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Generalized Derivations and C^*-Algebras: Comments and Some Open Problems

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Abstract

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{L}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. For $A, B \in \mathcal{L}(\mathcal{H})$, let $\delta_{A,B} : \mathcal{L}(\mathcal{H}) \mapsto \mathcal{L}(\mathcal{H})$ be the generalized derivation defined by $\delta_{A,B}(X) = AX - XB$. In this paper we will present some generalized finite operators and we discuss new C^*-algebras generated by generalized finite pairs of operators (A, B). Some open questions are also given.

Keywords: Generalized finite operator, Generalized derivation, C^*-algebra.

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1 Introduction

Let $\mathcal{H}$ be a separable infinite dimensional complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. Let $A, B \in \mathcal{L}(\mathcal{H})$ we define the generalized derivation $\delta_{A,B} : \mathcal{L}(\mathcal{H}) \mapsto \mathcal{L}(\mathcal{H})$ by

$$\delta_{A,B}(X) = AX - XB,$$

we note $\delta_{A,A} = \delta_A$. We say that the operator $A \in \mathcal{L}(\mathcal{H})$ is finite if $||I - (AX - XA)|| \geq 1$ (*) for all $X \in \mathcal{L}(\mathcal{H})$. J.P.Williams [9] has shown that the class of finite operators, $\mathcal{F}$, contains every normal, hyponormal operators. In [5], J.P.Williams results are generalized to more classes of operators containing the classes of normal and hyponormal operators. The well-known inequality (*) due to J.P.Williams [9] is the starting point of the topic of commutator approximation (a topic which has its roots in quantum theory [11]). This topic deals with minimizing the distance, measured by some norms or other, between a varying commutator $XX^* - X^*X$ and some fixed operator [7]. In [5] the author initiates the study of a more general class of finite operators defined by

$$F(\mathcal{H}) = \{(A, B) \in \mathcal{L}(\mathcal{H}) \times \mathcal{L}(\mathcal{H}) : \ ||I - (AX - XB)|| \geq 1\}$$

for all $X \in \mathcal{L}(\mathcal{H})$.

Such operators are called generalized finite operators. Recall that the class of generalized finite operators contains a large classes of operators containing the classes of normal and hyponormal operators [5]. In this paper we present some generalized finite operators. We also discuss some new C^*-algebras generated by some generalized finite pairs (A, B) of operators. Some open questions are also given.

2 Preliminaries

We begin by the following definitions which will be used for the sequel.

Definition 2.1. *A state f on a complex Banach Algebra $\mathcal{A}$, with identity e, is a functional, i.e, $f \in \mathcal{A}'$; the dual space such that $f(e) = 1 = \|f\|$. The set of all such functionals is denoted by $\mathcal{P}(\mathcal{A})$.*

Definition 2.2. *If $X \in \mathcal{A}$, then the numerical range of X is the set:*

$$W_\circ(X) = \{f(X) : f \in P(\mathcal{A})\}$$

We use the notation $W_\circ(A, \mathcal{A}) = \{f(A) : f \in \mathcal{P}(\mathcal{A})\} = W_\circ(A)$.

It is known that [10] if $\mathcal{A} = \mathcal{L}(\mathcal{H})$, then $W_\circ(A) = \overline{W(A)}$, where

$$W(A) = \{(Ah, h) : h \in H, \|h\| = 1\}.$$

Definition 2.3. *A linear form is said to be positive on a complex involution algebra $\mathcal{A}$, if $f(x^*x) \geq 0, \forall x \in \mathcal{A}$*

Definition 2.4. *If f is a linear form of $\mathcal{A}$, then the adjoint of f is the linear form f^* defined by $f^*(x) = \overline{f(x^*)}, \forall x \in \mathcal{A}$.*

Definition 2.5. *A linear form f is A-central if*

$$f(R(\delta_A)) = f(AX - XA) = 0, \forall X \in \mathcal{L}(\mathcal{H})$$

Theorem 2.1. *[5] $\mathcal{GF}(\mathcal{A})$ is norm closed in $\times \mathcal{A}$.*

Theorem 2.2. *[5] For $a, b \in$ the following statements are equivalent*
(i) $\|ax - xb - e\| \geq 1$ for all $x \in$.
(ii) There exists a state f such that $f(ax) = f(xb)$, for all $x \in$.
(iii) $0 \in W_0(ax - xb)$, $\forall x \in$.

3 Main results

The ideal $C_1(\mathcal{H})$ of $\mathcal{L}(\mathcal{H})$ admits a trace function $tr(T)$, given by $tr(T) = \sum_n \langle Te_n, e_n \rangle$ for any complete orthonormal system $\{e_n\}$ in $\mathcal{H}$. As a Banach space $C_1(\mathcal{H})$ can be identified with the dual of ideal $K(\mathcal{H})$ of compact operators by means of the linear isometry $T \mapsto f_T$, where $f_T(X) = tr(XT)$. Moreover $\mathcal{L}(\mathcal{H})$ is the dual of $C_1(\mathcal{H})$. The ultraweakly continuous linear functionals on $\mathcal{L}(\mathcal{H})$ are those of the form f_T for $T \in C_1(\mathcal{H})$ and the weakly continuous ones are those of the form f_T with T is of finite rank (see J.B. Conway [3]).

In the following theorem we will give a generalized finite pair (A, B) of operators.

Theorem 3.1. *If A and B are roots of a quadratic polynomial, then the pair (A, B) is generalized finite.*

Proof. Assume that (A, B) is not generalized finite. Then there exists
$X \in \mathcal{L}(\mathcal{H})$ such that $0 \notin \overline{W(AX - XB)}$. According to the definition of $W_\circ(A)$ we have,

$$W_\circ(A) = \{f(A) : f \in \mathcal{A}', \|f\| = f(I) = 1\}.$$

Since f is a linear form and $e^{i\theta} \in \mathbb{C}$, we conclude that $f(e^{i\theta}A) = e^{i\theta}f(A)$. This gives

$$W_\circ(AX - XB) = e^{-i\theta}W_\circ(e^{i\theta}(AX - XB).$$

By using a suitable rotation we can assume that

$$\overline{W(AX - XB)} \subset H_+ = \{z \in \mathbb{C} : Re(z) > 0\}.$$

Define

$$\tau : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$$

to be

$$\tau(C) = (AX - XB)C + C(AX - XB).$$

Since $\sigma(\tau) \subset \sigma(AX - XB) + \sigma(AX - XB) \subset H_+$, the operator τ is invertible [8]. If $(A - \alpha)(A - \beta) = 0$ and $(B - \alpha)(B - \beta) = 0$ for certain $\alpha\,\beta \in \mathbb{C}$, then $-\alpha\beta = A^2 - (\alpha + \beta)A$, $-\alpha\beta = B^2 - (\alpha + \beta)B$ and we have $\tau(A + B -$

$(\alpha+\beta)) = (AX-XB)(A+B-(\alpha+\beta)) + (A+B-(\alpha+\beta))(AX-XB)$ $= (A^2-(\alpha+\beta)A)X - X(A^2-(\alpha+\beta)A) + (B^2-(\alpha+\beta)B)X - X(B^2-(\alpha+\beta)B)$ $= -\alpha\beta+\alpha\beta-\alpha\beta+\alpha\beta = 0$. Since $A+B-(\alpha+\beta)\neq 0$, this contradicts the fact that τ is invertible. Hence the pair (A,B) is generalized finite. □

In the following theorem we will generalize a result given by J.P.Williams [10, Corollary 2]

Theorem 3.2. *Let $A,B\in\mathcal{L}(\mathcal{H})$. Then the following statements are equivalent*
(i) $ker(\delta_{B,A})$ contains a non-null compact normal operator.
(ii)$\rho=[C^\star(A,B),\mathcal{L}(\mathcal{H})\times\mathcal{L}(\mathcal{H})]$ is not dense for the ultra weak topology in $\mathcal{L}(\mathcal{H})$.

Proof. $(i)\Leftrightarrow(ii)$. It is known that ρ is w^*-dense in $\mathcal{L}(\mathcal{H})$ if and only if there exists a non null linear form, w^*-continuous and self-adjoint such that $f_T|_\rho=0$, that is, $f_T|_{R(\delta_{A,B})}=0$ and T is a finite self-adjoint trace operator. This completes the proof. □

Now we will recall the following well known theorems.

Theorem 3.3. *[2] Let $A\in\mathcal{L}(\mathcal{H})$. If there is a linear form $f\in\mathcal{L}(\mathcal{H})'$, which is normal, self-adjoint and A-central, that is, $f(R(\delta_A))=0$, then A is finite.*

Theorem 3.4. *[2] If $A\in\mathcal{L}(\mathcal{H})$, then the following statements are equivalent :*
(1) A is finite.
(2) $\rho=[C^\star(A),\mathcal{L}(\mathcal{H})]$ is not norm dense in $\mathcal{L}(\mathcal{H})$
(3)The vector space generated by $(R(\delta_A)\cup R(\delta_{A^\star}))$ is not norm dense.

In the proof of Theorem 3.3 and Theorem 3.4 the author has used the fact that $\beta(f)=\{B\in\mathcal{L}(\mathcal{H}) : f(BX)=f(XB),\forall X\in\mathcal{L}(\mathcal{H})\}$ is a $C^\star$- algebra containing $C^\star(B)$.

A reasonable question is the following:

Question 3.1. *Does $\alpha(f)=\{(A,B)\in\mathcal{L}(\mathcal{H})\times\mathcal{L}(\mathcal{H}) : f(AX)=f(XB),\forall X\in\mathcal{L}(\mathcal{H})\}$ is a $C^\star$- algebra?*

The following example shows that the answer is negative

Example 3.1. *Let A be the unilateral shift on $\mathcal{L}(\mathcal{H})$, defined by $Ae_n=e_{n+1},\forall n\geq 1$, where $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{L}(\mathcal{H})$. Let B be the projection on the space spanned by e_2, thus $Be_i=0,\forall i\neq 2$.*

We have $A^\star e_n=e_{n-1},\forall n\geq 2$ and $A^\star e_1=0$. Since B is a projection, we have $B^\star=B$. Thus B is self adjoint.

Define the functional $f\in\mathcal{L}(\mathcal{H})^{'}$, to be

$$f(X)=(Xe_1,e_1),\forall X\in\mathcal{L}(\mathcal{H}).$$

We have,

(i) Since

$$f^\star(X) = \overline{f(X^\star)} = \overline{(X^\star e_1, e_1)} = (Xe_1, e_1) = f(X),$$

f is self-adjoint

(ii) Since $f(I) = \|f\| = 1$, f is a state. Now consider

$$f(AX) = (AXe_1, e_1) = (Xe_1, A^\star e_1) = (Xe_1, 0) = 0$$

and

$$f(XB) = (XBe_1, e_1) = (0, e_1) = 0.$$

Thus $f(AX) = f(XB), \forall X \in \mathcal{L}(\mathcal{H})$. On the other hand, we have

$$f(A^\star A) = (A^\star Ae_1, e_1) = (Ae_1, Ae_1) = (e_2, e_2) = 1.$$

But $f(AB^\star) = (AB^\star e_1, e_1) = (0, e_1) = 0$. Thus $f(A^\star X) \neq f(XB^\star)$. Hence $(A^\star, B^\star)$ is not in $\alpha(f)$. This means that $\alpha(f)$ is not a $C^\star$-algebra.

4 Open Questions

Since $\alpha(f)$ is not in general a $C^\star$-algebra, we can ask the following question

Question 4.1. *For which $A, B \in \mathcal{L}(\mathcal{H})$, $\alpha(f)$ is a $C^\star$-algebra ?*

By the same techniques used in the proof of Theorem 3.3 and Theorem 3.4 and according to Example 3.1, we can not generalize these theorems for the case of generalized finite operators. Since we don't know more about properties of generalized finite operators, it is interesting to develop the study of such pairs of operators and to generalize the mentioned theorems for pairs (A, B) of operators in $\mathcal{L}(\mathcal{H})$. So, we can pose the following questions.

Question 4.2. *Let $A, B \in \mathcal{L}(\mathcal{H})$. If there is a linear form $f \in \mathcal{L}(\mathcal{H})'$, which is normal, self-adjoint and $f(R(\delta_{A,B})) = 0$. Does the pair (A, B) generalized finite ?*

Question 4.3. *Let $A, B \in \mathcal{L}(\mathcal{H})$. Does the following statements equivalent:*

(1) (A, B) is generalized finite.

(2) $\rho = [C^\star(A, B), \mathcal{L}(\mathcal{H}) \times \mathcal{L}(\mathcal{H})]$ is not norm dense

(3)The vector space generated by $(R(\delta_{A,B}) \cup R(\delta_{A^\star,B^\star}))$ is not norm dense?

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References

[1] J.H.Anderson, J.W.Bunce, J.A.Deddens and J.P.Williams, C^*-algebras and derivation ranges, Acta sci. Math.(Szeged) 40(1978), 211 - 227.

[2] J.W. Bunce, Finite operators and amenable C^*-algebras, Proc. Amer. Math. Soc, 66(1976), 149-151.

[3] J.B.Conway, Subnormal operators, Putnam Advanced Publishing Program, Boston, London, Melbourne, 1981.

[4] S. Mecheri, On the range of elementary operators, Integr. equ. oper. theory, 53(2005), 403-409 .

[5] S.Mecheri, *Finite operators*, Demonstratio Math, **35**(2002), 355-366.

[6] S.Mecheri, *Non-normal derivations and orthogonality*, Proc. Amer. Math. Soc, **133**(2005), 759-762.

[7] P.J.Maher, *Self commutator approximants*, Proc. Amer. Math. Soc, **134**(2006), 157-165.

[8] M.Rosenblum, On the operator equation $AX - XB = Q$, Duke Math. J, 23(1956)263-269.

[9] J.P.Williams, *Finite operators*, Proc. Amer. Math. Soc, **26**(1970), 129-135.

[10] J.P.Williams, On the range of derivation, Pacific.J.Math., 38(1971), 273-279.

[11] H. Wielandt, *ber die Unbeschrnktheit der Operatoren der Quantenmechanik*(German), Math. Ann, **121**(1949), 21.