In tables 1- 5, for service rates 1.5, 2.0, 2.5, 3.0 & 3.5, the mean system size at departure epoch, at an arbitrary epoch, mean length of idle period and mean length of busy period are compared with varying values of the arrival rates. It is observed that, if the arrival rate increases, the system size at departure epoch, at an arbitrary epoch and length of the busy period increase whereas length of the idle period decreases. Moreover, if the service rate increases, the system size at departure epoch, at an arbitrary epoch and length of the busy period decrease.

Also from the tables 1-5, it is clear that, if we allot the server to some other secondary job, the idle time is properly utilized and hence the total average cost is minimized.

In figure 1, for various service rates, the total average cost with effective utilization of the idle time is presented for different arrival rates. In figure 2, for various service rates, the total average cost without utilization of idle time is discussed for different arrival rate. From the figures 1 & 2, it is clear that the total average cost is optimized (minimized), if we allot the server to some other secondary job.

In tables 6 – 9, for different arrival rates and varying values of service rates, the mean system size at departure epoch and at an arbitrary epoch are compared with different values of failure rates. It is observed that, the mean system size at departure epoch and at an arbitrary epoch are increased if the failure rate is increased. Also, if the arrival rate and service rate are increased, the mean system size at departure epoch and at an arbitrary epoch are increased for different failure rates.

In figure 3 and 4, for various failure rates and for different service rates, the expected number of customers in the system at departure epoch and at an arbitrary epoch are presented for different arrival rates.

Table 1: Service rate=1.5

λ	ρ	L	E(N)	E(I)	E(B)	TAC when server is assigned for secondary job	TAC when server is not assigned for secondary job
0.2	0.2907	1.7914	1.7131	5.0039	1.9376	1.4447	2.8859
0.3	0.4240	2.4391	2.3942	3.3392	2.3885	2.8495	4.0150
0.4	0.5573	3.4767	3.4517	2.5078	3.1121	4.3316	5.2232
0.5	0.6907	5.4072	5.3962	2.0097	4.4611	6.1484	6.7678
0.6	0.8240	10.2579	10.2579	1.6782	7.8570	9.3166	9.6641

Table 2: Service rate=2.0

λ	ρ	L	E(N)	E(I)	E(B)	TAC when server is assigned for secondary job	TAC when server is not assigned for secondary job
0.2	0.2180	1.5389	1.4651	5.0039	1.3182	0.8724	2.4550
0.3	0.3180	1.9129	1.8714	3.3392	1.5129	1.9737	3.3498
0.4	0.4180	2.4160	2.3928	2.5078	1.7753	3.0494	4.2200
0.5	0.5180	3.1265	3.1166	2.0097	2.1473	4.1439	5.1103
0.6	0.6180	4.2084	4.2084	1.6782	2.7150	5.3406	6.1038

Table 3: Service rate=2.5

λ	ρ	L	E(N)	E(I)	E(B)	TAC when server is assigned for secondary job	TAC when server is not assigned for secondary job
0.2	0.1744	1.4087	1.3376	5.0039	0.9989	0.5400	2.2069
0.3	0.2544	1.6690	1.6295	3.3392	1.1071	1.4844	2.9862
0.4	0.3344	1.9921	1.9707	2.5078	1.2418	2.3878	3.7243
0.5	0.4144	2.4036	2.3944	2.0097	1.4139	3.2636	4.4383
0.6	0.4944	2.9449	2.9449	1.6782	1.6410	4.1385	5.1507

Table 4: Service rate=3.0

λ	ρ	L	E(N)	E(I)	E(B)	TAC when server is assigned for secondary job	TAC when server is not assigned for secondary job
0.2	0.1453	1.3293	1.2600	5.0039	0.8041	0.3221	2.0450
0.3	0.2120	1.5283	1.4901	3.3392	0.8729	1.1691	2.7546
0.4	0.2787	1.7644	1.7438	2.5078	0.9549	1.9722	3.4211
0.5	0.3453	2.0488	2.0401	2.0097	1.0540	2.7402	4.0535
0.6	0.4120	2.3977	2.3977	1.6782	1.1759	3.4845	4.6630

Table 5: Service rate=3.5

λ	ρ	L	E(N)	E(I)	E(B)	TAC when server is assigned for secondary job	TAC when server is not assigned for secondary job
0.2	0.1246	1.2759	1.2078	5.0039	0.6729	0.1687	1.9313
0.3	0.1817	1.4367	1.3994	3.3392	0.7205	0.9484	2.5935
0.4	0.2389	1.6221	1.6022	2.5078	0.7757	1.6864	3.2144
0.5	0.2960	1.8380	1.8296	2.0097	0.8401	2.3880	3.8003
0.6	0.3531	2.0922	2.0922	1.6782	0.9162	3.0598	4.3569

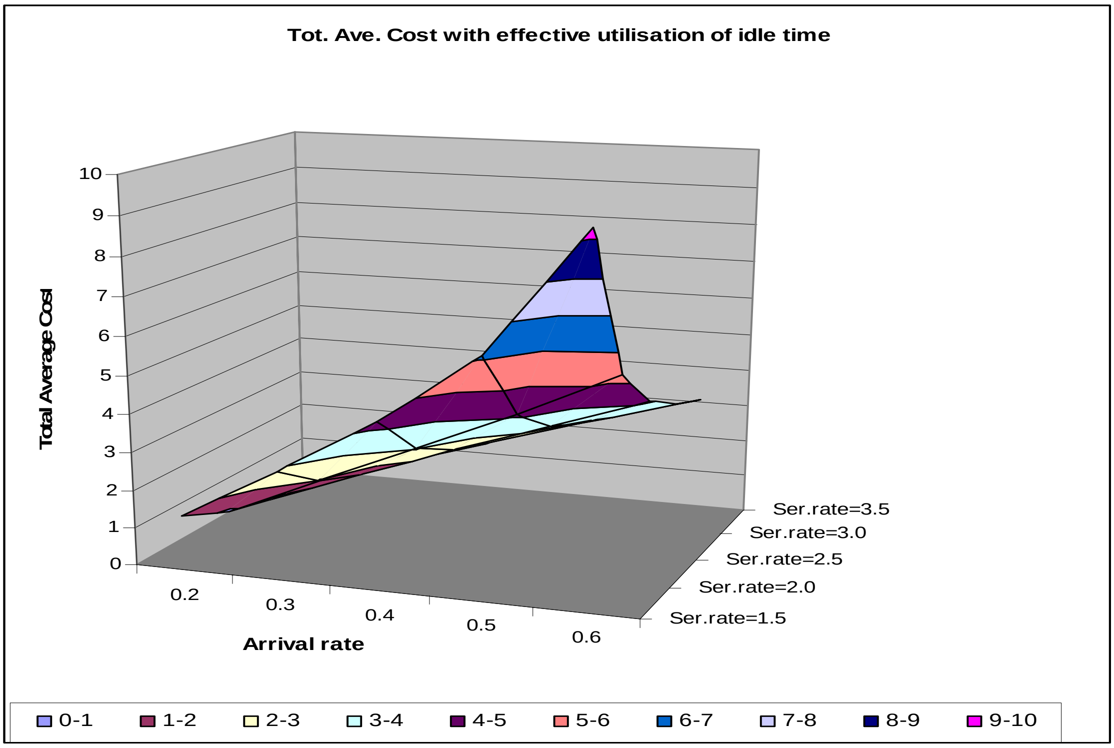


Fig : 1

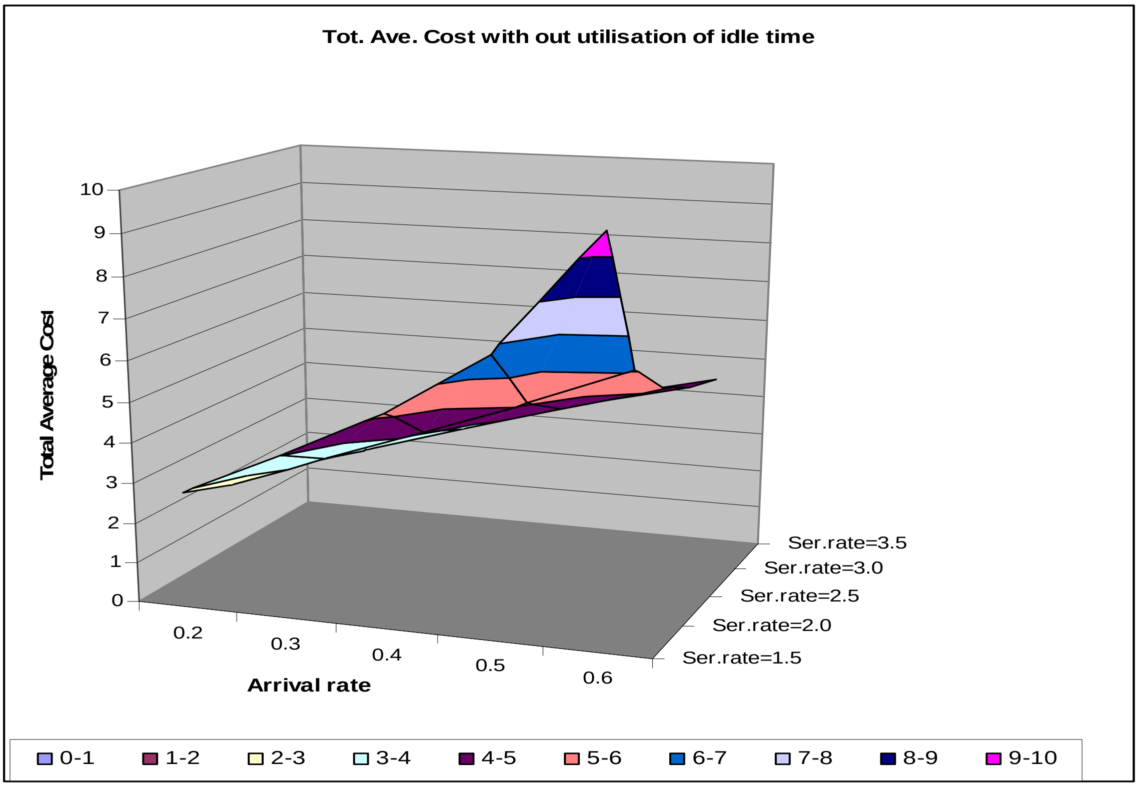


Fig :2

Table 6: Arrival rate=0.2 & Service rate=1.5

γ	ρ	L	E(N)
0.4	0.2987	1.8132	1.7122
0.5	0.3067	1.8354	1.7132
0.6	0.3147	1.8583	1.7160
0.7	0.3227	1.8816	1.7204
0.8	0.3307	1.9055	1.7263

Table 7: Arrival rate=0.3 & Service rate=2.0

γ	ρ	L	E(N)
0.4	0.3240	1.9309	1.8770
0.5	0.3300	1.9493	1.8836
0.6	0.3360	1.9679	1.8911
0.7	0.3420	1.9869	1.8994
0.8	0.3480	2.0063	1.9086

Table 8: Arrival rate=0.4 & Service rate=2.5

γ	P	L	E(N)
0.4	0.3392	2.0074	1.9795
0.5	0.3440	2.0230	1.9888
0.6	0.3488	2.0387	1.9986
0.7	0.3536	2.0547	2.0089
0.8	0.3584	2.0709	2.0197

Table 9: Arrival rate=0.5 & Service rate=3.0

γ	ρ	L	E(N)
0.4	0.3493	2.0621	2.0507
0.5	0.3533	2.0755	2.0615
0.6	0.3573	2.0891	2.0727
0.7	0.3613	2.1029	2.0841
0.8	0.3653	2.1168	2.0958

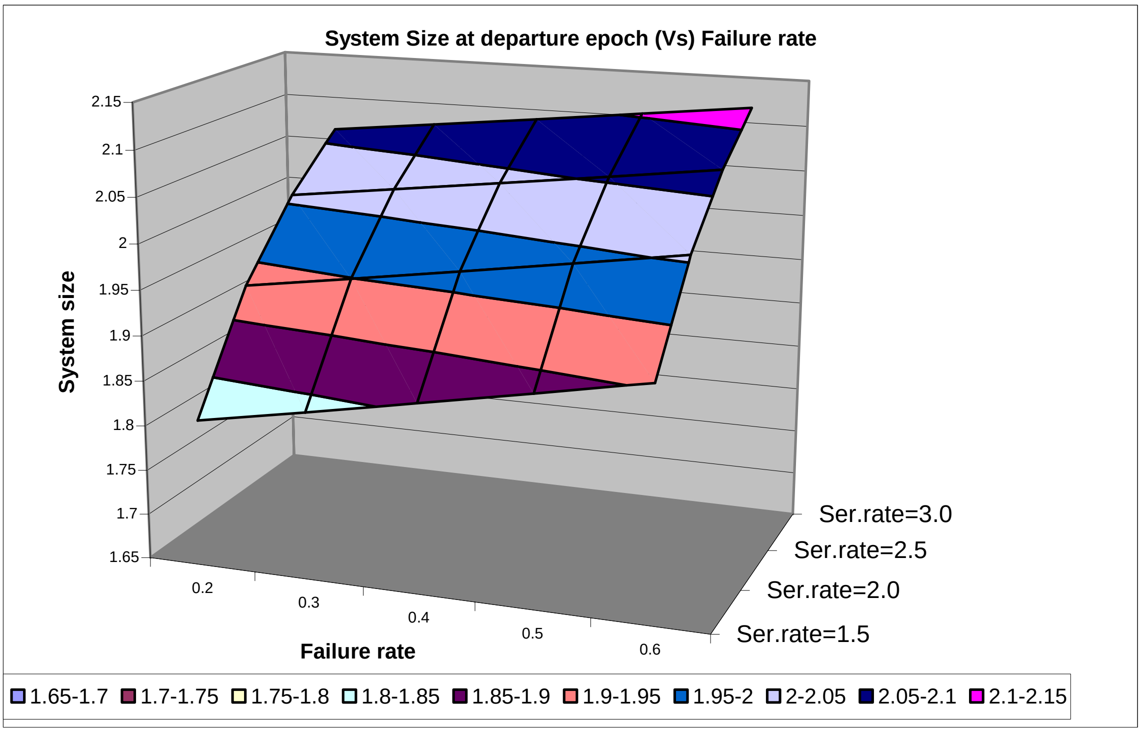


Fig : 3

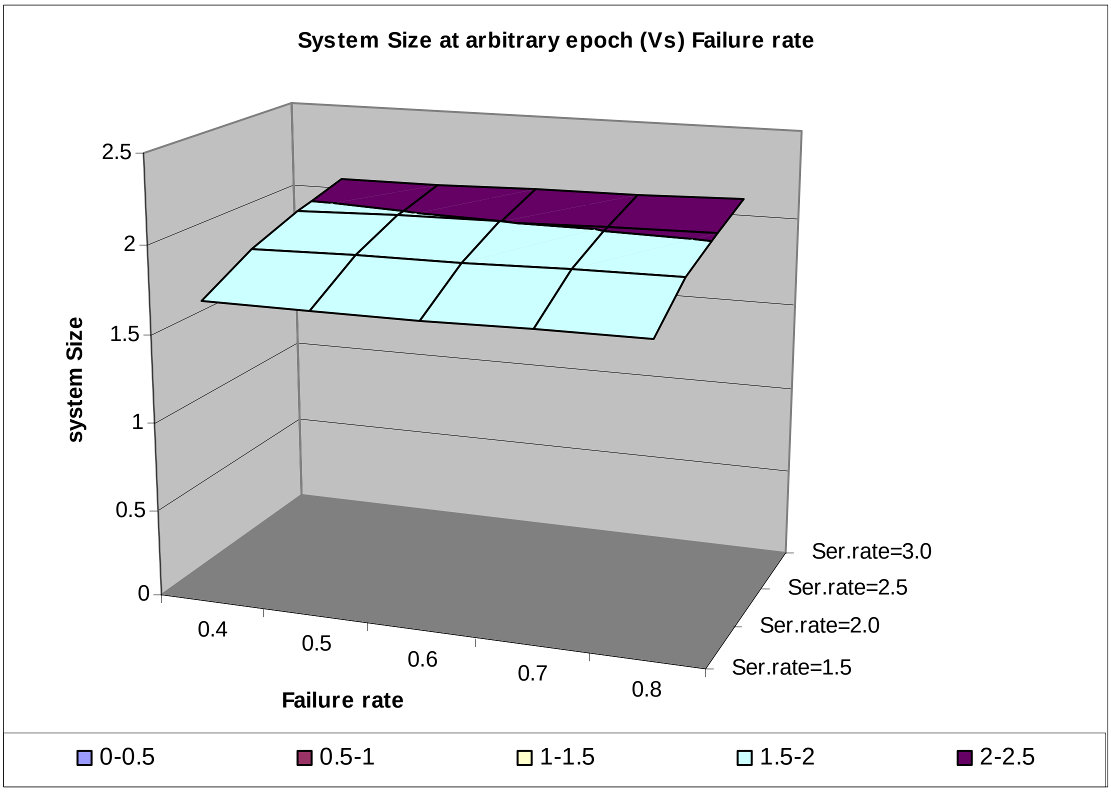


Fig : 4

13 Conclusion

In this paper, an $M^{[x]}/G/1$ queueing system with unreliable server and single vacation policy is analyzed. A cost model is discussed with numerical illustration. An example from manufacturing system is also given for the model.

14 Open Problem

This model can be extended to analyze other related models such as time dependent failures, bulk service models, multiple vacation models, optional re-service models, etc.

Acknowledgement

The authors would like to thank the referees for their valuable suggestions that led to the improvement of this paper.

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