Int. J. Open Problems Compt. Math., Vol. 2, No. 4, December 2009
ISSN 1998-6262; Copyright ©ICSRS Publication, 2009
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Multiple Positive Solutions For Impulsive Singular Boundary Value Problems With Integral Boundary Conditions

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Abstract

By constructing an available integral operator and combining fixed point index theory with properties of Green's function, this paper shows the existence of multiple positive solutions for a class of impulsive singular boundary value problems with integral boundary conditions. Our results extend some recent work in the literature on boundary value problems of ordinary differential equations. We illustrate our results by one example, which can not be handled using the existing results.

Keywords: *impulsive differential equations; integral boundary conditions; positive solution; fixed point index theory; singularity.*

1 Introduction and Preliminaries

The theory of impulsive differential equations describes processes which experience a sudden change of their state at certain moments. Processes with such a character arise naturally and often, especially in phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. For an introduction of the basic theory of impulsive differential equations, see

Lakshmikantham et al. [l], Bainov and Simeonov [2], and Samoilenko and Perestyuk [3] and the references therein. The theory of impulsive differential equations has become an important area of investigation in recent years and is much richer than the corresponding theory of differential equations (see, for instance, [4-16] and references cited therein).

Moreover, the theory of boundary-value problems with integral boundary conditions for ordinary differential equations arises in different areas of applied mathematics and physics. For example, heat conduction, chemical engineering, underground water flow, thermo-elasticity, and plasma physics can be reduced to the nonlocal problems with integral boundary conditions. For boundary-value problems with integral boundary conditions and comments on their importance, we refer the reader to the papers by Gallardo [17], Karakostas and Tsamatos [18], Lomtatidze and Malaguti [19] and the references therein. For more information about the general theory of integral equations and their relation with boundary-value problems we refer to the books by Corduneanu [20] and Agarwal and O'Regan [21].

On the other hand, recently, boundary-value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multipoint and nonlocal boundary-value problems as special cases. The existence and multiplicity of positive solutions for such problems have received a great deal of attention in the literature. To identify a few, we refer the reader to [4-8, 10-12, 16, 18, 22-35] and references therein.

Motivated by the works mentioned above, we intend in this paper to study the existence of multiple positive solutions for a class of singular boundary value problems with integral boundary conditions of second order impulsive differential equations:

$$\begin{cases} -\lambda y''(t) = w(t)f(t, y(t)), & t \in J, \quad t \neq t_k, \\ \Delta y'|_{t=t_k} = -I_k(y(t_k)), & k = 1, 2, \ldots, m, \\ y(0) = y(1) = \int_0^1 g(t)y(t)dt. \end{cases} \tag{1.1}$$

Here $J = [0, 1]$, $\lambda > 0$, $w : (0, 1) \to [0, +\infty)$ is continuous, and may be singular at $t = 0$ and (or) $t = 1$, $f \in C(J \times R^+, R^+)$, $I_k \in C(R^+, R^+)$, $R^+ = [0, +\infty)$, $t_k(k = 1, 2, \ldots, m)$ (where m is fixed positive integer) are fixed points with $0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < 1$, $\Delta y'\big|_{t=t_k} = y'(t_k^+) - y'(t_k^-)$, where $y'(t_k^+)$ and $y'(t_k^-)$ represent the right-hand limit and left-hand limit of $y'(t)$ at $t = t_k$, respectively, and $g \in L^1[0, 1]$ is nonnegative.

For the case of $I_k \neq 0$, $k = 1, 2, \ldots, m$, $g = 0$, problem (1.1) reduces to the problem studied by Lin and Jiang in [15]. By using the fixed point index theory in cones, the authors obtained some sufficient conditions for the existence of multiple positive solutions.

For the case of $I_k = 0, k = 1, 2, \ldots, m$, $g \neq 0$, problem (1.1) reduces to the problem studied by Feng, Ji and Ge in [22]. By using the fixed point theorem of strict-set-contractions, the authors obtained some sufficient conditions for the existence of at least one or two positive solutions in Banach spaces.

For the case of $I_k = 0,\ k = 1, 2, \ldots, m,\quad g = 0$, problem (1.1) is related to two-points boundary value problem of ODE. Guo and Lakshmikantham [33] obtained some sufficient conditions for the existence of at least one or two positive solutions to the two-point boundary-value problem in Banach spaces by using the fixed point theorem of strict set contractions. Erbe and Hu [34] have applied a fixed point index theorem in cones to establish the existence of multiple positive solutions to problem (1.1). Liu and Li [35] have proved that there exist at least two positive solutions by applying a fixed point index theorem in cones.

On the other hand, as far as second order nonlocal boundary value problems are concerned, a great deal of existence and uniqueness results have been established up to now. For details, see, for example, [4-8, 10-12, 16, 22-25, 28, 33-35] and references cited therein. However, among the existing results of [4-8, 10-12, 16, 22-25, 28, 33-35] no one can be applied to our problem. This is another reason why we study problem (1.1).

It is well known that fixed point index theorems have been applied to various boundary value problems to show the existence of multiple positive solutions. An overview of such results can be found in Guo and Lakshmikantham V., [36] and in Guo and Lakshmikantham V., Liu X.Z., [37].

Lemma 1.1. [36, 37] Let E be a real Banach space and K be a cone in E. For $r > 0$, define $K_r = \{x \in K : \|x\| < r\}$. Assume that $T : \bar{K}_r \to K$ is completely continuous such that $Tx \neq x$ for $x \in \partial K_r = \{x \in K : \|x\| = r\}$.

(i) If $\|Tx\| \geq \|x\|$ for $x \in \partial K_r$, then

$$i(T, K_r, K) = 0.$$

(ii) If $\|Tx\| \leq \|x\|$ for $x \in \partial K_r$, then

$$i(T, K_r, K) = 1.$$

The paper is organized in the following fashion. In Section 2, we provide some necessary background. In particular, we state some properties of the Green's function associated with problem (1.1). In Section 3, the main results will be stated and proved. Finally, in Section 4, one example is also included to illustrate the main results.

2 Preliminaries

In order to define the solution of problem (1.1), we shall consider the following space.

Let $J^{'} = J\backslash\{t_1, t_2, \ldots, t_m\}$, and

$PC^1[0,1] = \Big\{x \in C[0,1] : x^{'}|_{(t_k,t_{k+1})} \in C(t_k, t_{k+1}),\ x^{'}(t_k^-) = x^{'}(t_k),\ \exists\, x^{'}(t_k^+),\ k =$

$1,2,\ldots,m\Big\}$. Then $PC^1[0,1]$ is a real Banach space with norm

$$\|x\|_{PC^1} = \max\Big\{\|x\|_\infty, \|x'\|_\infty\Big\},$$

where $\|x\|_\infty = \sup_{t\in J}|x(t)|$, $\|x'\|_\infty = \sup_{t\in J}|x'(t)|$.

A function $x \in PC^1[0,1]\cap C^2(J')$ is called a solution of problem (1.1) if it satisfies (1.1).

To establish the existence of multiple positive solutions in $PC^1[0,1]\cap C^2(J')$ of problem (1.1), let us list the following assumptions:

(H_1) $w : (0,1) \to [0,+\infty)$ is continuous, and may be singular at $t = 0$ and (or) $t = 1$, and $0 < \int_0^1 w(t)dt < +\infty$;

(H_2) $f \in C(J \times R^+, R^+)$, $I_k \in C(R^+, R^+)$;

(H_3) $g \in L^1[0,1]$ is nonnegative and $\mu \in [0,1)$, where

$$\mu = \int_0^1 g(t)dt. \tag{2.1}$$

In our main results, we will make use of the following lemmas.

Lemma 2.1. If $(H_1) - (H_3)$ hold, then $y \in PC^1[0,1]\cap C^2(J')$ is a solution of (1.1) if and only if y is a solution of the following impulsive integral equation:

$$y(t) = \frac{1}{\lambda}\int_0^1 H(t,s)w(s)f(s,y(s))ds + \sum_{k=1}^{m} H(t,t_k)I_k(y(t_k)), \tag{2.2}$$

where

$$H(t,s) = G(t,s) + \frac{1}{1-\mu}\int_0^1 G(s,\tau)g(\tau)d\tau, \tag{2.3}$$

$$G(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1. \end{cases} \tag{2.4}$$

Proof. First suppose that $y \in PC^1[0,1]\cap C^2(J')$ is a solution of problem (1.1). It is easy to see by integration of BVP (1.1) that

$$y'(t) = y'(0) - \tfrac{1}{\lambda}\textstyle\int_0^t w(s)f(s,y(s))ds - \sum_{t_k<t} I_k(y(t_k)).$$

Integrating again, we can get

$$y(t) = y(0) + y'(0)t - \tfrac{1}{\lambda}\textstyle\int_0^t (t-s)w(s)f(s,y(s))ds - \sum_{t_k<t} I_k(y(t_k))(t-t_k). \tag{2.5}$$

Letting $t = 1$ in (2.5), we find

$$y'(0) = \tfrac{1}{\lambda}\textstyle\int_0^1 (1-s)w(s)f(s,y(s))ds + \sum_{t_k<1} I_k(y(t_k))(1-t_k). \tag{2.6}$$

Substituting $y(0) = \int_0^1 g(t)y(t)dt$ and (2.6) into (2.5), we obtain

$$\begin{aligned} y(t) &= y(0) + \tfrac{1}{\lambda}\textstyle\int_0^1 t(1-s)w(s)f(s,y(s))ds + t\sum_{t_k<1} I_k(y(t_k))(1-t_k) \\ &\quad -\tfrac{1}{\lambda}\textstyle\int_0^t (t-s)w(s)f(s,y(s))ds - \sum_{t_k<t} I_k(y(t_k))(t-t_k) \\ &= \tfrac{1}{\lambda}\textstyle\int_0^1 G(t,s)w(s)f(s,y(s))ds + \int_0^1 g(t)y(t)dt + \sum_{k=1}^m G(t,t_k)I_k(y(t_k))), \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} \textstyle\int_0^1 g(t)y(t)dt &= \textstyle\int_0^1 g(t)\Bigg[\int_0^1 g(t)y(t)dt + \frac{1}{\lambda}\int_0^1 G(t,s)w(s)f(s,y(s))ds \\ &\quad + \textstyle\sum_{k=1}^m G(t,t_k)I_k(y(t_k))\Bigg]dt \\ &= \textstyle\int_0^1 g(t)dt \times \int_0^1 g(t)y(t)dt + \frac{1}{\lambda}\int_0^1\int_0^1 G(t,s)g(s)w(s)f(s,y(s))dsdt \\ &\quad + \textstyle\int_0^1 g(t)\Bigg(\sum_{k=1}^m G(t,t_k)I_k(y(t_k))\Bigg)dt. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \textstyle\int_0^1 g(s)y(s)ds &= \frac{1}{1-\int_0^1 g(s)ds}\Bigg[\frac{1}{\lambda}\int_0^1\Bigg(\int_0^1 G(s,r)g(r)dr\Bigg)w(s)f(s,y(s))ds \\ &\quad + \textstyle\int_0^1 g(s)\Bigg(\sum_{k=1}^m G(s,t_k)I_k(y(t_k))\Bigg)ds\Bigg] \end{aligned}$$

and

$$\begin{aligned} y(t) &= \tfrac{1}{\lambda}\textstyle\int_0^1 G(t,s)w(s)f(s,y(s))ds + \sum_{k=1}^m G(t,t_k)I_k(y(t_k)) \\ &\quad + \tfrac{1}{1-\mu}\Bigg[\tfrac{1}{\lambda}\textstyle\int_0^1\Bigg(\int_0^1 G(s,r)g(r)dr\Bigg)w(s)f(s,y(s))ds \\ &\quad + \textstyle\int_0^1 g(s)\Bigg(\sum_{k=1}^m G(s,t_k)I_k(y(t_k))\Bigg)ds\Bigg]. \end{aligned}$$

Let

$$H(t,s) = G(t,s) + \frac{1}{1-\mu}\int_0^1 G(s,r)g(r)dr.$$

Then,

$$y(t) = \frac{1}{\lambda}\int_0^1 H(t,s)w(s)f(s,y(s))ds + \sum_{k=1}^m H(t,t_k)I_k(y(t_k)),$$

and the proof of sufficient is complete.

Conversely, if y is a solution of (2.2).

Direct differentiation of (2.2) implies, for $t \neq t_k$

$$\begin{aligned} y'(t) &= \tfrac{1}{\lambda}\textstyle\int_0^1 (1-s)w(s)f(s,y(s))ds + \sum_{k=1}^m I_k(y(t_k))(1-t_k) \\ &\quad -\tfrac{1}{\lambda}\textstyle\int_0^t w(s)f(s,y(s))ds - \sum_{t_k<t} I_k(y(t_k))(1-t_k). \end{aligned}$$

Evidently,

$$\lambda y''(t) = -w(t)f(t, y(t)).$$

$$\Delta y'|_{t=t_k} = -I_k(y(t_k)), \quad (k = 1, 2, \dots, m), \ y(0) = y(1) = \int_0^1 g(t)y(t)dt.$$

The Lemma is proved. $\square$

From (2.3) and (2.4), we can prove that $H(t,s), G(t,s)$ have the following properties.

Proposition 2.1. If (H_3) holds, then we have

$$H(t,s) > 0, \quad G(t,s) > 0, \quad for \quad t, s \in (0,1), \tag{2.8}$$

$$H(t,s) \geq 0, \quad G(t,s) \geq 0, \quad for \quad t, s \in J. \tag{2.9}$$

Proof. From the definitions of $H(t,s)$ and $G(t,s)$, it is easy to obtain the results of (2.8) and (2.9). $\square$

Proposition 2.2. For $t, s \in [0,1]$, we have

$$e(t)e(s) \leq G(t,s) \leq G(s,s) = s(1-s) = e(s) \leq \bar{e} = \max_{t\in[0,1]} e(s) = \frac{1}{4}. \tag{2.10}$$

Proof. In fact, for $t \in J, \quad s \in (0,1)$, we have

Case 1. If $0 < t \leq s < 1$, then

$$\frac{G(t,s)}{G(s,s)} = \frac{t(1-s)}{s(1-s)} = \frac{t}{s} \leq 1.$$

Case 2. If $0 < s \leq t < 1$, then

$$\frac{G(t,s)}{G(s,s)} = \frac{s(1-t)}{s(1-s)} = \frac{1-t}{1-s} \leq \frac{1-s}{1-s} \leq 1.$$

In addition, by the definition of $G(t,s)$, it is easy to obtain that

$$G(t,s) \leq G(s,s), \quad \forall \ t \in J, \quad s \in \{0,1\}.$$

Therefore,

$$G(t,s) \leq G(s,s) = e(s), \quad \forall \ t, \ s \in J.$$

Similarly, we can prove that

$$G(t,s) \geq e(t)e(s).$$

In fact, for all $t, \ s \in J$, we have

Case 1. If $t \leq s$, then

$$\frac{G(t,s)}{G(s,s)} = \frac{t(1-s)}{s(1-s)} = \frac{t}{s} \geq t \geq t(1-t).$$

Case 2. If $s \leq t$, then

$$\frac{G(t,s)}{G(s,s)} = \frac{s(1-t)}{s(1-s)} = \frac{1-t}{1-s} \geq 1-t \geq t(1-t).$$

So, we have

$$G(t,s) \geq e(t)e(s), \ \forall\, t,\ s \in J. \qquad \square$$

Proposition 2.3. If (H_3) holds, then for $t, s \in [0,1]$, we have

$$\rho e(s) \leq H(t,s) \leq \gamma s(1-s) = \gamma e(s) \leq \frac{1}{4}\gamma, \tag{2.11}$$

where

$$\gamma = \frac{1}{1-\mu}, \quad \rho = \frac{\int_0^1 e(\tau)g(\tau)d\tau}{1-\mu}. \tag{2.12}$$

Proof. By (2.3) and (2.10), we have

$$\begin{aligned} H(t,s) &= G(t,s) + \tfrac{1}{1-\mu}\textstyle\int_0^1 G(s,\tau)g(\tau)d\tau \\ &\geq \tfrac{1}{1-\mu}\textstyle\int_0^1 G(s,\tau)g(\tau)d\tau \\ &\geq \tfrac{\int_0^1 e(\tau)g(\tau)d\tau}{1-\mu}s(1-s) \\ &= \rho e(s), \quad t \in [0,1]. \end{aligned}$$

On the other hand, noticing $G(t,s) \leq s(1-s)$, we obtain

$$\begin{aligned} H(t,s) &= G(t,s) + \tfrac{1}{1-\mu}\textstyle\int_0^1 G(s,\tau)g(\tau)d\tau \\ &\leq s(1-s) + \tfrac{1}{1-\mu}\textstyle\int_0^1 s(1-s)g(\tau)d\tau \\ &\leq s(1-s)[1 + \tfrac{1}{1-\mu}\textstyle\int_0^1 g(\tau)d\tau] \\ &\leq s(1-s)\tfrac{1}{1-\mu} \\ &= \gamma e(s), \quad t \in [0,1]. \end{aligned}$$

The proof of Proposition 2.3 is complete. $\square$

To establish the existence of positive solutions to problem (1.1), we construct a cone K by

$$K = \left\{ y \in PC^1[0,1] : y(t) \geq 0, \quad t \in J \right\}. \tag{2.13}$$

Define an operator T by

$$(Ty)(t) = \frac{1}{\lambda}\int_0^1 H(t,s)w(s)f(s,y(s))ds + \sum_{k=1}^{m} H(t,t_k)I_k(y(t_k)). \tag{2.14}$$

From Lemma 2.1, we can obtain the following results.

Lemma 2.2. (i) If $y \in PC^1[0,1] \cap C^2(J^{'})$ is a solution of problem (1.1), then y is a fixed point of T;

(ii) If y is a fixed point of T, then $y \in PC^1[0,1] \cap C^2(J^{'})]$ is a solution of problem (1.1).

Lemma 2.3. Suppose that $(H_1) - (H_3)$ hold. Then $T(K) \subset K$ and $T : K_{r,R} \to K$ is completely continuous.

Proof. In fact, from $(H_1)-(H_3)$ and (2.9), we have $Ty \geq 0,\ \forall y \in K$, which implies that $T(K) \subset K$.

Now we prove operator T is completely continuous. For $n \geq 2$ define w_n by

$$w_n(t) = \begin{cases} \inf\limits_{0 \leq s \leq \frac{1}{n}} w(s), t \in (0, \frac{1}{n}]; \\ w(t), t \in (\frac{1}{n}, 1 - \frac{1}{n}); \\ \inf\limits_{1-\frac{1}{n} \leq s \leq 1} w(s), t \in [1 - \frac{1}{n}, 1), \end{cases}$$

and $T_n : K \to K$ by

$$(T_n x)(t) = \frac{1}{\lambda} \int_0^1 H(t,s) w_n(s) f(s, x(s)) ds + \sum_{k=1}^{m} H(t, t_k) I_k(y(t_k)).$$

As proven above, $T_n : K \to K$. Since $w_n : [0,1] \to [0, +\infty)$ is a piecewise continuous function, we can see that $T_n : K \to K$ is completely continuous (see [38]).

Let $R > 0$ and $M_R = \max\{f(t,x) : (t,x) \in J \times [0,R]\}$, then $M_R < +\infty$. Since $0 < \int_0^1 w(s)ds < +\infty$, by the absolute continuity of integral, we have

$$\lim_{n \to \infty} \int_{e(n)} w(s) ds \to 0, \quad n \to +\infty,$$

where $e(n) = [0, \frac{1}{n}] \cup [1 - \frac{1}{n}, 1]$. So,

$$\begin{aligned} &\sup \left\{ |T_n - Tx| : x \in K, \|x\| \leq R \right\} \\ &= \sup \left\{ \max_{t \in J} \tfrac{1}{\lambda} \textstyle\int_0^1 H(t,s) |w_n(s) - w(s)| f(s, x(s)) ds : x \in K, \|x\| \leq R \right\} \\ &\leq \tfrac{1}{\lambda} M_R \left\{ \max_{t \in J} \textstyle\int_0^1 H(t,s) |w_n(s) - w(s)| ds \right\} \\ &\leq \tfrac{1}{4} \gamma M_R \textstyle\int_0^1 |w_n(s) - w(s)| ds \\ &\leq \tfrac{1}{4} \gamma M_R \textstyle\int_{e(n)} w(s) ds \to 0, \quad n \to +\infty. \end{aligned}$$

It implies that the completely continuous operators T_n uniformly approximate T on any bounded subset of K. Therefore, $T : K \to K$ is completely continuous. The proof is complete. □

3 Main results

In this section, we apply Lemma 1.1 to establish the existence of positive solutions of problem (1.1), and we begin by introducing some notation:

$$a = \gamma \int_0^1 w(s)ds, \quad b = \frac{1}{\gamma m}.$$

Theorem 3.1. Assume that $(H_1) - (H_3)$ hold. In addition, letting f satisfy the following conditions

(H_4) $0 \leq f^0 = \limsup_{x \to 0} \max_{t \in J} \frac{f(t,y)}{y} < a$ and

$$0 \leq \limsup_{y \to 0} \frac{I_k(y)}{y} < b, \ k = 1, 2, \cdots, m;$$

(H_5) $0 \leq f^\infty = \limsup_{x \to \infty} \max_{t \in J} \frac{f(t,y)}{y} < a$ and

$$0 \leq \limsup_{y \to \infty} \frac{I_k(y)}{y} < b, \ k = 1, 2, \cdots, m;$$

(H_6) There exists $\nu > 0$, for $y \geq \nu$, $t \in J$ such that $f(t,y) \geq \eta$, where $\eta > 0$, then there exists $\delta > 0$ such that, for

$$\max\left\{af^0, \ af^\infty\right\} < \lambda < \delta, \tag{3.1}$$

problem (1.1) has at least two positive solutions $y_\lambda^{(1)}(t)$, $y_\lambda^{(2)}(t)$ and $\max_{t \in J} y_\lambda^{(1)}(t) > \nu$.

Proof. Letting λ satisfies (3.1) and $\delta = t_1(1-t_m)\rho\eta \int_{t_1}^{t_m} w(s)ds\nu^{-1}$. Choosing $\varepsilon > 0$ such that $f^0 + \varepsilon > 0$, $f^\infty + \varepsilon > 0$ and

$$\max\left\{a(f^0 + \varepsilon), \ a(f^\infty + \varepsilon)\right\} \leq \lambda < \delta.$$

Considering (H_4), for the ε mentioned above, then there exists $0 < r < \nu$ such that

$$f(t,y) \leq (f^0 + \varepsilon)y \leq (f^0 + \varepsilon)r, \quad \forall 0 \leq y \leq r, \quad t \in J,$$

and

$$I_k(y) \leq by, \quad \forall 0 \leq y \leq r, \ k = 1, 2, \cdots, m.$$

Therefore, for $y \in \partial K_r$, by (2.11), we have

$$\begin{aligned}(Tx)(t) &= \tfrac{1}{\lambda}\int_0^1 H(t,s)w(s)f(s,y(s))ds + \textstyle\sum_{k=1}^m H(t,t_k)I_k(y(t_k))\\ &\le \tfrac{1}{\lambda}\int_0^1 \tfrac{1}{4}\gamma w(s)f(s,y(s))ds + \textstyle\sum_{k=1}^m \tfrac{1}{4}\gamma I_k(y(t_k))\\ &\le \tfrac{1}{\lambda}\int_0^1 \tfrac{1}{4}\gamma w(s)ds(f^0+\varepsilon)\|y\| + \tfrac{1}{4}\gamma m\tfrac{1}{\gamma m}\|y\|\\ &\le \tfrac{1}{\lambda}\tfrac{1}{4}\gamma\int_0^1 w(s)ds(f^0+\varepsilon)\|y\| + \tfrac{1}{4}\|y\|\\ &\le \tfrac{1}{\lambda}\tfrac{1}{4}a(f^0+\varepsilon)\|y\| + \tfrac{1}{4}\|y\|\\ &\le \tfrac{1}{4}\|y\| + \tfrac{1}{4}\|y\|\\ &= \tfrac{1}{2}\|y\|\\ &< \|y\| = r.\end{aligned}$$

Consequently, for $y \in \partial K_r$, we have $\|Ty\| < \|y\|$, i.e., by Lemma 1.1,

$$i(T, K_r, K) = 1. \tag{3.2}$$

Now turning to (H_5), there exists $l > 0$, for $t \in J,\ y > l$, such that

$$f(t,y) \le (f^\infty + \varepsilon)y,$$

and

$$I_k(y) \le by,\ k = 1, 2, \cdots, m.$$

Letting

$$L = \max_{t\in J,\ 0\le y\le l} f(t,y), \quad L_k = \max_{0\le y\le l} I_k(y),\ k = 1, 2, \cdots, m,$$

then

$$0 \le f(t,y) \le (f^\infty+\varepsilon)y + L, \quad 0 \le I_k(y) \le by + L_k,\ k = 1, 2, \cdots, m. \tag{3.3}$$

Choosing

$$R > \max\Big\{\nu,\ \ 2\gamma(\frac{1}{\lambda}\int_0^1 w(s)ds + m)L^*\Big\}, \tag{3.4}$$

where $L^* = \max\{L, L_k\}$, $k = 1, 2, \cdots, m$.

So, for $y \in \partial K_R$, by (2.11), (3.3) and (3.4), we have

$$\begin{aligned}(Tx)(t) &= \tfrac{1}{\lambda}\int_0^1 H(t,s)w(s)f(s,y(s))ds + \textstyle\sum_{k=1}^m H(t,t_k)I_k(y(t_k))\\ &\le \tfrac{1}{\lambda}\int_0^1 \tfrac{1}{4}\gamma w(s)f(s,y(s))ds + \textstyle\sum_{k=1}^m \tfrac{1}{4}\gamma I_k(y(t_k))\\ &\le \tfrac{1}{\lambda}\int_0^1 \tfrac{1}{4}\gamma w(s)ds((f^\infty+\varepsilon)\|y\| + L) + \tfrac{1}{4}\gamma m(b\|y\| + L_k)\\ &\le \tfrac{1}{2}\|y\| + \tfrac{1}{4\lambda}\gamma\int_0^1 w(s)dsL + \tfrac{1}{4}\gamma mL_k\\ &\le \tfrac{1}{2}\|y\| + \tfrac{1}{4}\gamma(\tfrac{1}{\lambda}\int_0^1 w(s)ds + m)L^*\\ &< \tfrac{1}{2}\|y\| + \tfrac{1}{2}R\\ &= R\end{aligned}$$

i.e., by Lemma 1.1,

$$i(T,K_R,K)=1. \tag{3.5}$$

On the other hand, for $y\in \bar{K}_\nu^R=\Big\{y\in K: \|y\|\le R, \min\limits_{t\in[t_1,t_m]} y(t)\ge \nu\Big\}$, $t\in J$, (2.14) yields that

$$\|Ty\|\le \frac{1}{2}\|y\|+\frac{1}{4}\gamma(\frac{1}{\lambda}\int_0^1 w(s)ds+m)L^*<R. \tag{3.6}$$

Furthermore, for $y\in \bar{K}_\nu^R$, from (2.11), (2.16) and (H_6), we obtain

$$\begin{aligned}\min_{t\in[t_1,t_m]}(Tx)(t) &= \min_{t\in[t_1,t_m]}\tfrac{1}{\lambda}\textstyle\int_0^1 H(t,s)w(s)f(s,y(s))ds+\sum_{k=1}^m H(t,t_k)I_k(y(t_k))\\ &\ge \min_{t\in[t_1,t_m]}\tfrac{1}{\lambda}\textstyle\int_0^1 H(t,s)w(s)f(s,y(s))ds\\ &\ge \min_{t\in[t_1,t_m]}\tfrac{1}{\lambda}\textstyle\int_{t_1}^{t_m} H(t,s)w(s)f(s,y(s))ds\\ &\ge \tfrac{1}{\lambda}\rho\eta\textstyle\int_{t_1}^{t_m} e(s)w(s)ds\\ &\ge \tfrac{1}{\lambda}\rho\eta t_1(1-t_m)\textstyle\int_{t_1}^{t_m} w(s)ds\\ &> \tfrac{1}{\delta}\rho\eta t_1(1-t_m)\textstyle\int_{t_1}^{t_m} w(s)ds\\ &= \nu.\end{aligned} \tag{3.7}$$

Letting $y_0\equiv\frac{\mu+R}{2}$ and $\phi(t,y)=(1-t)Ty+ty_0$, then $\phi:[0,1]\times\bar{K}_\nu^R\to K$ is completely continuous, and from the analysis above, we obtain for $(t,y)\in[0,1]\times\bar{K}_\nu^R$

$$\phi(t,y)\in K_\nu^R. \tag{3.8}$$

Therefore, for $t\in J, y\in\partial K_\nu^R$, we have $\phi(t,y)\ne y$. Hence, by the normality property and the homotopy invariance property of the fixed point index, we obtain

$$i(T,K_\nu^R,K)=i(y_0,K_\nu^R,K)=1. \tag{3.9}$$

Consequently, by the solution property of the fixed point index, T has a fixed point $y_\lambda^{(1)}$ and $y_\lambda^{(1)}\in K_\nu^R$. By Lemma 2.1, it follows that $y_\lambda^{(1)}$ is a solution to problem (1.1), and

$$\max_{t\in J} y_\lambda^{(1)}\ge\min_{t\in[t_1,t_m]} y_\lambda^{(1)}>\nu.$$

On the other hand, from (3.2), (3.3) and (3.7) together with the additivity of the fixed point index, we get

$$\begin{aligned}&i(T,K_R\backslash(\bar{K}_r\cup\bar{K}_\nu^R),K)\\ &=i(T,K_R,K)-i(T,K_\nu^R,K)-i(T,K_r,K)\\ &=1-1-1=-1.\end{aligned} \tag{3.10}$$

Hence, by the solution property of the fixed point index, T has a fixed point $y_\lambda^{(2)}$ and $y_\lambda^{(2)} \in K_R \backslash (\bar{K}_r \cup \bar{K}_\nu^R)$. By Lemma 2.1, it follows that $y_\lambda^{(2)}$ is also a solution to problem (1.1), and $y_\lambda^{(1)} \neq y_\lambda^{(2)}$. The proof is complete. $\square$

4 Example

To illustrate how our main results can be used in practice we present an example.

Now we consider the following boundary value problem

$$\begin{cases} -\lambda y''(t) = \frac{1}{2\sqrt{t}}[kty + y^{\frac{1}{3}} \tanh y], & 0 < t < 1, \\ \Delta y'|_{t=\frac{1}{3}} = \frac{1}{5} y(\frac{1}{3}), \\ \Delta y'|_{t=\frac{1}{2}} = \frac{1}{5} y(\frac{1}{2}), \\ y(0) = y(1) = \int_0^1 ty(s)ds, \end{cases}$$

where $\lambda > 0$, $w(t) = \frac{1}{2\sqrt{t}}$, $f(t,y) = kty + y^{\frac{1}{3}} \tanh y, 0 \leq k < \frac{19}{14342}, I_1(y) = \frac{1}{5}y, I_2(y) = \frac{1}{5}y, g(t) = t$. By calculations we obtain that $\mu = \frac{1}{2}$, $\gamma = 2$, $\rho = \frac{1}{6}$, $a = 2$, $b = \frac{1}{4}$. Hence, the conditions $(H_1) - (H_3)$ hold. In addition, it is not difficult to see that

$$\lim_{y \to 0} \max_{t \in J} \frac{f(t,y)}{y} = k < 2 \ ;$$

$$\lim_{y \to +\infty} \max_{t \in J} \frac{f(t,y)}{y} = k < 2.$$

Choosing $\nu = 1$, $\eta = \frac{e^2-1}{e^2+1}$, we obtain $f(t,y) \geq \frac{e^2-1}{e^2+1} = \eta$ for $t \in [0,1], y \geq \nu$. So the conditions of the Theorem 3.1 are satisfied, then for $2k < \lambda < \delta = t_1(1-t_m)\rho\eta \int_{t_1}^{t_m} w(s)ds\nu^{-1} = \frac{19}{7171}$, problem (1.1) has at least two positive solutions $y_\lambda^{(1)}(t)$, $y_\lambda^{(2)}(t)$ and $\max_{t \in J} y_\lambda^{(1)}(t) > 1$. $\square$

Remark The example implies that there is a large number of functions that satisfy the conditions of Theorem 3.1. In addition, the conditions of Theorem 3.1 are also easy to check.

5 Open Problem

In this paper, by using the fixed point index theory, we have investigated the existence of multiple positive solutions for a class of impulsive singular boundary value problems with integral boundary conditions and have obtained some easily verifiable sufficient criteria which extend previous results. The methodology which we employed in studying the second order differential equations without impulses in [22] can be modified to establish similar sufficient criteria for second order impulsive differential equations. It is worth mentioning that there are still many problems that remain open in this vital field except for the results obtained in this paper: for example, whether or not our concise criteria can guarantee the stability of positive solutions. More efforts are still needed in the future.

Acknowledgement. This work is sponsored by the National Natural Science Foundation of China (10671023), the Scientific Creative Platform Foundation of Beijing Municipal Commission of Education (PXM2008-014224-067420) and the Science Foundation of Ministry of Education of Beijing(KM200810772010). The authors thank the referee for his/her valuable suggestions. These have greatly improved this paper.

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