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Subordinations For Certain Analytic Functions

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Abstract

For analytic functions $f(z)$ in the open unit disk $\mathbb{U}$, two subclasses $\mathcal{S}_\delta(\alpha)$ and $\mathcal{T}_\delta(\alpha)$ of $f(z)$ are introduced. The object of the present paper is to discuss some sufficient conditions for $f(z)$ to be in the classes $\mathcal{S}_\delta(\alpha)$ and $\mathcal{T}_\delta(\alpha)$. Furthermore, two examples for our results are considered.

Keywords: *analytic function, close-to-convex function, subordination.*

1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad and \quad |z| < 1\}.$$

For analytic functions $f(z)$ and $g(z)$ in $\mathbb{U}$, we say that $f(z)$ is subordinate to $g(z)$, written by $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$, such that $f(z) = g(w(z))$ (see [1]).
In particular, if $g(z)$ is univalent in $\mathbb{U}$, then the subordination $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.
A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{S}_\delta(\alpha)$ if it satisfies

$$\left(f'(z)\right)^\delta \prec \frac{\alpha(1-z)}{\alpha - z} \quad (z \in \mathbb{U}) \tag{1.2}$$

for some real $\alpha\,(\alpha > 1)$ and $\delta\,(\delta > 0)$. Also, a function $f(z) \in \mathcal{A}$ is said to be a member of the class $\mathcal{T}_\delta(\alpha)$ if it satisfies

$$\left(\frac{1}{f'(z)}\right)^\delta \prec \frac{\alpha(1-z)}{\alpha - z} \quad (z \in \mathbb{U}) \tag{1.3}$$

for some real $\alpha\,(\alpha > 1)$ and $\delta\,(\delta > 0)$.
To discuss our problem, we have to recall here the following lemma due to Jack [2] (also, due to Miller and Mocanu [3]).

Lemma 1 *Let $w(z)$ be non-constant and analytic in $\mathbb{U}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r$ at the point $z_0 \in \mathbb{U}$, then we have*

$$z_0 w'(z_0) = k w(z_0),$$

where k is real and $k \geqq 1$.

2 The class $\mathcal{S}_\delta(\alpha)$

Let us consider the sufficient condition for $f(z) \in \mathcal{A}$ to be in $\mathcal{S}_\delta(\alpha)$. Our first result is contained in

Theorem 1 *If $f(z) \in \mathcal{A}$ satisfies*

$$\mathrm{Re}\left(\frac{zf''(z)}{f'(z)}\right) < \frac{\alpha - 1}{2\delta(\alpha+1)} \quad (z \in \mathbb{U}) \tag{2.1}$$

for some $\alpha > 1$ and $\delta > 0$, then $f(z) \in \mathcal{S}_\delta(\alpha)$.

Proof Let us define $w(z)$ by

$$(f'(z))^\delta = \frac{\alpha(1 - w(z))}{\alpha - w(z)} \quad (w(z) \neq \alpha). \tag{2.2}$$

Then $w(z)$ is analytic in $\mathbb{U}$ with $w(0) = 0$. It follows that

$$\delta \mathrm{Re}\left(\frac{zf''(z)}{f'(z)}\right) = \mathrm{Re}\left(\frac{zw'(z)}{\alpha - w(z)} - \frac{zw'(z)}{1 - w(z)}\right) < \frac{\alpha - 1}{2(\alpha+1)} \quad (z \in \mathbb{U}) \tag{2.3}$$

for $\alpha > 1$. Now, we suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$\mathrm{Max}_{|z| \leqq |z_0|} |w(z)| = |w(z_0)| = 1.$$

Then, by Lemma 1, we can write that $w(z_0) = e^{i\theta}$ and $z_0 w'(z_0) = k w(z_0) = k e^{i\theta}$. Thus we have that

$$\delta \mathrm{Re}\left(\frac{z_0 f''(z_0)}{f'(z_0)}\right) = \mathrm{Re}\left(\frac{k e^{i\theta}}{\alpha - e^{i\theta}} - \frac{k e^{i\theta}}{1 - e^{i\theta}}\right)$$

$$= \mathrm{Re}\left(\frac{k\alpha}{\alpha - e^{i\theta}} - \frac{k}{1-e^{i\theta}}\right)$$
$$= \frac{k\alpha(\alpha - \cos\theta)}{\alpha^2 + 1 - 2\alpha\cos\theta} - \frac{k}{2}.$$

If $\alpha > 1$, then we have that

$$\delta\mathrm{Re}\left(\frac{z_0 f''(z_0)}{f'(z_0)}\right) \geqq \frac{k(\alpha-1)}{2(\alpha+1)} \geqq \frac{\alpha-1}{2(\alpha+1)}$$

which contradicts (2.1). Therefore, we say that there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = 1$. This means that $|w(z)| < 1$ for all $z \in \mathbb{U}$, that is, that

$$(f'(z))^\delta \prec \frac{\alpha(1-z)}{\alpha - z} \quad (z \in \mathbb{U}).$$

This completes the proof of the theorem.

Corollary 1 *If $f(z) \in \mathcal{A}$ satisfies the condition in Theorem 1, then*

$$\left|(f'(z))^\delta - \frac{\alpha}{\alpha+1}\right| < \frac{\alpha}{\alpha+1} \quad (z \in \mathbb{U}) \tag{2.4}$$

and

$$|\arg(f'(z))| < \frac{\pi}{2\delta} \quad (z \in \mathbb{U}). \tag{2.5}$$

Proof Since $f(z) \in \mathcal{S}_\delta(\alpha)$, there exists an analytic function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$ such that

$$(f'(z))^\delta = \frac{\alpha(1-w(z))}{\alpha - w(z)} \quad (z \in \mathbb{U}).$$

Noting that

$$|w(z)| = \left|\frac{\alpha\left((f'(z))^\delta - 1\right)}{(f'(z))^\delta - \alpha}\right| < 1,$$

we obtain (2.4) and (2.5).

Remark 1 If $f(z) \in \mathcal{A}$ satisfies the inequality (2.5) with $\delta \geqq 1$, then we know that $f(z)$ is strongly close-to-convex of order $\frac{1}{\delta}$ in $\mathbb{U}$.

Example 1 Let us consider a function $f(z) \in \mathcal{A}$ given by

$$\frac{zf''(z)}{f'(z)} = \frac{\alpha-1}{2\delta(\alpha+1)}\left(1 - \frac{1+z}{1-z}\right)$$

$$= \frac{2(1-\alpha)z}{2\delta(\alpha+1)(1-z)} \qquad (z \in \mathbb{U}).$$

Since

$$\mathrm{Re}\left(\frac{z}{1-z}\right) > -\frac{1}{2} \qquad (z \in \mathbb{U}),$$

we see that

$$\mathrm{Re}\left(\frac{zf''(z)}{f'(z)}\right) < \frac{\alpha-1}{2\delta(\alpha+1)} \qquad (z \in \mathbb{U}).$$

Thus the function $f(z)$ satisfies the condition (2.1) of Theorem 1. Noting that

$$\frac{f''(z)}{f'(z)} = \frac{1-\alpha}{\delta(\alpha+1)(1-z)},$$

we have that

$$f'(z) = (1-z)^{\frac{\alpha-1}{\delta(\alpha+1)}},$$

that is, that

$$f(z) = \frac{1}{M+1}\left\{1-(1-z)^{M+1}\right\},$$

where $M = \dfrac{\alpha-1}{\delta(\alpha+1)}$. Since

$$(f'(z))^{\delta} = (1-z)^{\frac{\alpha-1}{\alpha+1}},$$

$f(z)$ satisfies

$$(f'(z))^{\delta} \prec \frac{\alpha(1-z)}{\alpha-z} \qquad (z \in \mathbb{U}).$$

3 The class $\mathcal{T}_{\delta}(\alpha)$

Next, we derive the sufficient condition for $f(z) \in \mathcal{A}$ to be in the class $\mathcal{T}_{\delta}(\alpha)$.

Theorem 2 *If $f(z) \in \mathcal{A}$ satisfies*

$$\mathrm{Re}\left(\frac{zf''(z)}{f'(z)}\right) > \frac{1-\alpha}{2\delta(\alpha+1)} \quad (z \in \mathbb{U}) \tag{3.1}$$

for some $\alpha > 1$ and $\delta > 0$, then $f(z) \in \mathcal{T}_{\delta}(\alpha)$.

Proof Defining the function $w(z)$ by

$$\left(\frac{1}{f'(z)}\right)^{\delta} = \frac{\alpha(1-w(z))}{\alpha-w(z)} \quad (w(z) \neq \alpha),$$

we see that $w(z)$ is analytic in $\mathbb{U}$ with $w(0)=0$. Therefore, we have that

$$\delta \mathrm{Re}\left(\frac{zf''(z)}{f'(z)}\right) = \mathrm{Re}\left(\frac{zw'(z)}{1-w(z)} - \frac{zw'(z)}{\alpha - w(z)}\right) > \frac{1-\alpha}{2\delta(\alpha+1)} \qquad (z \in \mathbb{U}) \tag{3.2}$$

Let us suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$\mathrm{Max}_{|z| \leqq |z_0|}|w(z)| = |w(z_0)| = 1.$$

Then, we can write $w(z_0) = e^{i\theta}$ and $z_0 w'(z_0) = kw(z_0) = ke^{i\theta}$. It follows that

$$\delta \mathrm{Re}\left(\frac{z_0 f''(z_0)}{f'(z_0)}\right) = \mathrm{Re}\left(\frac{ke^{i\theta}}{1-e^{i\theta}} - \frac{ke^{i\theta}}{\alpha - e^{i\theta}}\right)$$

$$= \frac{k}{2} - \frac{k\alpha(\alpha - \cos\theta)}{\alpha^2 + 1 - 2\alpha\cos\theta}.$$

If $\alpha > 1$, then we see that

$$\delta \mathrm{Re}\left(\frac{z_0 f''(z_0)}{f'(z_0)}\right) \leqq \frac{k(1-\alpha)}{2(\alpha+1)} \leqq \frac{1-\alpha}{2(\alpha+1)}$$

which contradicts (3.1). This implies that there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = 1$. Thus we conclude that $f(z) \in \mathcal{T}_\delta(\alpha)$.

Corollary 2 *If $f(z) \in \mathcal{A}$ satisfies the condition in Theorem 2, then we have*

$$\left|\left(\frac{1}{f'(z)}\right)^\delta - \frac{\alpha}{\alpha+1}\right| < \frac{\alpha}{\alpha+1} \quad (z \in \mathbb{U}) \tag{3.3}$$

and

$$|\arg(f'(z))| < \frac{\pi}{2\delta} \quad (z \in \mathbb{U}). \tag{3.4}$$

Remark 2 If $f(z) \in \mathcal{A}$ satisfies the inequality (3.4) with $\delta \geqq 1$, then $f(z)$ is strongly close-to-convex of order $\frac{1}{\delta}$ in $\mathbb{U}$.

Example 2 Let $f(z) \in \mathcal{A}$ be given by

$$\frac{zf''(z)}{f'(z)} = \frac{1-\alpha}{2\delta(\alpha+1)}\left(1 - \frac{1+z}{1-z}\right)$$

$$= \frac{2(\alpha-1)z}{2\delta(\alpha+1)(1-z)} \qquad (z \in \mathbb{U}).$$

Then $f(z)$ satisfies

$$\mathrm{Re}\left(\frac{zf''(z)}{f'(z)}\right) > \frac{1-\alpha}{2\delta(\alpha+1)} \qquad (z \in \mathbb{U}),$$

which is the condition (3.1) of Theorem 2. It follows that

$$\frac{f''(z)}{f'(z)} = \frac{\alpha-1}{\delta(\alpha+1)(1-z)},$$

so that,

$$f'(z) = (1-z)^{\frac{1-\alpha}{\delta(\alpha+1)}}.$$

Therefore, we have that

$$f(z) = \frac{1}{N+1}\left\{1-(1-z)^{N+1}\right\},$$

where $N = \dfrac{1-\alpha}{\delta(\alpha+1)}$. Thus, we conclude that $f(z)$ satisfies

$$\left(\frac{1}{f'(z)}\right)^{\delta} = (1-z)^{\frac{\alpha-1}{\alpha+1}},$$

which implies that

$$\left(\frac{1}{f'(z)}\right)^{\delta} \prec \frac{\alpha(1-z)}{\alpha-z} \qquad (z \in \mathbb{U}).$$

In view of Corollary 1 and Corollary 2, we consider the following open problem.

Open Problem Corollary 1 gives us that $f(z)$ satisfies

$$\mathrm{Re}\,(f'(z))^{\delta} > 0 \qquad (z \in \mathbb{U}).$$

Please find some conditions such that Theorem 1 to be

$$\mathrm{Re}\,(f'(z))^{\delta} > \beta \qquad (z \in \mathbb{U}),$$

where $0 \leqq \beta < 1$. Also, by means of Corollary 2, please find some conditions such that Theorem 2 to be

$$\mathrm{Re}\left(\frac{1}{f'(z)}\right)^{\delta} > \beta \qquad (z \in \mathbb{U}),$$

where $0 \leqq \beta < 1$.

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