Int. J. Open Problems Compt. Math., Vol. 2, No. 3, September 2009
ISSN 1998-6262; Copyright ©ICSRS Publication, 2009
www.i-csrs.org

An Iterative Algorithm for Two Asymptotically Pseudocontractive Mappings

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Abstract

Let K be a nonempty closed convex subset of a real Banach space E, $T_i : K \to K$, $i = 1, 2$ be two uniformly L-Lipschitzian asymptotically pseudocontractive mappings with sequence $\{k_n\}_{n\geq 0} \subset [1, \infty)$, $\sum_{n\geq 0}(k_n - 1) < \infty$ such that $F(T_1) \cap F(T_2) \neq \varphi$, where $F(T_i)$ is the set of fixed points of T_i in K and p be a point in $F(T_1) \cap F(T_2)$. Let $\{\alpha_n\}_{n\geq 0}, \{\beta_n\}_{n\geq 0} \subset [0, 1]$ be two sequences such that $\sum_{n\geq 0} \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0 = \lim_{n\to\infty} \beta_n$. For arbitrary $x_0 \in K$, let $\{x_n\}_{n\geq 0}$ be a sequence iteratively defined by

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)\, x_n + \alpha_n T_1^n y_n, \\ y_n &= (1 - \beta_n)\, x_n + \beta_n T_2^n x_n,\ n \geq 0. \end{aligned}$$

Suppose there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$, $\phi(0) = 0$ such that

$$\langle T_i^n x - p, j(x - p)\rangle \leq k_n ||x - p||^2 - \phi(||x - p||), \forall x \in K, i = 1, 2.$$

Then $\{x_n\}_{n\geq 0}$ converges strongly to $p \in F(T_1) \cap F(T_2)$. The results proved in this paper significantly improve the results of Chang et al. [1].

Keywords: *Modified Mann iterative scheme, Uniformly L-Lipschitzian mappings, Asymptotically pseudocontractive mappings, Banach spaces*

1 Introduction

Let E be a real normed space and K be a nonempty convex subset of E. Let J denote the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = ||x||^2 \text{ and } ||f^*|| = ||x||\},$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. We shall denote the single-valued duality mapping by j.

Let $T : D(T) \subset E \to E$ be a mapping with domain $D(T)$ in E.

Definition 1.1 *The mapping T is said to be uniformly L-Lipschitzian if there exists $L > 0$ such that for all $x, y \in D(T)$*

$$\|T^n x - T^n y\| \leq L \|x - y\|.$$

Definition 1.2 *T is said to be nonexpansive if for all x, $y \in D(T)$, the following inequality holds:*

$$\|Tx - Ty\| \leq \|x - y\| \text{ for all } x,\, y \in D(T).$$

Definition 1.3 *T is said to be asymptotically nonexpansive [6], if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\sum_{n \geq 0} (k_n - 1) < \infty$ such that*

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \text{ for all } x,\, y \in D(T),\, n \geq 1.$$

Definition 1.4 *T is said to be asymptotically pseudocontractive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\sum_{n \geq 0} (k_n - 1) < \infty$ and there exists $j(x-y) \in J(x-y)$ such that*

$$\langle T^n x - T^n y, j(x - y) \rangle \leq k_n \|x - y\|^2 \text{ for all } x,\, y \in D(T),\, n \geq 1.$$

Remark 1.5 *1. It is easy to see that every asymptotically nonexpansive mapping is uniformly L-Lipschitzian.*

2. If T is asymptotically nonexpansive mapping then for all x, $y \in D(T)$ there exists $j(x - y) \in J(x - y)$ such that

$$\begin{aligned} \langle T^n x - T^n y, j(x - y) \rangle &\leq \|T^n x - T^n y\| \|x - y\| \\ &\leq k_n \|x - y\|^2,\, n \geq 1. \end{aligned}$$

Hence every asymptotically nonexpansive mapping is asymptotically pseudocontractive.

3. Rhoades in [11] showed that the class of asymptotically pseudocontractive mappings properly contains the class of asymptotically nonexpansive mappings.

The asymptotically pseudocontractive mappings were introduced by Schu [12] who proved the following theorem:

Theorem 1.6 *Let K be a nonempty bounded closed convex subset of a Hilbert space $H, T : K \to K$ a completely continuous, uniformly L-Lipschitzian and asymptotically pseudocontractive with sequence $\{k_n\} \subset [1, \infty)$; $q_n = 2k_n - 1$, $\forall n \in N$; $\sum(q_n^2 - 1) < \infty$; $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$; $\epsilon < \alpha_n < \beta_n \leq b$, $\forall n \in N$, and some $\epsilon > 0$ and some $b \in (0, L^{-2}[(1 + L^2)^{\frac{1}{2}} - 1])$; $x_1 \in K$ for all $n \in N$, define*

$$x_{n+1} = (1 - \alpha_n)\, x_n + \alpha_n T^n x_n.$$

Then $\{x_n\}$ converges to some fixed point of T.

The recursion formula of Theorem 1.6 is a modification of the well-known Mann iteration process (see [9]).

Recently, Chang [1] extended Theorem 1.6 to real uniformly smooth Banach space. In fact, he proved the following theorem:

Theorem 1.7 *Let K be a nonempty bounded closed convex subset of a real uniformly smooth Banach space E, $T : K \to K$ an asymptotically pseudocontractive mapping with sequence $\{k_n\} \subset [1, \infty)$, $\lim\limits_{n\to\infty} k_n = 1$, and $x^* \in F(T) = \{x \in K : Tx = x\}$. Let $\{\alpha_n\} \subset [0, 1]$ satisfying the following conditions: $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum \alpha_n = \infty$. For arbitrary $x_0 \in K$ let $\{x_n\}$ be iteratively defined by*

$$x_{n+1} = (1 - \alpha_n)\, x_n + \alpha_n T^n x_n,\ n \geq 0.$$

If there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$, $\phi(0) = 0$ such that

$$\langle T^n x - x^*, j(x - x^*)\rangle \leq k_n ||x - x^*||^2 - \phi(||x - x^*||),\ \forall n \in N,$$

then $x_n \to x^* \in F(T)$.

Remark 1.8 *Theorem 1.7, as stated is a modification of Theorem 2.4 of Chang [1] who actually included error terms in his algorithm.*

In [10], E. U. Ofoedu proved the following results.

Theorem 1.9 *Let K be a nonempty closed convex subset of a real Banach space E, $T : K \to K$ a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with sequence $\{k_n\}_{n\geq 0} \subset [1, \infty)$, $\lim\limits_{n\to\infty} k_n = 1$ such that $x^* \in F(T) = \{x \in K : Tx = x\}$. Let $\{\alpha_n\}_{n\geq 0} \subset [0, 1]$ be a sequence such that*

$\sum_{n\geq 0} \alpha_n = \infty$, $\sum_{n\geq 0} \alpha_n^2 < \infty$ *and* $\sum_{n\geq 0} \alpha_n(k_n - 1) < \infty$. *For arbitrary* $x_0 \in K$ *let* $\{x_n\}_{n\geq 0}$ *be a sequence iteratively defined by*

$$x_{n+1} = (1 - \alpha_n)\, x_n + \alpha_n T^n x_n,\ n \geq 0.$$

Suppose there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$, $\phi(0) = 0$ *such that*

$$\langle T^n x - x^*, j(x - x^*)\rangle \leq k_n ||x - x^*||^2 - \phi(||x - x^*||),\ \forall x \in K.$$

Then $\{x_n\}_{n\geq 0}$ *is bounded.*

Theorem 1.10 *Let* K *be a nonempty closed convex subset of a real Banach space* E, $T : K \to K$ *a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with sequence* $\{k_n\}_{n\geq 0} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$ *such that* $x^* \in F(T) = \{x \in K : Tx = x\}$. *Let* $\{\alpha_n\}_{n\geq 0} \subset [0, 1]$ *be be a sequence such that* $\sum_{n\geq 0} \alpha_n = \infty$, $\sum_{n\geq 0} \alpha_n^2 < \infty$ *and* $\sum_{n\geq 0} \alpha_n(k_n - 1) < \infty$. *For arbitrary* $x_0 \in K$ *let* $\{x_n\}_{n\geq 0}$ *be a sequence iteratively defined by*

$$x_{n+1} = (1 - \alpha_n)\, x_n + \alpha_n T^n x_n,\ n \geq 0.$$

Suppose there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$, $\phi(0) = 0$ *such that*

$$\langle T^n x - x^*, j(x - x^*)\rangle \leq k_n ||x - x^*||^2 - \phi(||x - x^*||),\ \forall x \in K.$$

Then $\{x_n\}_{n\geq 0}$ *converges strongly to* $x^* \in F(T)$.

Theorem 1.11 *Let* K *be a nonempty closed convex subset of a real Banach space* E, $T : K \to K$ *a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with sequence* $\{k_n\}_{n\geq 0} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$ *such that* $x^* \in F(T) = \{x \in K : Tx = x\}$. *Let* $\{a_n\}_{n\geq 0}, \{b_n\}_{n\geq 0}, \{c_n\}_{n\geq 0}$ *be real sequences in* $[0, 1]$ *satisfying the following conditions:*

i) $a_n + b_n + c_n = 1$;

ii) $\sum_{n\geq 0} (b_n + c_n) = \infty$;

iii) $\sum_{n\geq 0} (b_n + c_n)^2 < \infty$;

iv) $\sum_{n\geq 0} (b_n + c_n)(k_n - 1) < \infty$; *and*

v) $\sum_{n\geq 0} c_n < \infty$.

For arbitrary $x_0 \in K$ *let* $\{x_n\}_{n\geq 0}$ *be iteratively defined by*

$$x_{n+1} = a_n x_n + b_n T^n x_n + c_n u_n,\ n \geq 0,$$

where $\{u_n\}_{n\geq 0}$ *is a bounded sequence of error terms in* K. *Suppose there exists a strictly increasing function* $\phi : [0,\infty) \to [0,\infty)$, $\phi(0) = 0$ *such that*

$$\langle T^n x - x^*, j(x - x^*)\rangle \leq k_n ||x - x^*||^2 - \phi(||x - x^*||),\ \forall x \in K.$$

Then $\{x_n\}_{n\geq 0}$ *converges strongly to* $x^* \in F(T)$.

In [1], Chang et al., proved the following results.

Theorem 1.12 *Let* K *be a nonempty closed convex subset of a real Banach space* E, $T_i : K \to K$, $i = 1, 2$ *be two uniformly L-Lipschitzian asymptotically pseudocontractive mappings with sequence* $\{k_n\}_{n\geq 0} \subset [1,\infty)$, $\lim_{n\to\infty} k_n = 1$ *such that* $F(T_1) \cap F(T_2) \neq \varphi$, *where* $F(T_i)$ *is the set of fixed points of* T_i *in* K *and* p *be a point in* $F(T_1) \cap F(T_2)$. *Let* $\{\alpha_n\}_{n\geq 0}, \{\beta_n\}_{n\geq 0} \subset [0,1]$ *be two sequences such that* $\sum_{n\geq 0} \alpha_n = \infty$, $\sum_{n\geq 0} \alpha_n^2 < \infty$, $\sum_{n\geq 0} \beta_n < \infty$ *and* $\sum_{n\geq 0} \alpha_n(k_n - 1) < \infty$. *For arbitrary* $x_0 \in K$, *let* $\{x_n\}_{n\geq 0}$ *be be a sequence iteratively defined by*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)\, x_n + \alpha_n T_1^n y_n, \\ y_n &= (1-\beta_n)\, x_n + \beta_n T_2^n x_n,\ n \geq 0. \end{aligned}$$

Suppose there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$, $\phi(0) = 0$ *such that*

$$\langle T_i^n x - p, j(x - p)\rangle \leq k_n ||x - p||^2 - \phi(||x - p||),\ \forall x \in K,\ i = 1, 2.$$

Then $\{x_n\}_{n\geq 0}$ *converges strongly to* $p \in F(T_1) \cap F(T_2)$.

In this paper our purpose is to improve the results of Chang et al. [1] in a significantly more general context by removing the conditions $\sum_{n\geq 0} \alpha_n^2 < \infty$ and $\sum_{n\geq 0} \alpha_n(k_n - 1) < \infty$ from the Theorem 1.12.

2 Main Results

The following lemmas are now well known.

Lemma 2.1 *Let* $J : E \to 2^E$ *be the normalized duality mapping. Then for any* $x, y \in E$, *we have*

$$||x + y||^2 \le ||x||^2 + 2\langle y, j(x + y)\rangle, \quad \forall j(x + y) \in J(x + y).$$

Suppose there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ *with* $\phi(0) = 0$.

Lemma 2.2 *Let* $\{\theta_n\}$ *be a sequence of nonnegative real numbers,* $\{\lambda_n\}$ *be a real sequence satisfying*

$$0 \le \lambda_n \le 1, \sum_{n=0}^{\infty} \lambda_n = \infty$$

and let $\phi \in \Phi$. *If there exists a positive integer* n_0 *such that*

$$\theta_{n+1}^2 \le \theta_n^2 - \lambda_n \phi(\theta_{n+1}) + \sigma_n,$$

for all $n \ge n_0$, *with* $\sigma_n \ge 0$, $\forall n \in \mathbf{N}$, *and* $\sigma_n = 0(\lambda_n)$, *then* $\lim_{n\to\infty} \theta_n = 0$.

Theorem 2.3 *Let* K *be a nonempty closed convex subset of a real Banach space* E, $T_i : K \to K$, $i = 1, 2$ *be two uniformly L-Lipschitzian asymptotically pseudocontractive mappings with sequence* $\{k_n\}_{n\ge0} \subset [1, \infty)$, $\sum_{n\ge0}(k_n - 1) < \infty$ *such that* $F(T_1) \cap F(T_2) \ne \varphi$, *where* $F(T_i)$ *is the set of fixed points of* T_i *in* K *and* p *be a point in* $F(T_1) \cap F(T_2)$. *Let* $\{\alpha_n\}_{n\ge0}, \{\beta_n\}_{n\ge0} \subset [0, 1]$ *be two sequences such that* $\sum_{n\ge0} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0 = \lim_{n\to\infty} \beta_n$. *For arbitrary* $x_0 \in K$, *let* $\{x_n\}_{n\ge0}$ *be a sequence iteratively defined by*

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)\, x_n + \alpha_n T_1^n y_n, \\ y_n &= (1 - \beta_n)\, x_n + \beta_n T_2^n x_n,\ n \ge 0. \end{aligned} \tag{1}$$

Suppose there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$, $\phi(0) = 0$ *such that*

$$\langle T_i^n x - p, j(x - p)\rangle \le k_n||x - p||^2 - \phi(||x - p||), \forall x \in K,\ i = 1, 2. \tag{2}$$

Then $\{x_n\}_{n\ge0}$ *converges strongly to* $p \in F(T_1) \cap F(T_2)$.

Proof. Since T_1 and T_2 are uniformly L-Lipschitzian mappings, so there exists $L_1, L_2 > 0$ such that for all $x, y \in K$,

$$\|T_i^n x - T_i^n y\| \leq L_i \|x - y\|, i = 1, 2.$$

Denote $L = \max\{L_1, L_2\}$, implies

$$\|T_i^n x - T_i^n y\| \leq L \|x - y\|, i = 1, 2.$$

By $\lim\limits_{n\to\infty} \alpha_n = 0 = \lim\limits_{n\to\infty} \beta_n$ and $\lim\limits_{n\to\infty} k_n = 1$, there exists $n_0 \in \mathbf{N}$ such that $\forall n \geq n_0$,

$$\alpha_n \leq \min\left\{\frac{1}{2+3L}, \frac{\phi(2\phi^{-1}(a_0))}{18(1+L)(2+3L)[\phi^{-1}(a_0)]^2}\right\},$$

$$\beta_n \leq \min\frac{1}{2}\left\{\frac{1}{1+L}, \frac{\phi(2\phi^{-1}(a_0))}{18L(1+L)[\phi^{-1}(a_0)]^2}\right\},$$

and

$$k_n - 1 \leq \frac{\phi(2\phi^{-1}(a_0))}{54[\phi^{-1}(a_0)]^2}.$$

Define $a_{0,i} := \|x_{n_0} - T_i^{n_0} x_{n_0}\|\|x_{n_0} - p\| + (k_{n_0} - 1)\|x_{n_0} - p\|^2$, $i = 1, 2$ and $a_0 = \max\{a_{0,1}, a_{0,2}\}$. Then from (2), we obtain that $\|x_{n_0} - p\| \leq \phi^{-1}(a_0)$.

CLAIM. $\|x_n - p\| \leq 2\phi^{-1}(a_0)$ $\forall n \geq n_0$.

The proof is by induction. Clearly, the claim holds for $n = n_0$. Suppose it holds for some $n \geq n_0$, i.e., $\|x_n - p\| \leq 2\phi^{-1}(a_0)$. We prove that $\|x_{n+1} - p\| \leq 2\phi^{-1}(a_0)$. Suppose that this is not true. Then $\|x_{n+1} - p\| > 2\phi^{-1}(a_0)$, so that $\phi(\|x_{n+1} - p\|) > \phi(2\phi^{-1}(a_0))$. Using the recursion formula (1), we have the following estimates

$$\begin{aligned}\|x_n - T_2^n x_n\| &\leq \|x_n - p\| + \|T_2^n x_n - p\| \\ &\leq (1+L)\|x_n - p\| \\ &\leq 2(1+L)\phi^{-1}(a_0),\end{aligned}$$

$$\begin{aligned}\|y_n - p\| &= \|(1-\beta_n)x_n + \beta_n T_2^n x_n - p\| \\ &= \|x_n - p - \beta_n(x_n - T_2^n x_n)\| \\ &\leq \|x_n - p\| + \beta_n\|x_n - T_2^n x_n\| \\ &\leq 2\phi^{-1}(a_0) + 2(1+L)\phi^{-1}(a_0)\beta_n \\ &\leq 3\phi^{-1}(a_0),\end{aligned}$$

$$\begin{aligned}\|x_n - T_1^n y_n\| &\leq \|x_n - p\| + \|T_1^n y_n - p\| \\ &\leq \|x_n - p\| + L\|y_n - p\| \\ &\leq 2\phi^{-1}(a_0) + 3L\phi^{-1}(a_0) \\ &= (2+3L)\phi^{-1}(a_0),\end{aligned}$$

$$\begin{aligned}\|x_{n+1}-p\| &= \|\left(1-\alpha_n\right)x_n+\alpha_n T_1^n y_n-p\| \\ &= \|x_n-p-\alpha_n(x_n-T_1^n y_n)\| \\ &\le \|x_n-p\|+\alpha_n\|x_n-T_1^n y_n\| \\ &\le 2\phi^{-1}(a_0)+(2+3L)\phi^{-1}(a_0)\alpha_n \\ &\le 3\phi^{-1}(a_0). \end{aligned} \tag{3}$$

With these estimates and again using the recursion formula (1), we obtain by Lemma 2.1 that

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\left(1-\alpha_n\right)x_n+\alpha_n T_1^n y_n-p\|^2 \\ &= \|x_n-p-\alpha_n(x_n-T_1^n y_n)\|^2 \\ &\le \|x_n-p\|^2-2\alpha_n\langle x_n-T_1^n y_n, j(x_{n+1}-p)\rangle \\ &= \|x_n-p\|^2+2\alpha_n\langle T_1^n x_{n+1}-p, j(x_{n+1}-p)\rangle \\ &\quad -2\alpha_n\langle x_{n+1}-p, j(x_{n+1}-p)\rangle \\ &\quad +2\alpha_n\langle T_1^n y_n-T_1^n x_{n+1}, j(x_{n+1}-p)\rangle \\ &\quad +2\alpha_n\langle x_{n+1}-x_n, j(x_{n+1}-p)\rangle \\ &\le \|x_n-p\|^2+2\alpha_n\Big(k_n||x_{n+1}-p||^2-\phi(||x_{n+1}-p||)\Big) \\ &\quad -2\alpha_n\left\|x_{n+1}-p\right\|^2+2\alpha_n\left\|T_1^n y_n-T_1^n x_{n+1}\right\|||x_{n+1}-p|| \\ &\quad +2\alpha_n\left\|x_{n+1}-x_n\right\|||x_{n+1}-p|| \\ &\le \|x_n-p\|^2+2\alpha_n(k_n-1)||x_{n+1}-p||^2-2\alpha_n\phi(||x_{n+1}-p||) \\ &\quad +2\alpha_n L\left\|y_n-x_{n+1}\right\|||x_{n+1}-p|| \\ &\quad +2\alpha_n\left\|x_{n+1}-x_n\right\|||x_{n+1}-p||, \end{aligned} \tag{4}$$

where

$$\begin{aligned}\|y_n-x_{n+1}\| &\le \|y_n-x_n\|+\|x_{n+1}-x_n\| \\ &= \beta_n\|x_n-T_2^n x_n\|+\alpha_n\|x_n-T_1^n y_n\| \\ &\le 2(1+L)\phi^{-1}(a_0)\beta_n+(2+3L)\phi^{-1}(a_0)\alpha_n. \end{aligned} \tag{5}$$

Substituting (5) in (4), we get

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \|x_n-p\|^2-2\alpha_n\phi(||x_{n+1}-p||) \\ &\quad +2\alpha_n(k_n-1)||x_{n+1}-p||^2 \\ &\quad +4L(1+L)\phi^{-1}(a_0)\alpha_n\beta_n||x_{n+1}-p|| \\ &\quad +2(1+L)(2+3L)\phi^{-1}(a_0)\alpha_n^2||x_{n+1}-p|| \qquad (6) \\ &\le \|x_n-p\|^2-2\alpha_n\phi(2\phi^{-1}(a_0)) \\ &\quad +18\left[\phi^{-1}(a_0)\right]^2\alpha_n(k_n-1) \end{aligned}$$

$$
\begin{aligned}
&+12L(1+L)\left[\phi^{-1}(a_0)\right]^2\alpha_n\beta_n\\
&+6(1+L)(2+3L)\left[\phi^{-1}(a_0)\right]^2\alpha_n^2\\
\leq\;& \|x_n-p\|^2-2\alpha_n\phi(2\phi^{-1}(a_0))+\alpha_n\phi(2\phi^{-1}(a_0))\\
=\;& \|x_n-p\|^2-\alpha_n\phi(2\phi^{-1}(a_0)).
\end{aligned}
$$

Thus

$$\alpha_n\phi(2\phi^{-1}(a_0))\leq\|x_n-p\|^2-\|x_{n+1}-p\|^2,$$

implies

$$
\begin{aligned}
\phi(2\phi^{-1}(a_0))\sum_{n=n_0}^{j}\alpha_n &\leq \sum_{n=n_0}^{j}(\|x_n-p\|^2-\|x_{n+1}-p\|^2)\\
&= \|x_{n_0}-p\|^2,
\end{aligned}
$$

so that as $j\to\infty$ we have

$$\phi(2\phi^{-1}(a_0))\sum_{n=n_0}^{\infty}\alpha_n\leq\|x_{n_0}-p\|^2<\infty,$$

which implies that $\sum\alpha_n<\infty$, a contradiction. Hence, $\|x_{n+1}-p\|\leq 2\phi^{-1}(a_0)$; thus $\{x_n\}$ is bounded.

Now from (6), we get

$$
\begin{aligned}
\|x_{n+1}-p\|^2 \leq\;& ||x_n-p||^2-2\alpha_n\phi(\|x_{n+1}-p\|)\\
&+8\left[\phi^{-1}(a_0)\right]^2\alpha_n(k_n-1)\\
&+8L(1+L)\left[\phi^{-1}(a_0)\right]^2\alpha_n\beta_n\\
&+4(1+L)(2+3L)\left[\phi^{-1}(a_0)\right]^2\alpha_n^2\\
=\;& ||x_n-p||^2-2\alpha_n\phi(\|x_{n+1}-p\|)\\
&+4\left[\phi^{-1}(a_0)\right]^2[2(k_n-1)+(1+L)[2L\beta_n+(2+3L)\alpha_n]]\,\alpha_n.
\end{aligned}
\tag{7}
$$

Denote

$$
\begin{aligned}
\theta_n &= ||x_n-p||,\\
\lambda_n &= 2\alpha_n,\\
\sigma_n &= 4\left[\phi^{-1}(a_0)\right]^2[2(k_n-1)+(1+L)[2L\beta_n+(2+3L)\alpha_n]]\,\alpha_n.
\end{aligned}
$$

Condition $\lim\limits_{n\to\infty}\alpha_n=0$ ensures the existence of a rank $n_0\in\mathbf{N}$ such that $\lambda_n=2\alpha_n\leq 1$, for all $n\geq n_0$. Now with the help of $\sum\limits_{n\geq 0}\alpha_n=\infty$, $\lim\limits_{n\to\infty}\alpha_n=0=$

$\lim_{n\to\infty}\beta_n$, $\lim_{n\to\infty}k_n = 1$ and Lemma 2.2, we obtain from (7) that

$$\lim_{n\to\infty}||x_n - p|| = 0,$$

completing the proof.

Remark 2.4 *1. Let* $\alpha_n = \frac{1}{n^\sigma}$; $0 < \sigma < \frac{1}{2}$, *then* $\sum \alpha_n = \infty$, *but also* $\sum \alpha_n^2 = \infty$. *Hence the results of Theorems 1.9-1.10 are not true in general.*

2. The same argument can be applied for the results of Chang et al. [1] and of Chidume and Chidume [5].

3 Open Problem

We propose that the results of Theorem 2.3 to be extended for the case of three mappings.

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