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On Some Sequence Spaces Generated By $\Delta_{(r)}$- and Δ_r-Difference of Infinite Matrices

Hemen Dutta

Department of Mathematics, A.D.P. College, Nagaon-782002, Assam, India
e-mail: hemen_dutta08@rediffmail.com

Abstract

In this article we introduced some sequence spaces generated by $\Delta_{(r)}$- and Δ_r-difference of infinite matrices. We investigate these spaces for some linear topological structures. This article also introduces the application of $\Delta_{(r)}$ and Δ_r operator to infinite matices.

Keywords: *Sequence space, Difference operator, Paranorm, Completeness.*

1 Introduction

Let w denote the space of all real or complex sequences. By c, c_0 and ℓ_∞, we denote the Banach spaces of convergent, null and bounded sequences $x = (xk)$, respectively normed by

$$\|x\| = \sup_k |x_k|.$$

A linear functional L on ℓ_∞ is said to be a Banach limit (see [1]) if it has the properties:
(i) $L(x) \geq 0$ if $x \geq 0$,
(ii) $L(e) = 1$, where $e = (1, 1, 1, \dots)$,
(iii) $L(Dx) = L(x)$, where D is the shift operator defined by $D(x_n) = (x_{n+1})$.

Let B be the set of all Banach limits on ℓ_∞. A sequence x is said to be almost convergent to a number l if $L(x) = l$ for all L in B. Let $\hat{c}$ denote the set of all almost convergent sequences. Lorentz [3] proved that

$$\hat{c} = \{x : \lim_{m\to\infty} \frac{1}{m+1} \sum_{i=0}^{m} x_{n+i} \text{ exists uniformly in n}\}.$$

Similarly $\hat{c_0}$ denote the set of all sequences which are almost convergent to Zero.

In [6] the spaces $\hat{c_0}$ and $\hat{c}$ were extended to $\hat{c_0}(p)$ and $\hat{c}(p)$ in the same manner as ℓ_∞, c and c_0 are extended to $\ell_\infty(p)$, $c(p)$ and $c_0(p)$ respectively (see for instance Maddox [5]).

Several authors including Lorentz [3], King [2] and Nanda [7, 8] have studied almost convergent sequences.

The notion of difference sequence space was introduced by Kizmaz [4], who studied the difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Tripathy and Esi [9] by introducing the spaces $\ell_\infty(\Delta_r)$, $c(\Delta_r)$ and $c_0(\Delta_r)$.

Let r be non- negative integers, then for Z a given sequence space we have

$$Z(\Delta_r) = \{x = (x_k) \in w : (\Delta_r x_k) \in Z\},$$

where $\Delta_r x = (\Delta_r x_k) = (x_k - x_{k+r})$ and $\Delta_0 x_k = 0$ for all $k \in N$.
Taking $r = 1$, we get the spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ introduced and studied by Kizmaz[4].

Let $A = (a_{nk})$ be an infinite matrix of non-negative real numbers and (p_k) be a bounded sequence of positive real numbers. We write

$$B_{mn}(x) = \sum_{k=1}^{\infty} a_{mk} x_{k+n}, \text{ if the series converges for each } m \text{ and } n.$$

Let r be a non-negative integer. Then we define the following sequence spaces:

$$(\hat{A}, p, \Delta_{(r)})_0 = \{x : \lim_{m\to\infty} |\Delta_{(r)B_{mn}(x)}|^{p_m} = 0 \text{ uniformly in } n\},$$

$$(\hat{A}, p, \Delta_{(r)}) = \{x : \lim_{m\to\infty} |\Delta_{(r)B_{mn}(x-le)}|^{p_m} = 0 \text{ for some } l \text{ uniformly in } n\},$$

$$(\hat{A}, p, \Delta_{(r)})_\infty = \{x : \sup_{m,n} |\Delta_{(r)B_{mn}(x)}|^{p_m} < \infty\},$$

where $\Delta_{(r)} B_{mn}(x) = B_{mn}(x) - B_{m-r,n}(x) = \sum_{k=1}^{\infty} \Delta_{(r)} a_{mk} x_{k+n}$, $\Delta_{(r)} a_{mk} = a_{mk} - a_{m-r,k}$ and $\Delta_{(0)} a_{mk} = a_{mk}$ for all $m \in N$. (e.g., $\Delta_{(2)} a_{mk} = a_{mk} - a_{m-2,k}$). In this definition it is important to note that we take $a_{m-r,k} = 0$, for non-positive values of $m - r$. (e.g., $\Delta_{(2)} a_{13} = a_{13} - a_{-1,3} = a_{13}$ etc.).

If in the definition of the spaces we take $r = 0$, then $(\hat{A}, p, \Delta_{(r)})_0 = (\hat{A}, p)_0$, $(\hat{A}, p, \Delta_{(r)}) = (\hat{A}, p)$ and $(\hat{A}, p, \Delta_{(r)})_\infty = (\hat{A}, p)_\infty$. The spaces $(\hat{A}, p)_0$, $(\hat{A}, p)$ and $(\hat{A}, p)_\infty$ are studied by Nanda [8]. If we take $p_k = 1$, for all $k \in N$, then $(\hat{A}, p, \Delta_{(r)})_0 = (\hat{A}, \Delta_{(r)})_0$ etc.

If in the definitions of the spaces we take $A = (C, 1)$ and $r = 0$, then $(\hat{A}, p, \Delta_{(r)})_0 = \hat{c_0}(p)$, $(\hat{A}, p, \Delta_{(r)}) = \hat{c}(p)$ and $(\hat{A}, p, \Delta_{(r)})_\infty = \hat{m}(p)$, which can be found in Nanda [7].

Similarly using the difference operator Δ_r, we can define the spaces $(\hat{A}, p, \Delta_r)_0$, $(\hat{A}, p, \Delta_r)$ and $(\hat{A}, p, \Delta_r)_\infty$.

The following inequality will be used in the article.

Let $p = (p_k)$ be a positive sequence of real numbers with $0 < p_k \leq \sup p_p = G$, $D = \max\{1, 2^{G-1}\}$. Then for all $a_k, b_k \in C$ for all $k \in N$, we have

$$|a_k + b_k|^{p_k} \leq D\{|a_k|^{p_k} + |b_k|^{p_k}\} \quad (1)$$

and for all $\lambda \in C$,

$$|\lambda|^{p_k} \leq \max(1, |\lambda|^G) \quad (2)$$

2 Main Results

In this section we study some linear topological structures of the spaces $(\hat{A}, p, \Delta_{(r)})_0$, $(\hat{A}, p, \Delta_{(r)})$, $(\hat{A}, p, \Delta_{(r)})_\infty$, $(\hat{A}, p, \Delta_r)_0$, $(\hat{A}, p, \Delta_r)$ and $(\hat{A}, p, \Delta_r)_\infty$.

Without loss of generality, we may assume that $0 < p_m \leq 1$, for if $0 < p_m < \infty$ and $\sup p_m < \infty$, then $0 < \frac{p_m}{\sup p_m} \leq 1$.

Theorem 2.1 *(i)* $(\hat{A}, p, \Delta_{(r)})_0 \subset (\hat{A}, p, \Delta_{(r)})$,
(ii) $(\hat{A}, p, \Delta_{(r)})_0 \subset (\hat{A}, p, \Delta_{(r)})_\infty$,
(iii) $(\hat{A}, p, \Delta_{(r)}) \subset (\hat{A}, p, \Delta_{(r)})_\infty$, *if*

$$\sup_m |\sum_{k=1}^{\infty} \Delta_{(r)} a_{mk}|^{p_m} < \infty. \quad (3)$$

Proof: Proof of (i) and (ii) are easy and so omitted. We give the proof of (iii) only.

Let $x \in (\hat{A}, p, \Delta_{(r)})$ and $\sup_m |\sum_{k=1}^{\infty} \Delta_{(r)} a_{mk}|^{p_m} < \infty$.

Now

$$|\Delta_{(r)} B_{mn}(x)|^{p_m} = |\Delta_{(r)} B_{mn}(x - le + le)|^{p_m}$$

$$\leq |\Delta_{(r)}B_{mn}(x-le)|^{p_m} + |l\sum_{k=1}^{\infty}\Delta_{(r)}a_{mk}|^{p_m}, \text{ using (1)} \tag{4}$$

$$\leq |\Delta_{(r)}B_{mn}(x-le)|^{p_m} + \sup|l|^{p_m}|\sum_{k=1}^{\infty}\Delta_{(r)}a_{mk}|^{p_m}.$$

Hence $x \in (\hat{A}, p, \Delta_{(r)})_\infty$. This completes the proof.

Remark 2.2 *Similar results hold for the spaces* $(\hat{A}, p, \Delta_r)_0$, $(\hat{A}, p, \Delta_r)$ *and* $(\hat{A}, p, \Delta_r)_\infty$ *also.*

Theorem 2.3 *The spaces* $(\hat{A}, p, \Delta_{(r)})_0$, $(\hat{A}, p, \Delta_{(r)})$, $(\hat{A}, p, \Delta_{(r)})_\infty$, $(\hat{A}, p, \Delta_r)_0$, $(\hat{A}, p, \Delta_r)$ *and* $(\hat{A}, p, \Delta_r)_\infty$ *are linear.*

Proof: We give the proof only for the space $(\hat{A}, p, \Delta_{(r)})$ and for other spaces it will follow on applying similar arguments.

Let $x = (x_k)$ and $y = (y_k)$ be any two elements of $(\hat{A}, p, \Delta_{(r)})$. Then there exist l and l' such that

$$|\Delta_{(r)}B_{mn}(x-le)|^{p_m} \to 0 \text{ and } |\Delta_{(r)}B_{mn}(y-l'e)|^{p_m} \to 0,$$

as $m \to \infty$ uniformly in n.

$$|\Delta_{(r)}B_{mn}(\alpha x + \beta y - (\alpha l + \beta l')e)|^{p_m}$$

$$\leq \sup|\alpha|^{p_m}|\Delta_{(r)}B_{mn}(x-le)|^{p_m} + \sup|\beta|^{p_m}|\Delta_{(r)}B_{mn}(y-l'e)|^{p_m}, \text{ using (1).}$$

$$\to 0 \text{ as } m \to \infty \text{ uniformly in } n.$$

Thus $(\hat{A}, p, \Delta_{(r)})$ is linear.

Theorem 2.4 (i) *The space* $(\hat{A}, p, \Delta_{(r)})_0$ *is a paranormed space, paranormed by* g*, defined by*

$$g(x) = \sup_{m,n}|\Delta_{(r)}B_{mn}(x)|^{p_m} \tag{5}$$

(ii) *The space* $(\hat{A}, p, \Delta_{(r)})_\infty$ *is a paranormed space, paranormed by* g *if* $\inf p_m > 0$,

(iii) *The space* $(\hat{A}, p, \Delta_{(r)})$ *is a paranormed space, paranormed by* g *if* (3) *holds.*

Proof: We give the proof only for (i) and proof of (ii) and (iii) follow similarly.

Clearly $g(x) = g(-x)$; $x = \theta$ implies $g(x) = 0$.

Let $x = (x_k)$ and $y = (y_k)$ any two elements of $(\hat{A}, p, \Delta_{(r)})_0$. Then

$$g(x+y) = \sup_{m,n} |\Delta_{(r)} B_{mn}(x+y)|^{p_m}$$
$$\leq \sup_{m,n} |\Delta_{(r)} B_{mn}(x)|^{p_m} + \sup_{m,n} |\Delta_{(r)} B_{mn}(y)|^{p_m}, \text{ using (1).}$$
$$= g(x) + g(y)$$

Thus $g(x+y) \leq g(x) + g(y)$.

The continuity of the scalar multiplication follows from the following equality:

$$g(\alpha x) = \sup_{m,n} |\Delta_{(r)} B_{mn}(\alpha x)|^{p_m} = \sup_{m} |\alpha|^{p_m} g(x)$$

Theorem 2.5 (*i*) *The space* $(\hat{A}, p, \Delta_r)_0$ *is a paranormed space, paranormed by* g*, defined by*

$$g(x) = \sup_{m,n} |\Delta_r B_{mn}(x)|^{p_m}$$

(*ii*) *The space* $(\hat{A}, p, \Delta_r)_\infty$ *is a paranormed space, paranormed by* g *if* $\inf p_m > 0$.
(*iii*) *The space* $(\hat{A}, p, \Delta_r)$ *is a paranormed space, paranormed by* g *if* $\sup_m |\sum_{k=1}^{\infty} \Delta_r a_{mk}|^{p_m} < \infty$ *holds.*

Proof: This is routine verification and so omitted.

For the following results we shall assume that $A = (a_{nk})$ be an infinite matrix of non-negative real numbers such that $\alpha_{i1} = \Delta_{(r)} a_{i1} \neq 0$ and $\beta_{ij} = |\Delta_{(r)} a_{ij} - \Delta_{(r)} a_{i,j-1}| = 0$, for every i, j

Theorem 2.6 (*i*) *The space* $(\hat{A}, p, \Delta_{(r)})_\infty$ *is a complete paranormed space under the paranormed* g*, defined by* (5) *if* $\inf p_m > 0$.
(*ii*) *The space* $(\hat{A}, p, \Delta_{(r)})_0$ *is a complete paranormed space under the paranormed* g*, defined by* (5).
(*iii*) *The space* $(\hat{A}, p, \Delta_{(r)})$ *is a complete paranormed space under the paranormed* g*, defined by* (5) *if* $|\sum_{k=1}^{\infty} \Delta_{(r)} a_{mk}|^{p_m} \to 0$ *as* $m \to \infty$.

Proof: (*i*) Let (x^i) be any Cauchy sequence in $(\hat{A}, p, \Delta_{(r)})_\infty$. Then for $\varepsilon (0 < \varepsilon < 1)$, there exists a positive integer n_0 such that

$$g(x^i - x^j) < \varepsilon, \text{ for all } i, j \geq n_0.$$

using (5), we have

$$\sup_{m,n} |\Delta_{(r)} B_{mn}(x_k^i - x_k^j)|^{p_m} < \varepsilon, \text{ for all } i, j \geq n_0.$$

Hence

$$|\Delta_{(r)}B_{mn}(x_k^i - x_k^j)|^{p_m} < \varepsilon, \text{ for all } i,j \geq n_0 \text{ and for each } m,n.$$

It follows that

$$|\Delta_{(r)}B_{mn}(x_k^i - x_k^j)| < \varepsilon, \text{ for all } i,j \geq n_0 \text{ and for each } m,n.$$

Thus $(\Delta_{(r)}B_{mn}(x_k^i))$ is a Cauchy sequence in C, for each m,n. Therefore $(\Delta_{(r)}B_{mn}(x_k^i))$ is convergent in C, for each m,n.

i.e., $\{\Delta_{(r)}B_{mn}(x_k^1), \Delta_{(r)}B_{mn}(x_k^2) \ldots, \Delta_{(r)}B_{mn}(x_k^i), \ldots\}=$ $\{\sum_{k=1}^{\infty} \Delta_{(r)}a_{mk}x_{k+n}^1, \sum_{k=1}^{\infty} \Delta_{(r)}a_{mk}x_{k+n}^2, \ldots, \sum_{k=1}^{\infty} \Delta_{(r)}a_{mk}x_{k+n}^i, \ldots\}$ is convergent in C, for each m,n.

Let $\lim_{i\to\infty} \sum_{k=1}^{\infty} \Delta_{(r)}a_{mk}x_{k+n}^i = y_{mn}$, say for every m,n.

This implies that

$\lim_{i\to\infty}\{\Delta_{(r)}a_{m1}x_{1+n}^i + \Delta_{(r)}a_{m2}x_{2+n}^i + \Delta_{(r)}a_{m3}x_{3+n}^i + \ldots\} = y_{mn}$, for every m,n.

Replacing n by $n+1$, we get

$\lim_{i\to\infty}\{\Delta_{(r)}a_{m1}x_{2+n}^i + \Delta_{(r)}a_{m2}x_{3+n}^i + \Delta_{(r)}a_{m3}x_{4+n}^i + \ldots\} = y_{m,n+1}$, for every m,n.

Subtracting above two expressions, we have

$\lim_{i\to\infty}\{\Delta_{(r)}a_{m1}x_1^i + (\Delta_{(r)}a_{m2} - \Delta_{(r)}a_{m1})x_2^i + (\Delta_{(r)}a_{m3} - \Delta_{(r)}a_{m2})x_3^i + \ldots\} = y_{m,n} - y_{m,n+1}$

Hence by the assumptions that $\alpha_{i1} = \Delta_{(r)}a_{i1} \neq 0$ and $\beta_{ij} = |\Delta_{(r)}a_{ij} - \Delta_{(r)}a_{i,j-1}| = 0$, for every i,j, we have $\lim_{i\to\infty} x_1^i$ exists.

Proceeding in this way we can conclude that $\lim_{i\to\infty} x_k^i = x_k$, say exists for every $k \geq 1$.

Now we can have for all $i,j \geq n_0$,

$$\sup_{m,n} |\Delta_{(r)}B_{mn}(x_k^i - x_k^j)|^{p_m} < \varepsilon.$$

Then

$$\sup_{m,n} |\Delta_{(r)}B_{mn}(x_k^i - \lim_{j\to\infty} x_k^j)|^{p_m} < \varepsilon, \text{ for all } i \geq n_0.$$

Hence

$$\sup_{m,n} |\Delta_{(r)} B_{mn}(x_k^i - x_k)|^{p_m} < \varepsilon, \text{ for all } i \geq n_0.$$

This implies that $(x^i - x) \in (\hat{A}, p, \Delta_{(r)})_\infty$. Since $(\hat{A}, p, \Delta_{(r)})_\infty$ is linear, we have $x = x^i - (x^i - x) \in (\hat{A}, p, \Delta_{(r)})_\infty$. Hence $(\hat{A}, p, \Delta_{(r)})_\infty$ is complete.

(ii) This is same with part (i).

(iii) If $|\sum_{k=1}^{\infty} \Delta_{(r)} a_{mk}|^{p_m} \to 0$ as $m \to \infty$, then (3) holds and it follows from the inequality (4), that $(\hat{A}, p, \Delta_{(r)}) = (\hat{A}, p, \Delta_{(r)})_0$ and therefore the completeness of $(\hat{A}, p, \Delta_{(r)})$ follows from the completeness of $(\hat{A}, p, \Delta_{(r)})_0$.

Remark 2.7 *We get similar results as of Theorem 5 for the spaces* $(\hat{A}, p, \Delta_r)_0$, $(\hat{A}, p, \Delta_r)$ *and* $(\hat{A}, p, \Delta_r)_\infty$ *also.*

Theorem 2.8 *The spaces* $(\hat{A}, \Delta_{(r)})_0$ *and* $(\hat{A}, \Delta_{(r)})_\infty$ *are normed linear space, normed by*

$$\|x\| = \sup_{m,n} |\Delta_{(r)} B_{mn}(x)|$$

and $(\hat{A}, \Delta_{(r)})$ *is a normed linear space under the same norm if* $\sup_m |\sum_{k=1}^{\infty} \Delta_{(r)} a_{mk}| < \infty$.

Proof: We give the proof only for the space $(\hat{A}, \Delta_{(r)})_0$ and for the other spaces it will follow similarly.

It is obvious that $x = \theta$ implies $\|x\| = 0$, $\|\alpha x\| = |\alpha| \|x\|$, for any scalar α and $\|x + y\| \leq \|x\| + \|y\|$.

Let us assume that for any $x = (x_k) \in (\hat{A}, \Delta_{(r)})_0$, $\|x\| = 0$. Using the definition of norm, we have

$$\sup_{mn} |\Delta_{(r)} B_{mn}(x)| = 0$$

Then we have

$$\Delta_{(r)} B_{mn}(x) = \sum_{k=1}^{\infty} \Delta_{(r)} a_{mk} x_{k+n} = 0, \text{ for each } m, n$$

i.e.,

$$\Delta_{(r)} a_{m1} x_{1+n} + \Delta_{(r)} a_{m2} x_{2+n} + \Delta_{(r)} a_{m3} x_{3+n} + \cdots = 0, \text{ for each } m, n$$

Taking $n = 0, m = 1$, we get

$$\Delta_{(r)}a_{11}x_1 + \Delta_{(r)}a_{12}x_2 + \Delta_{(r)}a_{13}x_3 + \cdots = 0$$

Again taking $n = 1, m = 1$, we get

$$\Delta_{(r)}a_{11}x_2 + \Delta_{(r)}a_{12}x_3 + \Delta_{(r)}a_{13}x_4 + \cdots = 0$$

Subtracting above two expressions, we get

$$\Delta_{(r)}a_{11}x_1 + (\Delta_{(r)}a_{12} - \Delta_{(r)}a_{11})x_2 + (\Delta_{(r)}a_{13} - \Delta_{(r)}a_{12})x_3 + \cdots = 0$$

Hence by the assumptions that $\alpha_{i1} = \Delta_{(r)}a_{i1} \neq 0$ and $\beta_{ij} = |\Delta_{(r)}a_{ij} - \Delta_{(r)}a_{i,j-1}| = 0$, for every i, j, we have $x_1 = 0$.

Proceeding in this way we shall get $x_k = 0$, for every $k \in N$.
Thus $x = \theta$.

Remark 2.9 *We get similar results as of Theorem 6 for the spaces* $(\hat{A}, \Delta_r)_0$, $(\hat{A}, \Delta_r)$ *and* $(\hat{A}, \Delta_r)_\infty$ *also.*

Theorem 2.10 *Let* $A = (C, 1)$. *Then the spaces* $(\hat{A}, \Delta_{(r)})_\infty$ *and* $(\hat{A}, \Delta_r)_\infty$ *are isometrically isomorphic with the space* ℓ_∞.

Proof: We give the proof for the space $(\hat{A}, \Delta_{(r)})_\infty$ only and for the other space it will follow similarly.

In fact if $A = (C, 1)$, then $(\hat{A}, \Delta_{(r)})_\infty = \ell_\infty(\Delta_{(r)})$.
Let us consider the mapping $T : \ell_\infty(\Delta_{(r)}) \longrightarrow \ell_\infty$, defined by
$T(x) = y = (\Delta_{(r)}x_k)$, for every $x \in \ell_\infty(\Delta_{(r)})$.
Then $\|x\| = \sup_k |\Delta_{(r)}x_k| = \sup_k |y_k| = \|Tx\|$.
Also clearly T is bijective linear.
Hence $\ell_\infty(\Delta_{(r)})$ is isometrically isomorphic with ℓ_∞.

3 Open Problem

In this paper, the spaces $(\hat{A}, p, \Delta_{(r)})_\infty$, $(\hat{A}, p, \Delta_{(r)})_0$ and $(\hat{A}, p, \Delta_{(r)})$ are not always complete paranormed spaces. By imposing some conditions on the infinite matrix $A = (a_{nk})$, we have shown that they are complete paranormed spaces. Therefore it remains open to characterize all such infinite matrices $A = (a_{nk})$ for which the spaces $(\hat{A}, p, \Delta_{(r)})_\infty$, $(\hat{A}, p, \Delta_{(r)})_0$ and $(\hat{A}, p, \Delta_{(r)})$ are complete paranormed spaces.

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