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Steady-state Solution of the PTC Thermistor Problem Using a Quadratic Spline Finite Element Method

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The problem of heat transfer in a Positive Temperature Coefficient (PTC) thermistor, which may form one element of an electric circuit, is solved numerically by a finite element method. The approach used is based on Galerkin finite element using quadratic splines as shape functions. The resulting system of ordinary differential equations is solved by the finite difference method. Comparison is made with numerical and analytical solutions and the accuracy of the computed solutions indicates that the method is well suited for the solution of the PTC thermistor problem.

Keywords: PTC thermistor; Finite elements; Quadratic splines

AMS Subject Classification: 65N99, 80.65

1 INTRODUCTION: THE PROBLEM STATEMENT

Positive Temperature Coefficient (PTC) thermistors prepared from doped semiconducting BaTiO$_3$ ceramics are characterised by an increase in the electrical resistance with temperature. PTC thermistors can be used in various kinds of devices such as temperature sensors and current regulation [1–4].

This study deals with a dimensionless mathematical model of the PTC thermistor as follows:

Heat-flow is governed by the equation,

$$u_t = u_{xx} + \gamma \sigma(u) |\Phi_x|^2, \quad 0 < x < 1, \quad t > 0 \tag{1}$$

subject to boundary and initial conditions

$$u_x = 0, \quad x = 0, \quad t > 0 \tag{2}$$

$$u_x + \beta u = 0, \quad x = 1, \quad t > 0 \tag{3}$$

$$u(x, 0) = 0, \quad 0 \le x \le 1 \tag{4}$$

where u is the temperature, Φ is the electrical potential, γ is the ratio of the rate of heat production in the thermistor and β is the surface heat transfer coefficient.

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The electrical potential is governed by the equation

$$(\sigma(u)\Phi_x)_x = 0, \quad 0 < x < 1, \quad t > 0 \tag{5}$$

subject to boundary and initial conditions

$$\Phi(0, t) = 0, \quad t > 0 \tag{6}$$

$$\Phi(1, t) = 1, \quad t > 0 \tag{7}$$

$$\Phi(x, 0) = 0, \quad 0 \le x \le 1 \tag{8}$$

where the electrical conductivity $\sigma(u)$ was taken as

$$\sigma(u) = 1 + (\delta - 1)s \tag{9}$$

where s is the location of the interface and a typical values of δ is about 10^{-5}.

This problem has attracted attention recently both from the theoretical and experimental point of view. The numerical solution of the problem has been the subject of many papers over the last decade [5–14]. The engineering approach to the problem can be found in [15–17] and references quoted therein. The present work covers the numerical and analytical steady-state solution of the problem.

2 THE EXACT STEADY-STATE SOLUTIONS

Since the initial temperature of the device is equal to 0 and $\beta > 0$, assume monotonicity of the temperature profile so that the point $x = 0$ will always be the hottest and will also be the first point to reach the critical temperature $u_c = 1$. Finally, the rate of heat loss at $x = 1$ will equal the rate of internal heat generation and equilibrium will be attained. So, the steady state may be one of the three scenarios shown in Figure 1 [12].
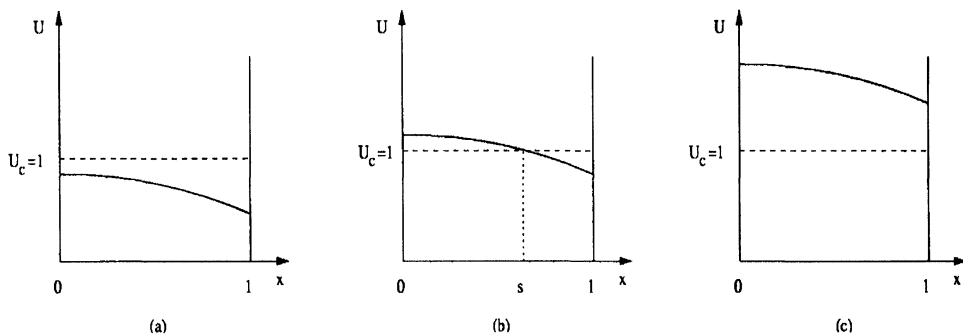

FIGURE 1   The steady-state configurations: (a) cold phase; (b) warm phase; (c) hot phase.

By virtue of Eq. (9) the evolving problem defined by Eqs. (1)–(4) will move sequentially through the cold, warm and hot phases. The values of β and γ defined the particular steady-state for cold, warm and hot phases, are [10]

$$\left(\frac{1}{\beta} + \frac{1}{2}\right) \leq \frac{1}{\gamma} \tag{10}$$

$$\gamma\delta < \beta < \gamma\left(1 + \frac{\beta}{2}\right) \tag{11}$$

$$\beta < \gamma\delta. \tag{12}$$

For each one of the cold, warm and hot phases Eqs. (5)–(8) have the exact solution

$$\Phi(x, t) = x, \quad 0 \leq x \leq 1, \quad t > 0. \tag{13}$$

The solution of the Eqs. (1)–(4) for cold and hot phases are the same as in [9].
In the cold phase $s = 0$ and so $\sigma = 1$ and the solution is

$$u(x) = \gamma\left[\frac{1}{\beta} + \frac{(1 - x^2)}{2}\right], \quad 0 \leq x \leq 1 \tag{14}$$

In the hot phase $s = 1$ and so $\sigma = \delta$ and the solution is

$$u(x) = \gamma\delta\left[\frac{1}{\beta} + \frac{(1 - x^2)}{2}\right], \quad 0 \leq x \leq 1 \tag{15}$$

In the warm phase the interface takes the values in the interval $0 < s < 1$ and thus the conductivity is $\sigma = 1 + (\delta - 1)s$. Hence, Eq. (1) can reduce the following steady-state form in the warm phase:

$$\frac{d^2u}{dx^2} + \gamma[1 + (\delta - 1)s] = 0, \quad 0 < x < 1 \tag{16}$$

subject to the conditions

$$\frac{du}{dx} = 0, \quad x = 0 \tag{17}$$

$$\frac{du}{dx} + \beta u = 0, \quad x = 1 \tag{18}$$

If we consider Eq. (11) we obtained the exact solution as

$$u(x) = \gamma[1 + (\delta - 1)s]\left[\frac{1}{\beta} + \frac{(1 - x^2)}{2}\right], \quad 0 \leq x \leq 1 \tag{19}$$

Here we need to determine the location of interface s. Since the boundary condition $u(x) = 1$ at $x = s$ we obtain a quadratic function of s

$$f(s) = as^2 + bs + c \tag{20}$$

where $a = \delta - 1$, $b = 1 - (1 + 2/\beta)(\delta - 1)$ and $c = 2/\gamma - (2/\beta) - 1$.

Using inequality (11) it can be seen that $b^2 - 4ac > 0$ *i.e.* function has two distinct roots. However, only one of them, which is $s = (-b + \sqrt{b^2 - 4ac})/2a$, in $(0, 1)$ since $-b/2a > 1$, $f(0) = c < 0$ and $f(1) = a + b + c > 0$.

3   THE QUADRATIC SPLINE FINITE ELEMENT SOLUTION

In the numerical solution of the problem, Galerkin approach is used with quadratic spline finite elements. For the construction of a finite-element solution of the problem we begin denoting the coordinates of the node points on the interval $[0, 1]$ by x_i, where $0 = x_1 < \cdots < x_{N-1} < x_N = 1$.

Let $\varphi(x)$ be shape function satisfying the boundary conditions and is a linear combination of the $N + 1$ shape functions expressed as

$$\varphi(x) = \sum_{i=1}^{N+1} a_i \varphi_i(x) \tag{21}$$

where a_i's are arbitrary real numbers and $\varphi_i(x)$ are the quadratic splines defined by

$$\varphi_i(x) = \frac{1}{h^2} \begin{cases} (x - x_{i-1})^2, & [x_{i-1}, x_i] \\ h^2 + 2h(x - x_i) - (x - x_i)^2, & [x_i, x_{i+1}] \\ [h - (x - x_{i+1})]^2, & [x_{i+1}, x_{i+2}] \\ 0, & \text{otherwise} \end{cases} \tag{22}$$

where $h = x_{i+1} - x_i$.

If the heat equation (1) is multiplied by $\varphi(x)$ and integrating over $[0, 1]$ with respect to x and utilizing the boundary conditions, we obtained the weak or variational form of the problem as

$$\int_0^1 \left(\varphi(x)u_t + \varphi_x u_x - \gamma\sigma(u)\varphi(x) \right) dx = -\varphi(1)\beta u(1, t) \tag{23}$$

Assuming that the temperature $u(x, t)$ can be approximated by

$$\sum_{i=1}^{N+1} b_i(t)\varphi_i(x) \tag{24}$$

where the functions $b_i(t)$'s are as yet undetermined differentiable functions of time.

Substituting (21) and (24) into (23) yields a system of linear first-order ordinary differential equations for unknown vector $\underline{b}(t)$ which can be written in matrix form as

$$C\frac{d\underline{b}}{dt} + K\underline{b} = \underline{d} \tag{25}$$

where we have set

$$b_i = b_i(t)$$

$$C_{ij} = \int_0^1 \varphi_i(x)\varphi_j(x)\,dx$$

$$K_{ij} = \int_0^1 \varphi_i'(x)\varphi_j'(x)\,dx + \beta\varphi_i(1)\varphi_j(1)$$

$$d_i = \gamma\sigma(u)\int_0^1 \varphi_i(x)\,dx$$

and where $1 \leq i, j \leq N + 1$.

Therefore, the system of initial-value problem described by (25) must be solved for each time level.

We can solve this equation system by a finite-difference method, using the forward difference for $d\underline{b}/dt$ and the weight average for $\underline{b}^{(m)}$. Hence, Eq. (25) takes the form

$$(C + \theta\Delta tK)\underline{b}^{(m+1)} = (C - \Delta t(1 - \theta)K)\underline{b}^{(m)} + \Delta t\underline{d} \tag{26}$$

in which $\underline{b}^{(m)} \cong \underline{b}(m\Delta t)$, $m = 0, 1, 2, \ldots$. Initially, $\underline{b}^{(0)}$ can be determined from the initial condition (4) and $0 \leq \theta \leq 1$ is the weight factor. The cases $\theta = 0$, $\theta = 0.5$ and $\theta = 1$ correspond, respectively, to the explicit, Crank–Nicolson and fully implicit finite difference methods.

4 NUMERICAL RESULTS AND DISCUSSION

In the numerical solution in the warm phase location of the interface was estimated in the following manner [13]. The predicted temperature at a node point is such that at a time level $j + 1$, $U_i^{j+1} > 1$ and $U_{i+1}^{j+1} < 1$. Thus moving boundary s is between adjoining values U_i^{j+1} and U_{i+1}^{j+1}. Since $u(x, t) = 1$ at $x = s(t)$, the location of the interface s can be estimated by using a linear interpolation as

$$s^j = x_i + \frac{1 - U_i^{j+1}}{U_i^{j+1} - U_{i+1}^{j+1}}(x_i - x_{i+1}).$$

All numerical computations were performed with the weight factor $\theta = 0.5$. In order to obtain the steady-state solutions we use the criteria

$$\frac{1}{\Delta t}\max\left\{\left|U_0^{(m+1)} - U_0^{(m)}\right|, \left|U_N^{(m+1)} - U_N^{(m)}\right|\right\} \leq 0.5 \times 10^{-8}, \quad m = 0, 1, 2, \ldots$$

for the method in which $U_i^{(m)}$ denotes the nodal value $u(ih, m\Delta t)$.

The values of β and γ were chosen satisfying the inequalities (10)–(12) for cold, warm and hot phases, respectively. $\beta = 0.2$ was taken for all phases, but for cold phase $\gamma = 0.1$, for warm phase $\gamma = 0.5$ and for hot phase $\gamma = 100\,000$.

It can be seen from the Table I that there is an excellent agreement between the numerical solution and the exact solution for cold and hot phases. In the warm phase, there is little difference between two solutions since the location of interface is calculated numerically. It is

TABLE I Numerical and exact solutions at steady-state ($\Delta x = 0.025$ and $\Delta t = 0.001$)

x	Cold phase (exact/numerical)	Warm phase		Hot phase (exact/numerical)
		Exact	Numerical	
0.0	0.55000	1.05920	1.03659	5.50000
0.1	0.54950	1.05824	1.03565	5.49500
0.2	0.54800	1.05535	1.03282	5.48000
0.3	0.54550	1.05054	1.02811	5.45500
0.4	0.54200	1.04380	1.02151	5.42000
0.5	0.53750	1.03513	1.01303	5.37500
0.6	0.53200	1.02454	1.00266	5.32000
0.7	0.52550	1.01202	0.99041	5.25500
0.8	0.51800	0.99758	0.97628	5.18000
0.9	0.50950	0.98121	0.96026	5.09500
1.0	0.50000	0.96291	0.94235	5.00000

TABLE II Steady-state times for cold phase

h	$\Delta t = 0.1$	$\Delta t = 0.01$
0.1	90.0	89.9
0.05	90.0	89.9
0.025	222.0	89.9
0.0125	249.0	89.9
0.01	223.6	89.9

TABLE III Steady-state times for hot phase

h	$\Delta t = 0.1$	$\Delta t = 0.01$
0.1	102.3	102.2
0.05	150.3	102.2
0.025	420.5	102.2
0.0125	963.8	102.2
0.01	1148.8	102.2

TABLE IV Steady-state times for warm phase

h	$\Delta t = 0.1$	$\Delta t = 0.01$	$\Delta t = 0.001$
0.1	48.5	6.3	6.4
0.05	145.4	6.2	6.3
0.025	398.2	6.2	6.3
0.0125	872.7	14.7	6.3
0.01	10008.6	19.5	6.3

also observed that, the time step $\Delta t = 0.1$ is not enough to obtain the stable steady-state times for different values of h for all three phases. In the hot and cold phases, if the time step is reduced to $\Delta t = 0.01$ we obtain the same steady-state times for the different values of h as seen in Tables II and III, but for the warm phase we needed to use a finer time step
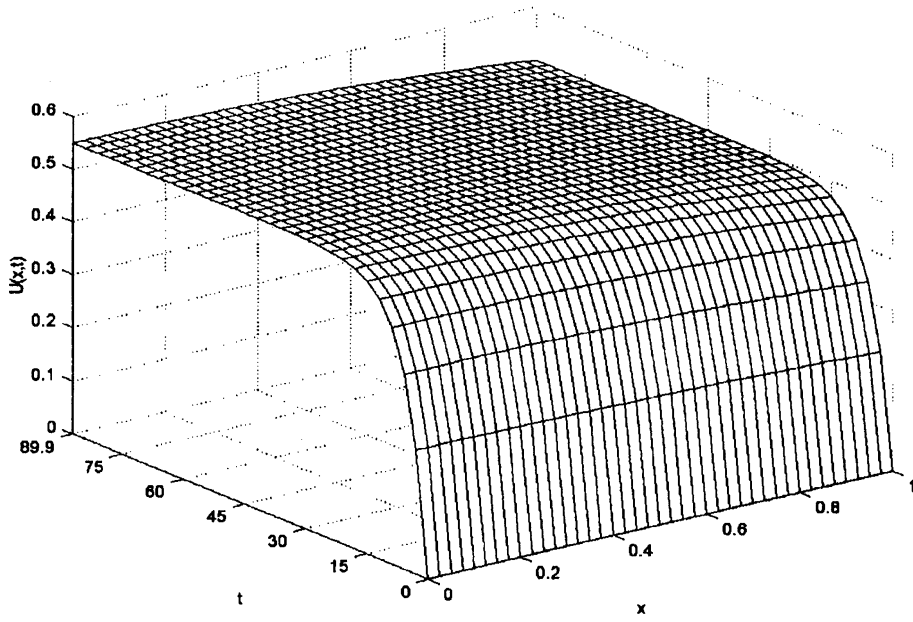
FIGURE 2   The evolution temperature in the cold phase ($\Delta x = 0.025$ and $\Delta t = 0.01$).
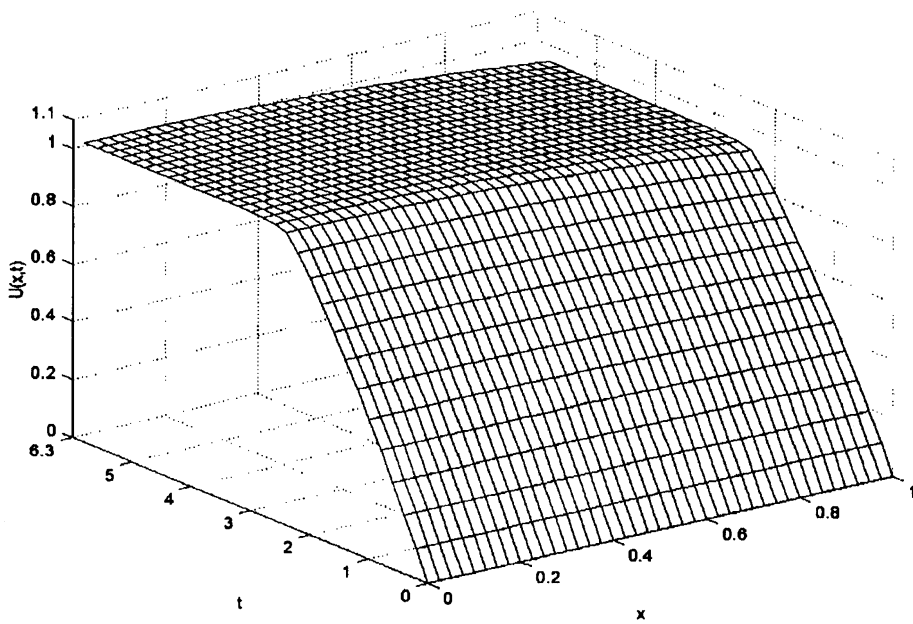

FIGURE 3   The evolution temperature in the warm phase ($\Delta x = 0.025$ and $\Delta t = 0.001$).
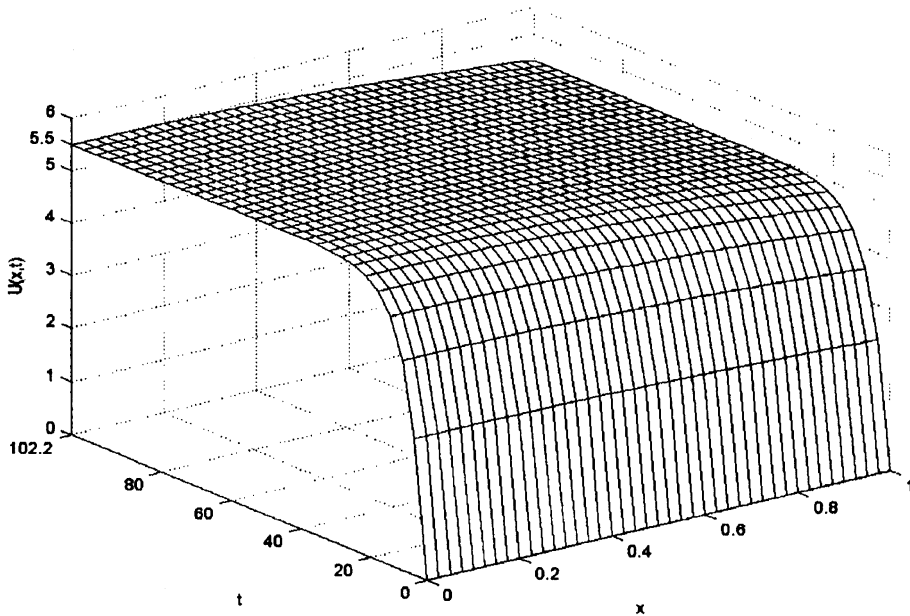
FIGURE 4   The evolution temperature in the hot phase ($\Delta x = 0.025$ and $\Delta t = 0.01$).

$\Delta t = 0.001$ as seen in Table IV. The evolving temperature profiles at the steady-state times are represented in Figures 2–4 for cold, warm and hot phases, respectively.

5   CONCLUSIONS

A finite element solution to the PTC thermistor problem has been constructed using quadratic spline spatial elements and finite difference approximation in time. The performance of this algorithm has been examined by comparing the analytical and the numerical results. It is observed that the numerical solution is shown to exhibit the correct physical characteristic of the PTC thermistor problem. Therefore, the scheme can be extended to the solution of two dimensional case of the problem.

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