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# Nonstandard Numerical Integrations of A Lotka-Volterra System

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**Abstract**

In this article, we consider a three dimensional Lotka-Volterra system. We have developed some nonstandard numerical integrations of the model which preserve all properties of real solutions, and they are consistent. We have shown some numerical results to support this methods.

**Keywords:** *Lotka-Voltera model, predator and prey system, nonstandard integration, approximate solution, system of differential equation, iterative methods, trajectory.*

# 1 Introduction

Predator prey models are one of the most important, interesting and challenging models in biological systems. These are very exiting to model, analyze and understand substantially. There are a lot of mathematical models to represent predator prey interactions, one of the most important models is Lotka-Volterra (L-V) representing predator and prey system, for detail discussion please have a look on [1, 2, 4, 5, 3] and references there in. In general, for all $i = 1, 2, \cdots, n$, this system can be written as

$$\frac{dx_i}{dt} = x_i \left( c_i - \sum_{j=1}^{n} \gamma_{ij} x_j \right) \tag{1}$$

where $x_i$, $i = 1, 2, \cdots, n$, represents population of different predator or prey, $\gamma_{ij}$ represents interference coefficient between $x_i$ and $x_j$, $c_i$ represents the individual growth rates of each species and $n$ represents number of species involved with the model. System of the form (1) is fully non-linear, but has an important advantage of getting fixed point of the system pretty easily as $\frac{dx}{dt} = 0$ gives a system of $n$ linear equations with $n$ unknowns. Specially, in [1], author discussed the system with different aspect of stability issues. They discussed stability of such multi-species monomorphic and polymorphic models. In [4, 7], authors introduced an nonstandard integration scheme for a special normalized system, whereas in the book [6], the author discussed various standard and nonstandard integration schemes for differential equations.

In particular, we have considered the following 3-dimensional system

$$\left.\begin{array}{rcl} \frac{dx}{dt} &=& Ax - Bxy \\ \frac{dy}{dt} &=& -Cy + Dxy - Eyz \\ \frac{dz}{dt} &=& -Fz + Gyz \end{array}\right\} \tag{2}$$

where $A$, $B$, $C$, $D$, $E$, $F$, and $G$ are positive constants. The system (2) has periodic solutions for all $(x_0, y_0, z_0) > (0, 0, 0)$ except we start with fixed points $(0, 0, 0)$ or $(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{C}{D}, \frac{A}{B}, 0\right)$. Our goal is to find an explicit iterative technique for (2) so that we don't need to solve a system of nonlinear equations in each iterations. To be specific we develop the two dimensional system approximated in [5] to three dimensional system introducing a third species $z$ to the system where species $z$ is a predator that feeds exclusively on the predator population $y$. We extended Micken's method [5] and also introduced some other unconventional integration schemes for the three dimensional system (2).

In this short article, we show that nonstandard integration schemes can be applied and extended for $n \geq 2$ dimensional Lotka-Volterra system introduced in [5] and references there in. We show that all nonstandard integration schemes that we introduced can easily generate solutions that are periodic. We also show that solutions form a closed trajectories in $(y, x)$ and $(y, z)$ phase plane in all the cases.

Here is the roadmap of the article. We start with introducing various nonstandard finite differences schemes for (2), and then we have shown some numerical results with various choice of $(x_0, y_0, z_0)$ and parameters, and we finish discussing results and future research direction and extension.

## 2 Numerical integration schemes

In this study, we have numerically integrated the 3 dimensional system

$$\frac{dx}{dt} = Ax - Bxy \tag{3}$$

$$\frac{dy}{dt} = -Cy + Dxy - Eyz \tag{4}$$

$$\frac{dz}{dt} = -Fz + Gyz \tag{5}$$

with some nonstandard schemes. Here we introduced three different schemes for the system (3)-(5). To approximate solutions in $[0, t]$, we use step size $h = \Delta t$ so that $t_i = ih$ where $i = 1 : N$ with $N = \frac{t}{h}$. In [6], author discussed various choice of replacing standard finite difference integrator by some nonstandard one keeping structure and properties of approximation correct. Using the fact that $\lim_{h\to 0} \sin h = h$, $\lim_{h\to 0} \sinh h = h$, $\lim_{h\to 0} \tan h = h$ one can replace $h$ by $\phi = \sin h$, $\phi = \sinh h$, $\phi = \tan h$ or some other function with the same properties which gives approximations as

$$\frac{dx}{dt} \to \frac{x_{i+1} - x_i}{\phi},$$

$$\frac{dy}{dt} \to \frac{y_{i+1} - y_i}{\phi},$$

$$\frac{dz}{dt} \to \frac{z_{i+1} - z_i}{\phi}.$$

We approximate $x$, $y$, $z$, $xy$ and $yz$ for each of (3) - (5) differently. We also use upgraded approximate solutions $x_{i+1}$ and $y_{i+1}$ in each iterations.

## 2.1 Case 1

In (3), we replace

$$x = 2x - x = 2x_i - x_{i+1},$$

$$xy = x_{i+1}y_i.$$

We replace all terms of right hand side of (4) by

$$y = -y + 2y \to -y_i + 2y_{i+1}$$

$$xy = 2xy - xy \to 2x_{i+1}y_i - x_iy_{i+1}$$

and

$$yz = 2yz - yz \to 2y_{i+1}z_i - y_iz_i.$$

We replace all terms of right hand side of (5) by

$$z = 2z - z \to 2z_{i+1} - z_i$$

$$yz = 2yz - yz \to 2y_{i+1}z_i - y_iz_{i+1}.$$

With the representations above the Lotka-Volterra system can be written as

$$\frac{x_{i+1}-x_i}{\phi} = A\left(2x_i - x_{i+1}\right) - Bx_{i+1}y_i,$$

$$\frac{y_{i+1}-y_i}{\phi} = -C\left(-y_i + 2y_{i+1}\right) + D\left(2x_{i+1}y_i - x_iy_{i+1}\right) - E\left(2y_{i+1}z_i - y_iz_i\right),$$

and

$$\frac{z_{i+1}-z_i}{\phi} = -F\left(2z_{i+1} - z_i\right) + G\left(2y_{i+1}z_i - y_iz_{i+1}\right)$$

which is equivalent to

$$x_{i+1} = \frac{1+A\phi}{1+2\phi(A+By_i)}x_i,$$

$$y_{i+1} = \frac{1+C\phi+2\phi Dx_{i+1}+E\phi z_i}{1+2C\phi+D\phi x_i+2\phi Ez_i}y_i, \quad \text{and}$$

$$z_{i+1} = \frac{1+\phi F+2G\phi y_{i+1}}{1+2\phi F+G\phi y_i}z_i$$

gives explicit expression in terms of $x_i$, $y_i$, $z_i$ if we use $x_{i+1}$ and $y_{i+1}$ as needed in $y_{i+1}$ and $z_{i+1}$ respectively.

## 2.2 Case 2

We approximate and represent $x_{i+1}$ in terms of $x_i$, $y_i$ with the same scheme of **Case 1** and replace $Cy \to Cy_{i+1}$ and $Fz \to Fz_{i+1}$ in (4) - (5) and keeping $xy$, $yz$ same as approximated in **Case 1** for (4) - (5). With the above representation we get

$$\frac{y_{i+1}-y_i}{\phi} = -Cy_{i+1} + D\left(2x_{i+1}y_i - x_iy_{i+1}\right) - E\left(2y_{i+1}z_i - y_iz_i\right),$$

and

$$\frac{z_{i+1}-z_i}{\phi} = -Fz_{i+1} + G\left(2y_{i+1}z_i - y_iz_{i+1}\right).$$

Above expressions are equivalent to

$$x_{i+1} = \frac{1+A\phi}{1+2\phi(A+By_i)}x_i,$$

$$y_{i+1} = \frac{1+2\phi Dx_{i+1}+\phi Ez_i}{1+C\phi+D\phi x_i+2E\phi z_i}y_i,$$

$$z_{i+1} = \frac{1+2G\phi y_{i+1}}{1+F\phi+G\phi y_i}z_i.$$

## 2.3 Case 3

We keep all terms of (3) by same approximation as before and replace

$$y \to y_{i+1},$$

$$xy = 2xy - xy \to 2x_{i+1}y_i - x_{i+1}y_{i+1},$$

$$yz \to y_{i+1}z_i$$

in (4) and

$$z \to z_{i+1},$$

$$yz = 2yz - yz \to 2y_{i+1}z_i - y_{i+1}z_{i+1}$$

in (5). With above expressions of real values we get

$$\frac{y_{i+1} - y_i}{\phi} = -Cy_{i+1} + D\left(2x_{i+1}y_i - x_{i+1}y_{i+1}\right) - Ey_{i+1}z_i,$$

and

$$\frac{z_{i+1} - z_i}{\phi} = -Fz_{i+1} + G\left(2y_{i+1}z_i - y_{i+1}z_{i+1}\right)$$

which gives

$$x_{i+1} = \frac{1 + A\phi}{1 + 2\phi(A + By_i)}x_i,$$

$$y_{i+1} = \frac{1 + 2D\phi x_{i+1}}{1 + C\phi + D\phi x_{i+1} + E\phi z_i}y_i,$$

and

$$z_{i+1} = \frac{1 + 2G\phi y_{i+1}}{1 + F\phi + G\phi y_{i+1}}z_i.$$

To get a numerical result for the integration schemes discussed above we start with different choice of $(x_0, y_0, z_0)$ and then one can get $(x_i, y_i, z_i)$ successively. Using any of the approximation schemes presented above and replacing $x_{i+1}$, $y_{i+1}$ as needed one can easily get an explicit expression of $x_{i+1}$, $y_{i+1}$, and $z_{i+1}$ in terms of $(x_i, y_i, z_i)$ as

$$x_{i+1} = \mathcal{G}_1(x_i, y_i, z_i)$$

$$y_{i+1} = \mathcal{G}_2(x_i, y_i, z_i)$$

$$z_{i+1} = \mathcal{G}_3(x_i, y_i, z_i)$$

which generates solutions for each choice of $x_0 > 0$, $y_0 > 0$ and $z_0 > 0$ and a suitable choice of $h = \Delta t > 0$ and $\Delta t \to 0$.

# 3 Results

The 3-dimensional Lotka-Volterra system (3) - (5) has been numerically integrated by three different schemes respectively with various choice of parameter values, initial function and step size $\Delta t$ to get solutions $(x_i, y_i, z_i)$ at $t_i = i\Delta t$, for $1 \leq i \leq N$. In all three schemes we plotted $x_i$, $y_i$, $z_i$ versus i, and $x_i$, $z_i$ with respect to $y_i$. Figures 1 - 3 show results. We observe that solutions by the approximations we used are periodic and positive, and they form closed orbits, that is, solutions preserve properties of the problem modeled. Here from these experimental results shown in Figures 1 - 3 we find that schemes we developed preserve positivity condition and solutions found are bounded as well.

# 4 Conclusion and Open problem

Numerical schemes used to solve differential equations where real solution are sometimes not easily integrable and in most cases are impossible to integrate. The difficulty of an approximation scheme is to find accurate enough one so that solutions preserve same properties of the problem represented by the system. Our target was to find a scheme that can produce explicit iterations that can produce solutions faster than solutions obtaining from any implicit techniques or Newton's type method for nonlinear system. Here from the schemes introduced above we find that they preserve periodicity which is one of the most important property of the model and form closed orbit in $(y, x)$, $(y, z)$ plane showing the dependence of species. There are number of standard finite difference schemes that form solutions near to the actual solution which don't form close orbits whereas our scheme described above and extended version of scheme [5] form close trajectories in all phase space. We find solutions are nonnegative and bounded.

The scheme developed here can be extended to more general 3-dimensional system

$$\left.\begin{array}{rl} \frac{dx}{dt} = & x\,(a_1 - b_1x - c_1y - d_1z) \\ \frac{dy}{dt} = & y\,(a_2 - b_2x - c_2y - d_2z) \\ \frac{dz}{dt} = & z\,(a_3 - b_3x - c_3y - d_3z) \end{array}\right\}$$

where parameters $a_i, b_i, c_i, d_i$ can be with positive or negation sign, and also nonstandard integration scheme can be updated to more general model (1) which is our next research goal.

The purpose this study was to extend and develop some nonstandard integration technique for the 3-dimensional Lotka-Volterra system (2) so that approximate solutions preserve properties of real one. From the explicit and linearized approximations shown in 3 different cases we observe that they produce solutions very easily, correctly, and they preserve properties of the real system. From the figures we find that solutions have the correct dynamic properties, getting closed orbits on the phase planes. Here solutions are non-negative follows from the schemes straightly and boundedness follows from the numerical results shown in Figures 1 - 3. Here in the integration techniques introduced we find difficulty of using $\Delta t$. Here to get correct behaviour and accurate enough solutions integrated by the methods introduced above one needs to choose small enough $\Delta t > 0$.

Our future research target is to develop and analyze more sophisticated techniques to solve such type of problems, also to extend these techniques for a general n-dimensional system, prove non-negativity and boundedness, find order of accuracy of solutions by such nonstandard integration schemes.

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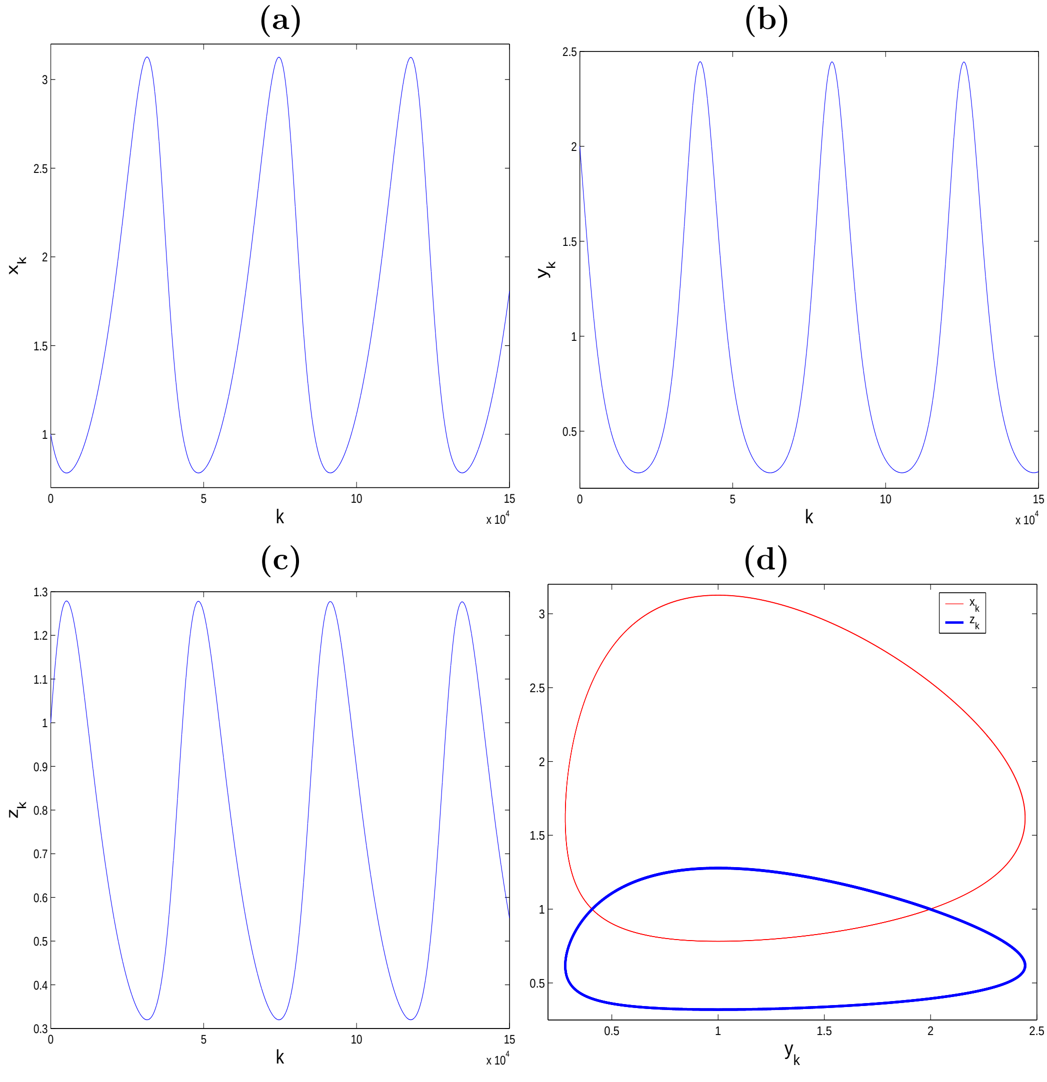


Figure 1: Solution with approximation scheme 1 with $A = B = C = D = E = F = G = 1$, $x(0) = 1$, $y(0) = 2$, $z(0) = 1$ and $\Delta t = 0.0001$.

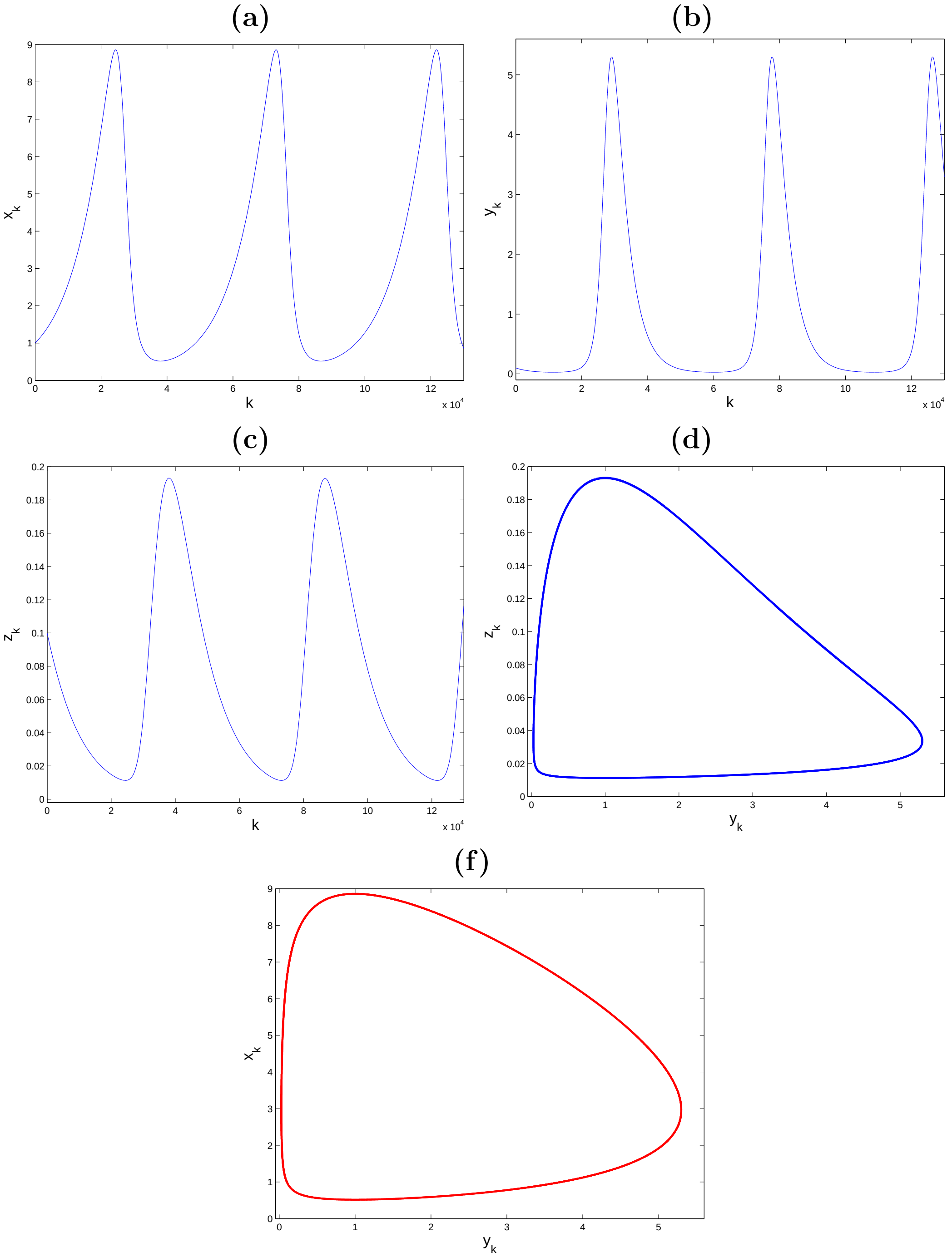


Figure 2: Solution with approximation scheme 2 with $A = B = C = D = E = F = G = 1$, $x(0) = 1$, $y(0) = .1$, $z(0) = .1$ and $\Delta t = 0.0001$.

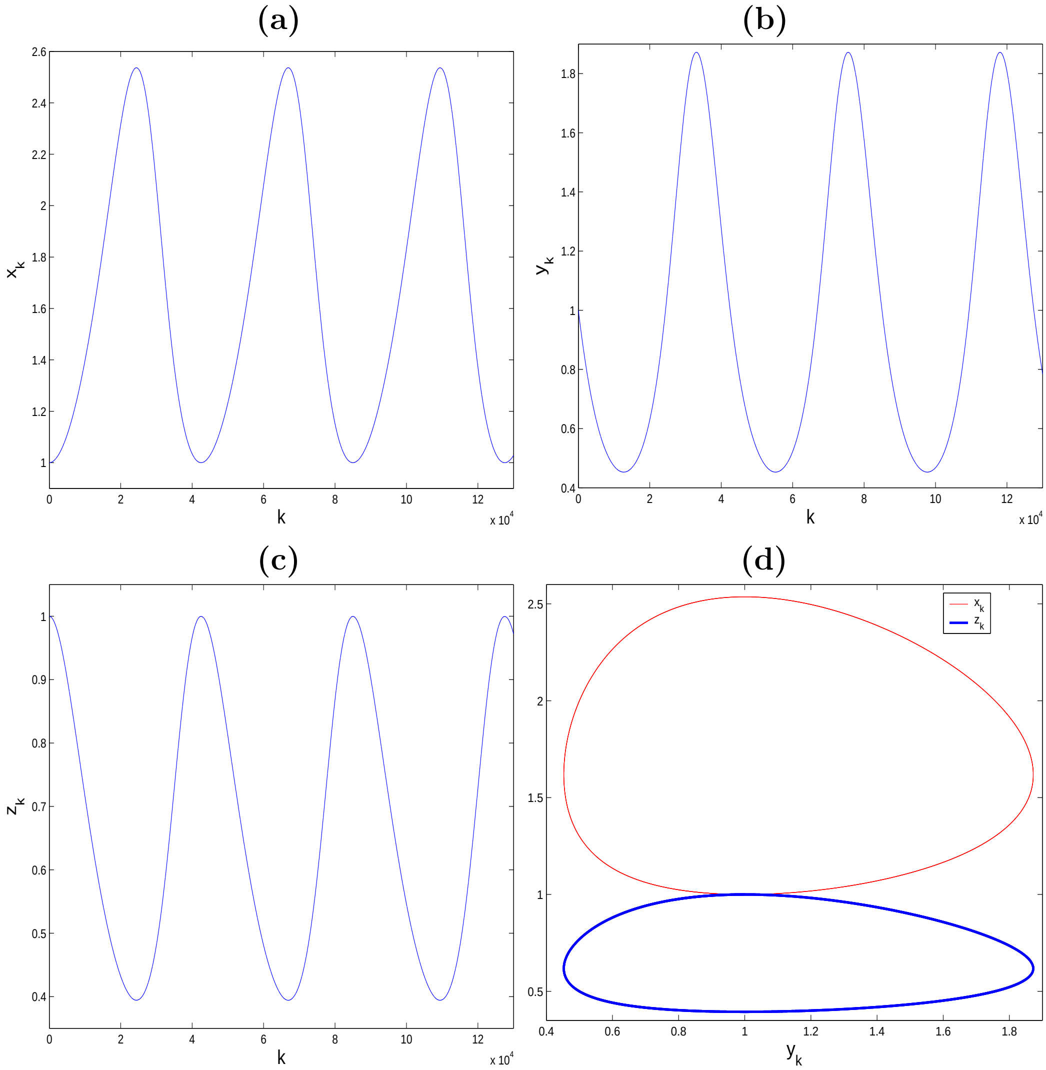


Figure 3: Solution with approximation scheme 3 with $A = B = C = D = E = F = G = 1$, $x(0) = 1$, $y(0) = 1$, $z(0) = 1$ and $\Delta t = 0.0001$.