Int. J. Open Problems Compt. Math., Vol. 2, No. 3, September 2009
ISSN 1998-6262;
www.i-csrs.org

Sharp bounds on the mathematical constant e

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Abstract

In this work, we construct sharp upper and lower bounds for the Euler constant e. For obtaining these bounds, we use Lobotto and Gauss-Legendre quadrature rules.

Keywords: *Mathematical constant, Euler constant, Lobotto, Gauss-Legendre, Quadrature, Bounds, quadrature.*

The number e is one of the most indispensable numbers in mathematics. This number is also referred to as Euler's number or Napier's constant. In this work, we develop sharp upper and lower bounds for this number. We derive these bounds from the Lobatto and Gauss-Legendre quadrature rules. Classically the number e is defined as follows:

$$e \stackrel{\textbf{def}}{=} \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \tag{1}$$

[see 1–4; 9, and references there in]. Figure 1 presents a graph of the function $1/x$. The area under the graph, and between the vertical lines $x = n$ and $x = n + 1$ is given by the integral:

$$\int_n^{n+1} \frac{1}{x}\,\mathrm{d}x$$

For obtaining upper and lower bounds for the number e, we approximate this integral by quadrature rules. Lobatto quadrature is used for forming a lower bound for the number e. While, Gauss-Legendre quadrature is used for forming a upper bound for the number e. The exact value of this integral is $\ln(1 + 1/n)$.

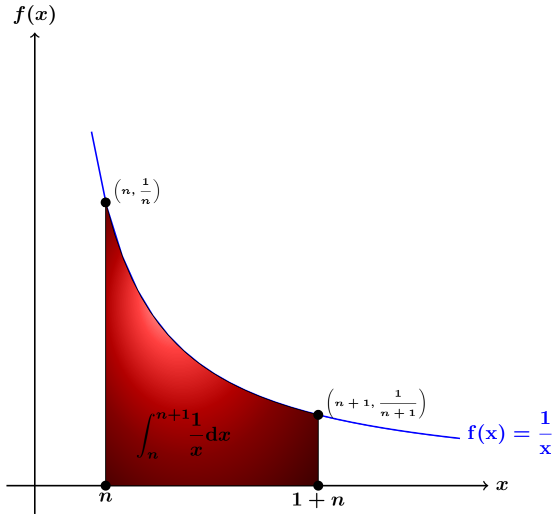

Figure 1: Graph of $f(x) = \dfrac{1}{x}$. The shaded area is equal to $\ln(1 + {}^1\!/\!_n)$.

Let us briefly discuss about the Lobatto quadrature [5–7]. Integral of a function $f(x)$ between the limits a and b through n points Lobatto quadrature is given as:

$$\int_a^b f(x)\,\mathrm{d}x = k\sum_{i=1}^{n} \omega_i\, f(c + k\, x_i) - \mathcal{E} \tag{2}$$

Here, ω_i, x_i and $\mathcal{E}$ are weights, abscissa and error of the quadrature, respectively. The error is given as:

$$\mathcal{E} = \frac{n(n-1)^3 2^{2n-1}\left[(n-2)!\right]^4}{(2n-1)\left[(2n-2)!\right]^3} f^{(2n-2)}(\xi) \tag{3}$$

Here, $\xi \in [a, b]$. For the function $f(x) = {}^1\!/\!_x$, the even order derivative $f^{(2n-2)}(\xi)$ is:

$$f^{(2n-2)}(\xi) = \frac{(2n-2)!}{\xi^{2n-1}}$$

and which is strictly positive for all $\xi > 0$. Thus error is positive for a positive interval of integration. Consequently for positive interval of integration the equation (2) results in the following inequality:

$$\int_a^b \frac{1}{x}\,\mathrm{d}x < k\sum_{i=1}^{n} \omega_i\, f(c + k\, x_i) \tag{4}$$

The constants k and c are defined from a and b as follows:

$$c = \frac{a+b}{2} \qquad \text{and} \qquad k = \frac{b-a}{2}$$

For our purpose $a = n$ and $b = n + 1$ (see figure 1), thus:

$$c = \frac{2n+1}{2} \qquad \text{and} \qquad k = \frac{1}{2}$$

For Lobatto quadrature, boundary abscissas are fixed. Thus,

$$x_1 = n \quad \text{and} \quad x_n = n + 1$$

The free abscissas x_i for $i = 2, 3, \ldots, n-1$ are the roots of $P'_{n-1}(x)$. Here, $P_n(x)$ is a Legendre polynomial of degree n [7]. We are using the **Maple** software package for finding the free abscissa [7] through the following commands:

1. First we specify the Legendre polynomial $P_n(x)$ of degree n:

```
Pn  := simplify(LegendreP(n,x));
```

2. Then we find the derivative $P'_n(x)$ of the above polynomial:

```
dPn := diff(Pn,x);
```

3. Finally the free abscissas x_i are obtained by solving $P'_n(x) = 0$:

```
solve(dPn=0,x);
```

The weights of the free abscissas are given as:

$$\omega_i = \frac{2}{n\,(n-1)P^2_{n-1}(x_i)}$$

while the weights for the fixed abscissas are:

$$\omega_i = \frac{2}{n\,(n-1)}$$

Let us now find a lower bound for the number e. For seven point Lobotto quadrature rule, the weights and abscissa for the interval of integation (see figure 1) are given as follows:

$$w_1 = \frac{256}{525} \qquad w_2 = \frac{43293}{175\,(3+7\sqrt{15})^2} \qquad w_3 = \frac{43293}{175\,(3+7\sqrt{15})^2}$$

$$w_4 = \frac{43293}{175\,(-3+7\sqrt{15})^2} \qquad w_5 = \frac{43293}{175\,(-3+7\sqrt{15})^2} \qquad w_6 = \frac{1}{21} \quad \text{and} \quad w_7 = \frac{1}{21}$$

$$x_1 = 0 \qquad x_2 = -\frac{\sqrt{495+66\sqrt{15}}}{33} \qquad x_3 = \frac{\sqrt{495+66\sqrt{15}}}{33}$$

$$x_4 = -\frac{\sqrt{495-66\sqrt{15}}}{33} \qquad x_5 = \frac{\sqrt{495-66\sqrt{15}}}{33} \qquad x_6 = 1 \quad \textbf{and} \quad x_7 = -1$$

Substituting these weights and abscissa in the Lobatto quadrature inequality (4) and replacing the left hand side by the exact integral gives:

$$\ln\left(1+\frac{1}{n}\right) < \frac{27720\,n^6 + 83160\,n^5 + 93030\,n^4 + 47460\,n^3 + 10689\,n^2 + 819\,n + 5}{27720\,n^7 + 97020\,n^6 + 132300\,n^5 + 88200\,n^4 + 29400\,n^3 + 4410\,n^2 + 210\,n}$$

$$\ln\left(1+\frac{1}{n}\right) < \frac{1}{n}\left[\frac{1}{1+\dfrac{13860\,n^5 + 39270\,n^4 + 40740\,n^3 + 18711\,n^2 + 3591\,n + 205}{27720\,n^6 + 83160\,n^5 + 93030\,n^4 + 47460\,n^3 + 10689\,n^2 + 819\,n + 5}}\right]$$

$$1 > \ln\left(1+\frac{1}{n}\right)\, n\, \left[1+\frac{13860\,n^5 + 39270\,n^4 + 40740\,n^3 + 18711\,n^2 + 3591\,n + 205}{27720\,n^6 + 83160\,n^5 + 93030\,n^4 + 47460\,n^3 + 10689\,n^2 + 819\,n + 5}\right]$$

Now using the property: $a > \ln b \Rightarrow e^a > b$, we get:

$$e > \left(1+\frac{1}{n}\right)^{n\left[1+\frac{13860\,n^5 + 39270\,n^4 + 40740\,n^3 + 18711\,n^2 + 3591\,n + 205}{27720\,n^6 + 83160\,n^5 + 93030\,n^4 + 47460\,n^3 + 10689\,n^2 + 819\,n + 5}\right]}$$

This is our lower bound for the number e.

For $n = 100$, the right hand side of the above inequality gives

2.71828182845904523536028747135239335

which is e accurate to thirty one decimal places.

For deriving an upper bound, we use the Gauss-Legendre quadrature. Let us now discuss about the Gauss-Legendre quadrature [8]. Integral of a function $f(x)$ between the limits a and b through n points Gauss-Legendre quadrature is given as follows:

$$\int_a^b f(x)\,\mathrm{d}x = k \sum_{i=1}^{n} \omega_i\, f(x_i) + \mathcal{E} \tag{5}$$

Here, ω_i, x_i and $\mathcal{E}$ are weights, abscissa and error of the quadrature, respectively. The error is given as:

$$\mathcal{E} = \frac{2^{2n+1}\,(n!)^4}{(2n+1)\;[(2n!)]^3}\, f^{2n}(\xi) \tag{6}$$

Here, $\xi \in [a, b]$. For the function $f(x) = {}^1\!/x$, $f^{(2n)}(\xi)$ is given as follows:

$$f^{(2n)}(\xi) = \frac{(2n)!}{\xi^{2n+1}}$$

and which is strictly positive for all $\xi > 0$. Thus the error is positive for a positive interval of integration. Consequently for a positive interval of integration, equation (5) results in the following inequality:

$$\int_a^b \frac{1}{x}\,\mathrm{d}x > k \sum_{i=1}^{n} \omega_i\, f(x_i) \tag{7}$$

Weights and abscissa for a five point Gauss-Legendre quadrature are taken from the literature [8]. The weights w_i and the points x_i are given as follows:

$$\begin{aligned}
w_1 &= \frac{128}{225} & x_1 &= n + \frac{1}{2}\\
w_2 &= \frac{161}{450} + \frac{13}{900}\sqrt{70} & x_2 &= n + \frac{1}{2} + \frac{1}{42}\sqrt{245 - 14\sqrt{70}}\\
w_3 &= \frac{161}{450} + \frac{13}{900}\sqrt{70} & x_3 &= n + \frac{1}{2} - \frac{1}{42}\sqrt{245 - 14\sqrt{70}}\\
w_4 &= \frac{161}{450} + \frac{13}{900}\sqrt{70} & x_4 &= n + \frac{1}{2} + \frac{1}{42}\sqrt{245 + 14\sqrt{70}}\\
w_5 &= \frac{161}{450} - \frac{13}{900}\sqrt{70} & x_5 &= n + \frac{1}{2} - \frac{1}{42}\sqrt{245 + 14\sqrt{70}}
\end{aligned}$$

Substituting these weights and abscissa in the Gauss-Legendre inequality (7) gives:

$$\ln\left(1+\frac{1}{n}\right) > \frac{7560\,n^4+15120\,n^3+9870\,n^2+2310\,n+137}{7560\,n^5+18900\,n^4+16800\,n^3+6300\,n^2+900\,n+30}$$
$$> \frac{1}{n}\left[\frac{1}{1+\frac{3780\,n^3+6930\,n^2+3990\,n+763+30\,n^{-1}}{7560\,n^4+15120\,n^3+9870\,n^2+2310\,n+137}}\right]$$

$$\Rightarrow e < \left(1+\frac{1}{n}\right)^{n\left[1+\frac{3780\,n^3+6930\,n^2+3990\,n+763+30\,n^{-1}}{7560\,n^4+15120\,n^3+9870\,n^2+2310\,n+137}\right]} \tag{8}$$

Which is our upper bound for the number e. For $n = 100$ the right hand side of the above inequality gives:

2.718281828459045235360287508375

and which is e accurate to twenty five decimal places.

Open Problem and Suggestions

The number e is one of the most fundamental numbers in mathematics. The number e is irrational. Thus it is not a ratio of integers. And, it is transcendental. Thus it is not a root of any polynomial with integer coefficients. It is not known whether the following numbers are transcendental:

$$e^e \quad \text{and} \quad \pi^e$$

Classically the number e is defined as [see 1–4; 9, and references there in]:

$$e \stackrel{\textbf{def}}{=} \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \tag{9}$$

Let us define the number e through the limit:

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n\left[1+\frac{13860\,n^5+39270\,n^4+40740\,n^3+18711\,n^2+3591\,n+205}{27720\,n^6+83160\,n^5+93030\,n^4+47460\,n^3+10689\,n^2+819\,n+5}\right]} \tag{10}$$

Let us see the motivation behind the above result. Let us compute e from these two definitions for $n = 100$. From the classical definition, we get $e =$ 2.704813829. Which is accurate only till 2 decimal places. While from the new definition (10), we get

$$2.71828182845904523536028747135239335$$

which is e accurate to thirty one decimal places.

ACKNOWLEDGEMENTS. The author thanks the referee and editor for useful comments.

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