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On L_1-Convergence of Certain Cosine Sums With Third Semi-Convex Coefficients

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Abstract

In this paper a criterion for L_1- convergence of a certain cosine sums with third semi-convex coefficients is obtained. Also a necessary and sufficient condition for L_1-convergence of the cosine series is deduced as a corollary.

Keywords: *cosine sums, L_1-convergence, third semi-convex null sequences.*

1 Introduction

It is well known that if a trigonometric series converges in L_1-metric to a function $f \in L_1$, then it is the Fourier series of the function f. Riesz [2] gave a counter example showing that in a metric space L_1 we cannot expect the converse of the above said result to hold true. This motivated the various authors to study L_1-convergence of the trigonometric series. During their investigations some authors introduced modified trigonometric sums as these sums approximate their limits better than the classical trigonometric series in the sense that they converge in L_1-metric to the sum of the trigonometric series whereas the classical series itself may not. In this contest we will introduce new modified cosine series given by relation:

$$N_n^{(3)}(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^6}\sum_{k=1}^{n}\sum_{j=k}^{n}\left(\Delta^5 a_{j-3} - \Delta^5 a_{j-2}\right)\cos kx - \frac{a_1(15 - 6\cos x + \cos 2x)}{\left(2\sin\frac{x}{2}\right)^6} + \frac{a_2(6 - \cos x)}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_3}{\left(2\sin\frac{x}{2}\right)^6} \quad (1)$$

and for this modified cosine series we will prove L_1-convergence, under conditions that coefficients (a_n) are third semi-convex. In the sequel we will briefly describe the notation and definitions which are used throughout the paper. In what follows we will denote by:

$$g(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx, \tag{2}$$

with partial sums defined by

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx, \tag{3}$$

and

$$g(x) = \lim_{n\to\infty} S_n(x). \tag{4}$$

In the sequel we will mention some results which are useful for the further work. Dirichlet's kernels are denoted by

$$D_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos kt = \frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}$$

$$\widetilde{D}_n(t) = \sum_{k=1}^{n} \cos kt$$

$$\overline{\overline{D}}_n(t) = \sum_{k=1}^{n} \sin kt = \frac{\cos\frac{t}{2} - \cos\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}$$

$$\overline{D}_n(t) = -\frac{1}{2}\cot\frac{t}{2} + \overline{\overline{D}}_n(t) = -\frac{\cos\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}$$

Definition 1.1 *A sequence of scalars (a_n) is said to be semi-convex if $a_n \to 0$ as $n \to \infty$, and*

$$\sum_{n=1}^{\infty} n|\Delta^2 a_{n-1} + \Delta^2 a_n| < \infty, (a_0 = 0), \tag{5}$$

where $\Delta^2 a_n = \Delta a_n - \Delta a_{n+1}$.

Definition 1.2 *A sequence of scalars (a_n) is said to be quasi-convex if $a_n \to 0$ as $n \to \infty$, and*

$$\sum_{n=1}^{\infty} n|\Delta^2 a_{n-1}| < \infty, (a_0 = 0). \tag{6}$$

Definition 1.3 *A sequence of scalars* (a_n) *is said to be twice semi-convex if* $a_n \to 0$ *as* $n \to \infty$, *and*

$$\sum_{n=1}^{\infty} n^2 |\Delta^4 a_{n-1} - \Delta^4 a_n| < \infty, (a_0 = a_{-1} = 0), \tag{7}$$

where $\Delta^4 a_n = \Delta^3 a_n - \Delta^3 a_{n+1}$.

Definition 1.4 *A sequence of scalars* (a_n) *is said to be twice quasi-convex if* $a_n \to 0$ *as* $n \to \infty$, *and*

$$\sum_{n=1}^{\infty} n^2 |\Delta^4 a_{n-1}| < \infty, (a_0 = a_{-1} = 0). \tag{8}$$

Definition 1.5 *A sequence of scalars* (a_n) *is said to be third semi-convex if* $a_n \to 0$ *as* $n \to \infty$, *and*

$$\sum_{n=1}^{\infty} n^3 |\Delta^5 a_{n-3} - \Delta^5 a_{n-2}| < \infty, (a_0 = a_{-1} = a_{-2} = 0), \tag{9}$$

where $\Delta^5 a_n = \Delta^4 a_n - \Delta^4 a_{n+1}$.

Definition 1.6 *A sequence of scalars* (a_n) *is said to be third quasi-convex if* $a_n \to 0$ *as* $n \to \infty$, *and*

$$\sum_{n=1}^{\infty} n^3 |\Delta^5 a_{n-3}| < \infty, (a_0 = a_{-1} = a_{-2} = 0). \tag{10}$$

Remark 1.7 *If* (a_n) *is a third quasi-convex null scalar sequence, then it is third semi-convex scalars sequence too.*

The L_1-convergence of cosine and sine sums was studied by several authors. Kolmogorov in [8], proved the following theorem:

Theorem 1.8 *If* (a_n) *is a quasi-convex null sequence, then for the* L_1-*convergence of the cosine series (1), it is necessary and sufficient that:*
$\lim_{n\to\infty} a_n \cdot \log n = 0$.

The case in which sequence (a_n) is convex, of this theorem was established by Young (see [13]).

Bala and Ram in [1] have proved that Theorem 1.8 holds true for cosine series with semi-convex null sequences in the following form:

Theorem 1.9 *If (a_n) is a semi-convex null sequence, then for the convergence of the cosine series (1) in the metric space L, it is necessary and sufficient that $a_{k-1}\log k = 0(1), k \to \infty$.*

Garret and Stanojevic in [5], have introduced modified cosine sums

$$G_n(x) = \frac{1}{2}\sum_{k=0}^{n}\Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n}(\Delta a_j)\cos kx. \tag{11}$$

The same authors (see [6]), Ram in [11] and Singh and Sharma in [12] studied the L_1-convergence of this cosine sum under different sets of conditions on the coefficients (a_n). Kumari and Ram in [10], introduced new modified cosine and sine sums as

$$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n}\Delta\left(\frac{a_j}{j}\right)\cos kx \tag{12}$$

and

$$G_n'(x) = \sum_{k=1}^{n}\sum_{j=k}^{n}\Delta\left(\frac{a_j}{j}\right)\sin kx, \tag{13}$$

and have studied their L_1-convergence under the condition that the coefficients (a_n) belong to different classes of sequences. Later one, Kulwinder in [9], introduced new modified sine sums as

$$K_n(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}\sum_{j=k}^{n}(\Delta a_{j-1} - \Delta a_{j+1})\sin kx, \tag{14}$$

and have studied their L_1-convergence under the condition that the coefficients (a_n) are semi-convex null. In [3], was introduced the following modified cosine sums:

$$N_n(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^2}\sum_{k=1}^{n}\sum_{j=k}^{n}(\Delta^2 a_{j-1} - \Delta^2 a_j)\cos kx + \frac{a_1}{\left(2\sin\frac{x}{2}\right)^2}.$$

For this cosine sums was studied L_1-convergence under the condition that the coefficients (a_n) are semi-convex null. In [4], was introduced the following modified cosine sums:

$$N_n^{(2)}(x) = \frac{1}{\left(2\sin\frac{x}{2}\right)^4}\sum_{k=1}^{n}\sum_{j=k}^{n}(\Delta^4 a_{j-2} - \Delta^4 a_{j-1})\cos kx + \frac{a_1(\cos x - 4)}{\left(2\sin\frac{x}{2}\right)^4} + \frac{a_2}{\left(2\sin\frac{x}{2}\right)^4}.$$

For this cosine sums was studied L_1-convergence under the condition that the coefficients (a_n) are twice semi-convex null.

2 Results

The aim of this paper is to study the L_1-convergence of this modified cosine sums with third semi-convex coefficients and to give necessary and sufficient condition for L_1-convergence of the cosine series defined by relation (1).

Theorem 2.1 *Let (a_n) be a third semi-convex null sequence, then $N_n^{(3)}(x)$ converges to $g(x)$ in L_1 norm.*

Proof. We have

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cdot \cos kx = \frac{1}{\left(2\sin\frac{x}{2}\right)^6} \cdot \sum_{k=1}^{n} a_k \cdot \cos kx \cdot \left(2\sin\frac{x}{2}\right)^6$$

$$= \frac{1}{\left(2\sin\frac{x}{2}\right)^6} \cdot \sum_{k=1}^{n} a_k[-\cos(k+3)x + 6\cos(k+2)x - 15\cos(k+1)x +$$

$$20\cos kx - 15\cos(k-1)x + 6\cos(k-2)x - \cos(k-3)x]$$

$$= -\frac{1}{\left(2\sin\frac{x}{2}\right)^6} \cdot \sum_{k=1}^{n} a_k[\cos(k+3)x - 6\cos(k+2)x + 15\cos(k+1)x - 20\cos kx +$$

$$15\cos(k-1)x - 6\cos(k-2)x + \cos(k-3)x] = -\frac{1}{\left(2\sin\frac{x}{2}\right)^6} \cdot$$

$$\sum_{k=1}^{n}(a_{k-3} - 6a_{k-2} + 15a_{k-1} - 20a_k + 15a_{k+1} - 6a_{k+2} + a_{k+3})\cos kx + \frac{a_{-2}\cos x}{\left(2\sin\frac{x}{2}\right)^6} +$$

$$\frac{a_{-1}\cos 2x}{\left(2\sin\frac{x}{2}\right)^6} + \frac{a_0}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_{n-2}\cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_{n-1}\cos(n+2)x}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_n\cos(n+3)x}{\left(2\sin\frac{x}{2}\right)^6} -$$

$$\frac{6a_{-1}\cos x}{\left(2\sin\frac{x}{2}\right)^6} - \frac{6a_0\cos 2x}{\left(2\sin\frac{x}{2}\right)^6} + \frac{6a_{n-1}\cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^6} + \frac{6a_n\cos(n+2)x}{\left(2\sin\frac{x}{2}\right)^6} + \frac{15a_0}{\left(2\sin\frac{x}{2}\right)^6} -$$

$$\frac{15a_n\cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^6} - \frac{15a_1}{\left(2\sin\frac{x}{2}\right)^6} + \frac{15a_{n+1}\cos nx}{\left(2\sin\frac{x}{2}\right)^6} + \frac{6a_1\cos x}{\left(2\sin\frac{x}{2}\right)^6} + \frac{6a_2}{\left(2\sin\frac{x}{2}\right)^6} -$$

$$\frac{6a_{n+1}\cos(n-1)x}{\left(2\sin\frac{x}{2}\right)^6} - \frac{6a_{n+2}\cos nx}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_1\cos 2x}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_2\cos x}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_3}{\left(2\sin\frac{x}{2}\right)^6} +$$

$$\frac{a_{n+1}\cos(n-2)x}{\left(2\sin\frac{x}{2}\right)^6} + \frac{a_{n+2}\cos(n-1)x}{\left(2\sin\frac{x}{2}\right)^6} + \frac{a_{n+3}\cos nx}{\left(2\sin\frac{x}{2}\right)^6}$$

Applying Abel's transformation, we have

$$S_n(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^6}\cdot\sum_{k=1}^{n-1}(\Delta^5 a_{k-3} - \Delta^5 a_{k-2})\widetilde{D}_k(x) + \frac{(\Delta^5 a_{n-3} - \Delta^5 a_{n-2})\cdot\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^4} -$$

$$\frac{a_{n-2}\cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_{n-1}\cos(n+2)x}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_n\cos(n+3)x}{\left(2\sin\frac{x}{2}\right)^6} + \frac{6a_{n-1}\cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^6} +$$

$$\frac{6a_n\cos(n+2)x}{\left(2\sin\frac{x}{2}\right)^6} - \frac{15a_n\cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^6} - \frac{15a_1}{\left(2\sin\frac{x}{2}\right)^6} + \frac{15a_{n+1}\cos nx}{\left(2\sin\frac{x}{2}\right)^6} + \frac{6a_1\cos x}{\left(2\sin\frac{x}{2}\right)^6} +$$

$$\frac{6a_2}{\left(2\sin\frac{x}{2}\right)^6} - \frac{6a_{n+1}\cos(n-1)x}{\left(2\sin\frac{x}{2}\right)^6} - \frac{6a_{n+2}\cos nx}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_1\cos 2x}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_2\cos x}{\left(2\sin\frac{x}{2}\right)^6}$$

$$-\frac{a_3}{\left(2\sin\frac{x}{2}\right)^6} + \frac{a_{n+1}\cos(n-2)x}{\left(2\sin\frac{x}{2}\right)^6} + \frac{a_{n+2}\cos(n-1)x}{\left(2\sin\frac{x}{2}\right)^6} + \frac{a_{n+3}\cos nx}{\left(2\sin\frac{x}{2}\right)^6}$$

Since $\widetilde{D}_n(x)$ is uniformly bounded on every segment $[\epsilon, \pi-\epsilon]$, for every $\epsilon > 0$,

$$g(x) = \lim_{n\to\infty} S_n(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^6}\cdot\sum_{k=1}^{\infty}(\Delta^5 a_{k-3} - \Delta^5 a_{k-2})\widetilde{D}_k(x) -$$

$$\frac{a_1(15 - 6\cos x + \cos 2x)}{\left(2\sin\frac{x}{2}\right)^6} + \frac{a_2(6-\cos x)}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_3}{\left(2\sin\frac{x}{2}\right)^6}$$

Also

$$N_n^{(3)}(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^6}\sum_{k=1}^{n}\Delta^5 a_{k-3}\widetilde{D}_k(x) + \frac{\Delta^5 a_{n-2}\cdot\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^6} -$$

$$\frac{a_1(15 - 6\cos x + \cos 2x)}{\left(2\sin\frac{x}{2}\right)^6} + \frac{a_2(6-\cos x)}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_3}{\left(2\sin\frac{x}{2}\right)^6}$$

Now applying Abel's transformation in the last relation we get the following:

$$N_n^{(3)}(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^6}\sum_{k=1}^{n-1}(\Delta^5 a_{k-3} - \Delta^5 a_{k-2})\widetilde{D}_k(x) + \frac{\Delta^5 a_{n-3}\cdot\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^6} +$$

$$\frac{\Delta^5 a_{n-2}\cdot\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_1(15 - 6\cos x + \cos 2x)}{\left(2\sin\frac{x}{2}\right)^6} + \frac{a_2(6-\cos x)}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_3}{\left(2\sin\frac{x}{2}\right)^6}$$

From above relations we will have:

$$g(x) - N_n^{(3)}(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^6}\sum_{k=n+1}^{\infty}(\Delta^5 a_{k-3} - \Delta^5 a_{k-2})\widetilde{D}_k(x) -$$

$$\frac{\Delta^5 a_{n-3}\cdot \widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^6} - \frac{\Delta^5 a_{n-2}\cdot \widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^6} \Rightarrow$$

$$g(x) - N_n^{(3)}(x) = -\lim_{m\to\infty}\left(\frac{1}{\left(2\sin\frac{x}{2}\right)^6}\sum_{k=n+1}^{m}(\Delta^5 a_{k-3} - \Delta^5 a_{k-2})\widetilde{D}_k(x)\right) -$$

$$\frac{\Delta^5 a_{n-3}\cdot \widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^6} - \frac{\Delta^5 a_{n-2}\cdot \widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^6}.$$

Thus, we have

$$\int_0^{\pi} |g(x) - N_n^{(3)}(x)|dx \to 0,$$

for $n \to \infty$.

Corollary 2.2 *Let* (a_n) *be a third quasi-convex null sequence, then* $N_n^{(2)}(x)$ *converges to* $g(x)$ *in* L_1 *norm.*

Proof. Proof of the corollary follows directly from Theorem 2.1 and Remark 1.7.

Corollary 2.3 *If* (a_n) *is a third semi-convex null sequence of scalars, then the necessary and sufficient condition for* L_1*-convergence of the cosine series (1) is* $\lim_{n\to\infty} a_n \log n = 0$.

Proof. Let us start from this estimation:

$$||S_n(x) - g(x)||_{L_1} \le ||S_n(x) - N_n^{(3)}(x)||_{L_1} + ||N_n^{(3)}(x) - g(x)||_{L_1} \le ||N_n^{(3)}(x) - g(x)||_{L_1} +$$

$$2\left\|\frac{-\Delta^5 a_{n-2}\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^6}\right\| + \left\|\frac{a_{n+1}\cos(n-2)x}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_{n-2}\cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^6}\right\| +$$

$$\left\|\frac{a_{n+2}\cos(n-1)x}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_{n-1}\cos(n+2)x}{\left(2\sin\frac{x}{2}\right)^6}\right\| + \left\|\frac{a_{n+3}\cos nx}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_n\cos(n+3)x}{\left(2\sin\frac{x}{2}\right)^6}\right\| +$$

$$6\left\|\frac{a_{n-1}\cos{(n+1)x}}{\left(2\sin\frac{x}{2}\right)^6}-\frac{a_{n+1}\cos{(n-1)x}}{\left(2\sin\frac{x}{2}\right)^6}\right\|+6\left\|\frac{a_n\cos{(n+2)x}}{\left(2\sin\frac{x}{2}\right)^6}-\frac{a_{n+2}\cos nx}{\left(2\sin\frac{x}{2}\right)^6}\right\|$$

$$+15\left\|\frac{a_{n+1}\cos nx}{\left(2\sin\frac{x}{2}\right)^6}-\frac{a_n\cos{(n+1)x}}{\left(2\sin\frac{x}{2}\right)^6}\right\|$$

On the other hand

$$\Delta^5 a_{n-2}=\sum_{k=n-2}^{\infty}(\Delta^5 a_k-\Delta^5 a_{k+1})=\sum_{k=n-2}^{\infty}\frac{k}{k}(\Delta^5 a_k-\Delta^5 a_{k+1})\leq$$

$$\frac{1}{n-2}\sum_{k=n-2}^{\infty}k(\Delta^5 a_k-\Delta^5 a_{k+1})=o\left(\frac{1}{n}\right).$$

Since

$$\int_0^{\pi}\frac{\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^6}=O(n),$$

therefore

$$\Delta^5 a_{n-2}\cdot\int_0^{\pi}\frac{\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^6}=o(1).$$

For the rest terms of the above relation we will get this estimation:

$$\int_0^{\pi}\left|\frac{a_{n+1}\cos{(n-2)x}}{\left(2\sin\frac{x}{2}\right)^6}-\frac{a_{n-2}\cos{(n+1)x}}{\left(2\sin\frac{x}{2}\right)^6}\right|dx\leq$$

$$C_1\int_0^{\pi}a_{n-2}\left|\frac{\cos{(n-2)x}}{\left(2\sin\frac{x}{2}\right)^2}-\frac{\cos{(n+1)x}}{\left(2\sin\frac{x}{2}\right)^2}\right|dx\leq$$

$$C_1\cdot C_2\int_0^{\pi}a_{n-2}\left|\widetilde{D}_n(x)-\frac{1}{2}\right|dx\sim C_1\cdot C_2(a_{n-2}\log n).$$

In similar way we can estimate this expressions:

$$\int_0^{\pi}\left|\frac{a_{n+2}\cos{(n-1)x}}{\left(2\sin\frac{x}{2}\right)^6}-\frac{a_{n-1}\cos{(n+2)x}}{\left(2\sin\frac{x}{2}\right)^6}\right|dx\sim C_3(a_{n-1}\log n),$$

$$\int_0^{\pi}\left|\frac{a_{n+3}\cos nx}{\left(2\sin\frac{x}{2}\right)^6}-\frac{a_n\cos{(n+3)x}}{\left(2\sin\frac{x}{2}\right)^6}\right|dx\sim C_4(a_n\log n),$$

$$6\int_0^{\pi}\left|\frac{a_{n-1}\cos{(n+1)x}}{\left(2\sin\frac{x}{2}\right)^6}-\frac{a_{n+1}\cos{(n-1)x}}{\left(2\sin\frac{x}{2}\right)^6}\right|dx\sim C_5(a_{n-1}\log n),$$

$$6\int_0^{\pi}\left|\frac{a_{n}\cos{(n+2)x}}{\left(2\sin\frac{x}{2}\right)^6}-\frac{a_{n+2}\cos{nx}}{\left(2\sin\frac{x}{2}\right)^6}\right|dx\sim C_6(a_{n}\log n),$$

$$15\int_0^{\pi}\left|\frac{a_{n+1}\cos{nx}}{\left(2\sin\frac{x}{2}\right)^6}-\frac{a_{n}\cos{(n+1)x}}{\left(2\sin\frac{x}{2}\right)^6}\right|dx\sim C_7(a_{n}\log n),$$

where C_i, $i=1,2,\cdots,7$ are constants. From Theorem 2.1 it follows that

$$||N_n^{(3)}(x)-g(x)||=o(1), n\to\infty.$$

Finally we get this estimation

$$\lim_{n\to\infty}\int_0^{\pi}|g(x)-S_n(x)|=o(1),$$

if and only if

$$\lim_{n\to\infty}a_n\log n=0,$$

with which was proved corollary.

Corollary 2.4 *If (a_n) is a third quasi-convex null sequence of scalars, then the necessary and sufficient condition for L_1-convergence of the cosine series (1) is* $\lim_{n\to\infty}a_n\log n=0$.

3 Open problem

It would be interesting to prove the L_1 convergence of the cosine series, in case where coefficients satisfies the higher order of convexity.

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