Improvability of Assembly Systems II: Improvability Indicators and Case Study*

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Based on the performance analysis technique developed in Part I, this paper presents improvability indicators for assembly lines with unreliable machines. In particular, it shows that assembly lines are unimprovable with respect to workforce re-distribution if each buffer is, on the average, close to being half full. These lines are unimprovable with respect to buffer capacity re-distribution if each machine is starved and blocked with almost equal frequency. In addition, the paper provides indicators for identification of bottleneck machines and bottleneck buffers. Finally, the paper reports on an application of these improvability indicators in a case study at an automotive components plant.

Keywords: Improvability; Assembly system; Bottleneck

1   INTRODUCTION

This work is devoted to the study of a system-theoretic property of assembly systems, referred to as improvability. The structure of the assembly system under consideration is shown in Fig. 1, where the
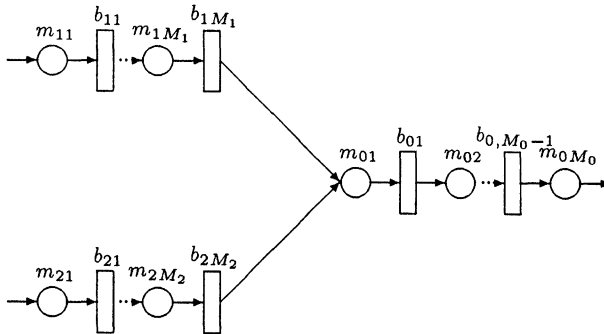

FIGURE 1   Assembly system.

circles represent the machines and the rectangles are the buffers. Such systems are widely used in large volume manufacturing environment, e.g., the automotive industry. Often, these systems produce at the level of 60–70% of their capacity, and therefore improvement of their performance is an important problem of production management. This justifies the utility of the problem of improvability from the practical point of view. From the theoretical perspective, this problem is also of importance since current literature offers no quantitative engineering methods for design of continuous improvement projects. This paper is intended to contribute to this end.

The notion of improvability was introduced in [1] in the context of serial production lines. Several applications were reported in [2–4]. Assembly systems have been addressed in [5] but only for three machines lines. The current work is devoted to the general case.

This work is being published in two parts, of which the current paper is the second. The first one, [6], has been devoted to the problem formulation and performance analysis. In the current paper, we use the results of [6] to derive improvability indicators, which are simple rules that can be used to design a continuous improvement project, and to report on their applications at an automotive component plant.

For the sake of completeness, we cite below the assumptions on the assembly system introduced in [6] and considered throughout this work:

(i) The system consists of component machines, m_{ij}, $i = 1, 2$, $j = 1, \ldots, M_i$, an assembly machine, m_{01}, additional processing

machines, $m_{0j}, j = 2, \ldots, M_0$, and buffers, $b_{ij}, i = 1, 2, j = 1, \ldots, M_i$, and $b_{0j}, j = 1, \ldots, M_0 - 1$, storing the parts produced by m_{ij}, respectively.

(ii) All the machines have an identical cycle time, T; the time axis is slotted with the slot duration T.

(iii) Each machine is characterized by the probability, $p_{ij}, i = 0, 1, 2, j = 1, \ldots, M_i$, to produce a part during a time slot; these probabilities are referred to as machines' *isolation production rate*.

(iv) Each buffer is characterized by its capacity, $N_{ij}, i = 1, 2, j = 1, \ldots, M_i$ and $N_{0j}, j = 1, \ldots, M_0 - 1$; the buffers capacity is assumed to be finite.

(v) Machine m_{ij} (except m_{01}) is starved during a time slot if buffer $b_{i,j-1}$ is empty at the beginning of this time slot. The assembly machine m_{01} is starved for parts, if at least one of the buffers $b_{iM_i}, i = 1, 2$, is empty at the beginning of this time slot. The first two component machines $m_{i1}, i = 1, 2$, are never starved.

(vi) Machine m_{ij} (except $m_{iM_i}, i = 1, 2$) is blocked during a time slot if buffer b_{ij} has N_{ij} parts at the beginning of this time slot and machine $m_{i,j+1}$ fails to take a part during this time slot. Machine $m_{iM_i}, i = 1, 2$, is blocked during a time slot if buffer b_{iM_i} is full and the assembly machine, m_{01}, fails to take parts from the buffers at the beginning of this time slot. Machine m_{0M_0} is never blocked.

Given this assembly system its performance measures of interest are:

- *Production rate*, $\widetilde{PR} = \widetilde{PR}(p_1, p_2, p_0, N_1, N_2, N_0)$, i.e., the average number of parts produced by the last machine, m_{0M_0}, per cycle time. Here p_i and $N_i, i = 0, 1, 2$, are vectors with components $[p_{i1}, \ldots, p_{iM_i}]$, and $[N_{i1}, \ldots, N_{iM_i}]$ (for $i = 0$, $N_0 = [N_{01}, \ldots, N_{0,M_0-1}]$), respectively.
- *Average steady state buffer occupancy*, $E[\tilde{h}_{ij}]$, $\forall ij \neq 0M_0$.
- *Probabilities of manufacturing starvations and blockages* of the machines,

$$\widetilde{mb}_{ij} = \mathrm{Prob}(\{m_{ij} \text{ is up during a time slot}\}$$

$$\cap \{b_{ij} \text{ is full at the beginning of this slot}\}$$

$$\cap \{\text{the immediate downstream machine of } m_{ij} \text{ fails to take}$$

$$\text{a part from } b_{ij} \text{ at the beginning of this slot}\}),$$

$$\forall ij \neq 0M_0, \tag{1.1}$$

$$\widetilde{ms}_{ij} = \text{Prob}(\{m_{ij} \text{ is up during a time slot}\}$$
$$\cap \{b_{i,j-1} \text{ is empty at the beginning of this slot}\}),$$
$$\forall ij \neq 11, 21, 01,$$
$$\widetilde{ms}_{01_i} = \text{Prob}(\{m_{01} \text{ is up during a time slot}\}$$
$$\cap \{b_{iM_i} \text{ is empty at the beginning of this slot}\}),$$
$$i = 1, 2. \tag{1.2}$$

These performance measures have been evaluated in [6], using the following recursive procedure:

$$\mu_{M_1+1}(s+1) = p_{01}[1 - X_{M_2''}(0, s)],$$
$$\mu_i^b(s+1) = \mu_i(s+1)[1 - Q(\mu_{i+1}^b(s+1), \mu_i^f(s+1), \Gamma_i)],$$
$$1 \leq i \leq M_1 + M_0 - 1,$$
$$\mu_i^f(s+1) = \mu_i(s+1)[1 - Q(\mu_{i-1}^f(s+1), \mu_i^b(s+1), \Gamma_{i-1})],$$
$$2 \leq i \leq M_1 + M_0,$$
$$X_{M_1'}(0, s+1) = Q(\mu_{M_1}^f(s+1), \mu_{M_1+1}^b(s+1), \Gamma_{M_1}),$$
$$\nu_{M_2+1}(s+1) = p_{01}[1 - X_{M_1'}(0, s+1)],$$
$$\nu_i^b(s+1) = \nu_i(s+1)[1 - Q(\nu_{i+1}^b(s+1), \nu_i^f(s+1), \Lambda_i)], \tag{1.3}$$
$$1 \leq i \leq M_2 + M_0 - 1,$$
$$\nu_i^f(s+1) = \nu_i(s+1)[1 - Q(\nu_{i-1}^f(s+1), \nu_i^b(s+1), \Lambda_{i-1})],$$
$$2 \leq i \leq M_2 + M_0,$$
$$X_{M_2''}(0, s+1) = Q(\nu_{M_2}^f(s+1), \nu_{M_2+1}^b(s+1), \Lambda_{M_2}),$$
$$\mu_i(s+1) = \mu_i, \quad i = 1, \ldots, M_1, M_1 + 2, \ldots, M_1 + M_0,$$
$$\nu_i(s+1) = \nu_i, \quad i = 1, \ldots, M_2, M_2 + 2, \ldots, M_2 + M_0,$$
$$s = 0, 1, 2, 3, \ldots,$$

with the initial conditions

$$X_{M_2''}(0, 0) = 0,$$

where

$$Q(x, y, N) = \begin{cases} \dfrac{(1-x)(1-\alpha)}{1 - (x/y)\alpha^N}, & x \neq y, \quad \alpha = \dfrac{x(1-y)}{y(1-x)}. \\ \dfrac{1-x}{N+1-x}, & x = y, \end{cases}$$

and

$$\mu_i = \begin{cases} p_{1i}, & i = 1, \ldots, M_1, \\ p_{0,i-M_1}, & i = M_1 + 2, \ldots, M_1 + M_0, \end{cases}$$

$$\Gamma_i = \begin{cases} N_{1i}, & i = 1, \ldots, M_1, \\ N_{0,i-M_1}, & i = M_1 + 1, \ldots, M_1 + M_0 - 1, \end{cases}$$

$$\nu_i = \begin{cases} p_{2i}, & i = 1, \ldots, M_2, \\ p_{0,i-M_2}, & i = M_2 + 2, \ldots, M_2 + M_0, \end{cases}$$

$$\Lambda_i = \begin{cases} N_{2i}, & i = 1, \ldots, M_2, \\ N_{0,i-M_2}, & i = M_2 + 1, \ldots, M_2 + M_0 - 1. \end{cases}$$

It has been shown in [6] that this procedure is convergent and, in terms of its steady states, the above performance measures can be evaluated as follows:

$$PR = \mu_{M_1+M_0}^f = \nu_{M_2+M_0}^f, \tag{1.4}$$

$$E[h_{1j}] = \sum_{k=0}^{N_{1j}} kQ(\mu_j^f, \mu_{j+1}^b, N_{1j}) \frac{1}{1 - \mu_{j+1}^b} \left(\frac{\mu_j^f(1 - \mu_{j+1}^b)}{\mu_{j+1}^b(1 - \mu_j^f)} \right)^k,$$
$$j = 1, \ldots, M_1,$$

$$E[h_{2j}] = \sum_{k=0}^{N_{2j}} kQ(\nu_j^f, \nu_{j+1}^b, N_{2j}) \frac{1}{1 - \nu_{j+1}^b} \left(\frac{\nu_j^f(1 - \nu_{j+1}^b)}{\nu_{j+1}^b(1 - \nu_j^f)} \right)^k,$$
$$j = 1, \ldots, M_2,$$

$$E[h_{0j}] = \sum_{k=0}^{N_{0j}} kQ(\mu_{M_1+j}^f, \mu_{M_1+j+1}^b, N_{0j}) \frac{1}{1 - \mu_{M_1+j+1}^b} \tag{1.5}$$

$$\times \left(\frac{\mu_{M_1+j}^f(1 - \mu_{M_1+j+1}^b)}{\mu_{M_1+j+1}^b(1 - \mu_{M_1+j}^f)} \right)^k$$

$$= \sum_{k=0}^{N_{0j}} kQ(\nu_{M_2+j}^f, \nu_{M_2+j+1}^b, N_{0j}) \frac{1}{1 - \nu_{M_2+j+1}^b}$$

$$\times \left(\frac{\nu_{M_2+j}^f(1 - \nu_{M_2+j+1}^b)}{\nu_{M_2+j+1}^b(1 - \nu_{M_2+j}^f)} \right)^k,$$

$$j = 1, \ldots, M_0 - 1,$$

$$ms_{1j} = \mu_j Q(\mu_{j-1}^f, \mu_j^b, \Gamma_{j-1}), \quad j = 2, \ldots, M_1,$$

$$ms_{2j} = \nu_j Q(\nu_{j-1}^f, \nu_j^b, \Lambda_{j-1}), \quad j = 2, \ldots, M_2,$$

$$ms_{01_1} = p_{01} Q(\mu_{M_1}^f, \mu_{M_1+1}^b, \Gamma_{M_1}),$$

$$ms_{01_2} = p_{01} Q(\nu_{M_2}^f, \nu_{M_2+1}^b, \Lambda_{M_2}),$$

$$ms_{0j} = \mu_{M_1+j} Q(\mu_{M_1+j-1}^f, \mu_{M_1+j}^b, \Gamma_{M_1+j-1})$$

$$= \nu_{M_2+j} Q(\nu_{M_2+j-1}^f, \nu_{M_2+j}^b, \Lambda_{M_2+j-1}), \quad j = 2, \ldots, M_0,$$

$$mb_{1j} = \mu_j Q(\mu_{j+1}^b, \mu_j^f, \Gamma_j), \quad j = 1, \ldots, M_1,$$

$$mb_{2j} = \nu_j Q(\nu_{j+1}^b, \nu_j^f, \Lambda_j), \quad j = 1, \ldots, M_2,$$

$$mb_{0j} = \mu_{M_1+j} Q(\mu_{M_1+j+1}^b, \mu_{M_1+j}^f, \Gamma_{M_1+j})$$

$$= \nu_{M_2+j} Q(\nu_{M_2+j+1}^b, \nu_{M_2+j}^f, \Lambda_{M_2+j}), \quad j = 1, \ldots, M_0 - 1.$$

(1.6)

The accuracy of these estimates has been shown to be $\mathcal{O}(\delta)$ – close to the exact values of these quantities, i.e.,

$$\widetilde{PR} = PR + \mathcal{O}(\delta), \quad E[\tilde{h}_{ij}] = E[h_{ij}] + \mathcal{O}(\delta), \quad \forall ij \neq 0M_0,$$

$$\widetilde{ms}_{ij} = ms_{ij} + \mathcal{O}(\delta), \quad \forall ij \neq 01, \quad \widetilde{mb}_{ij} = mb_{ij} + \mathcal{O}(\delta), \quad \forall ij,$$

$$\widetilde{ms}_{01_{(k)}} = ms_{01_{(k)}} + \mathcal{O}(\delta), \quad k = 1, 2,$$

where δ is a parameter, which characterizes couplings between the machines. In [6], it has been shown numerically that $\delta \ll 1$.

Using these results, the current paper develops methods for determining whether the production rate can be increased by re-allocating buffers capacities, N_{ij}'s, and workforce, p_{ij}'s (constrained improvability) and for identifying a machine and a buffer, which are most detrimental for the system performance (unconstrained improvability). These two problems are addressed, respectively, in Sections 2 and 3 below. The application is reported in Section 4, and the conclusions are formulated in Section 5. The proofs are included in the Appendix.

Concluding this introduction, it should be pointed out that current literature offers several results on structural properties of assembly systems, related to the improvability properties addressed in this work.

Specifically, papers [7] and [8] analyze the question of work distribution between the assembly machine and the components machines

and show that less work should be assigned to the former. This could be viewed as, perhaps, the first system-theoretic property of assembly systems described in the literature. Improvability theory provides additional results in this direction. In particular, the following improvability rules have been derived in this paper:

- An assembly system is unimprovable with respect to workforce (WF) re-allocation if each of its buffers is, on the average, close to being half full (Subsection 2.1).
- An assembly system is unimprovable with respect to workforce and work-in-process (WIP) re-allocation simultaneously, if all of its buffers are of equal capacity and, on the average, close to being half full (Subsection 2.2).
- An assembly system is unimprovable with respect to WIP re-allocation, if each machine is blocked and starved with almost equal frequency (Subsection 2.3).
- If the system is unimprovable with respect to WF, the worst machine is the bottleneck machine (BN-M); otherwise, any machine, including the best one, may be the BN-M (Subsection 3.1). The "worst" and the "best" are understood here in terms of machine's production rate in isolation.
- If the frequency of the manufacturing blockage of a machine is larger than the frequency of manufacturing starvation of the immediate down-stream machine, the BN-M is downstream; otherwise, it is upstream (Subsection 3.2). For each component line, machines in the other component lines are considered down-stream.
- The bottleneck buffer (BN-B) is one of the buffers surrounding the BN-M (Subsection 3.5).

Although these rules contain fuzzy terms "almost", "close to" etc., given the "fuzziness" of the performance data available on the factory floor, they still provide useful information for design and implementation of continuous improvement projects (Section 4).

2 CONSTRAINED IMPROVABILITY

In terms of the production rate estimate (1.4), the notion of constrained improvability, introduced in [6], can be re-formulated as

follows:

- Assembly system (i)–(vi) is *improvable with respect to WIP* if there exist N_1^*, N_2^* and N_0^* such that $\sum_{i=1}^{M_1} N_{1i}^* + \sum_{i=1}^{M_2} N_{2i}^* + \sum_{i=1}^{M_0-1} N_{0i}^* = N^*$ and

$$PR(p_1, p_2, p_0, N_1^*, N_2^*, N_0^*) > PR(p_1, p_2, p_0, N_1, N_2, N_0).$$

- Assembly system (i)–(vi) is *improvable with respect to WF* if there exist vectors p_0^*, p_1^*, p_2^* such that $\prod_{i=1}^{M_1} p_{1i}^* \prod_{i=1}^{M_2} p_{2i}^* \prod_{i=1}^{M_0} p_{0i}^* = p^*$ and

$$PR(p_1^*, p_2^*, p_0^*, N_1, N_2, N_0) > PR(p_1, p_2, p_0, N_1, N_2, N_0).$$

- Assembly system (i)–(vi) is *improvable with respect to WF and WIP* simultaneously if there exist vectors p_0^*, p_1^*, p_2^* and N_1^*, N_2^*, N_0^*, such that $\prod_{i=1}^{M_1} p_{1i}^* \prod_{i=1}^{M_2} p_{2i}^* \prod_{i=1}^{M_0} p_{0i}^* = p^*$ and $\sum_{i=1}^{M_1} N_{1i}^* + \sum_{i=1}^{M_2} N_{2i}^* + \sum_{i=1}^{M_0-1} N_{0i}^* = N^*$ and

$$PR(p_1^*, p_2^*, p_0^*, N_1^*, N_2^*, N_0^*) > PR(p_1, p_2, p_0, N_1, N_2, N_0).$$

Below, conditions when each of these properties take place are given.

2.1   Improvability with Respect to Workforce

2.1.1   Improvability Conditions

THEOREM 2.1   *Assembly system (i)–(vi) is unimprovable with respect to WF re-allocation if and only if*

$$\begin{aligned} \mu_i^f &= \mu_{i+1}^b, \quad i = 1, \ldots, M_1 + M_0 - 1, \\ \nu_i^f &= \nu_{i+1}^b, \quad i = 1, \ldots, M_2 + M_0 - 1, \end{aligned} \quad (2.1)$$

where μ_i^f, μ_i^b, ν_i^f and ν_i^b are the steady states of the recursive procedure (1.3).

Proof   See the Appendix.

To reformulate this result in terms of the performance measures available on the factory floor, consider the quantities

$$\widetilde{cs}_{ij} = \frac{\widetilde{ms}_{ij}}{p_{ij}}, \quad \forall ij \neq 01,$$

$$\widetilde{cs}_{01_k} = \frac{\widetilde{ms}_{01_k}}{p_{01}}, \quad k = 1, 2, \tag{2.2}$$

$$\widetilde{cb}_{ij} = \frac{\widetilde{cb}_{ij}}{p_{ij}}, \quad \forall ij.$$

These quantities characterize the so-called communication blockage and communication starvation of the machines. They differ from the manufacturing blockage (1.1) and starvation (1.2) by the fact that the machine status (up or down) is not taken into account. In terms of these quantities and in terms of the average buffer occupancy, improvability with respect to WF can be characterized as follows:

COROLLARY 2.1 *Under condition* (2.1),

(a)

$$\widetilde{cb}_{ij} = \widetilde{cs}_{i,j+1} + \mathcal{O}(\delta), \quad ij \neq 0M_0, 1M_1, 2M_2,$$

$$\widetilde{cb}_{iM_i} = \widetilde{cs}_{01_i} + \mathcal{O}(\delta), \quad i = 1, 2; \tag{2.3}$$

(b)

$$\bar{\bar{h}}_{1j} = E[\tilde{h}_{1j}] = \frac{N_{1j}}{2} \frac{N_{1j}+1}{N_{1j}+1-\mu_j^f} + \mathcal{O}(\delta) \approx \frac{N_{1j}}{2}, \quad j = 1, \ldots, M_1,$$

$$\bar{\bar{h}}_{2j} = E[\tilde{h}_{2j}] = \frac{N_{2j}}{2} \frac{N_{2j}+1}{N_{2j}+1-\nu_j^f} + \mathcal{O}(\delta) \approx \frac{N_{2j}}{2}, \quad j = 1, \ldots, M_2,$$

$$\bar{\bar{h}}_{0j} = E[\tilde{h}_{0j}] = \frac{N_{0j}}{2} \frac{N_{0j}+1}{N_{0j}+1-\mu_{M_1+j}^f} + \mathcal{O}(\delta) \tag{2.4}$$

$$\left(or = \frac{N_{0j}}{2} \frac{N_{0j}+1}{N_{0j}+1-\nu_{M_2+j}^f} + \mathcal{O}(\delta) \right)$$

$$\approx \frac{N_{0j}}{2}, \quad j = 1, \ldots, M_0 - 1.$$

Proof See the Appendix.

Since in most industrial situations, the multiplier of $N_{ij}/2$ in (2.4) is close to 1, Corollary 2.1 can be re-formulated as the following practical rule:

IMPROVABILITY INDICATOR 2.1   *An assembly system is unimprovable with respect to WF re-distribution if each of its buffers is, on the average, close to being half full.*

Remark 2.1   Improvability Indicator 2.1 can be rationalized as follows: Consider a part of the line consisting of two machines separated by the buffer. This buffer serves to protect the upstream machine from blockages and the downstream machine from starvations. From the point of view of the upstream machine, the buffer should always be empty. From the point of view of the downstream machine, the buffer should always be full. The compromise is buffer half full: Under this condition the performance of the machines is "aligned" appropriately, and the disturbance rejection capability of the buffer is utilized to its fullest. This leads to the largest production rate.

2.1.2   Unimprovable Distribution of p^*

Improvability Indicator 2.1 characterizes whether the system can be improved even if the parameters of the machines and buffers are unknown. If N_{ij}'s are known, the value of p_{ij}'s that render the system WF-unimprovable can be calculated as follows:

Let PR^* denote the system production rate under the best possible distribution of p^*, i.e.,

$$PR^* = \max_{p_1, p_2, p_0; \prod_{i=1}^{M_1} p_{1i} \prod_{i=1}^{M_2} p_{2i} \prod_{i=1}^{M_0} p_{0i} = p^*} PR(p_1, p_2, p_0, N_1, N_2, N_0). \quad (2.5)$$

Introduce the recursive procedure

$$x(s+1)$$

$$= (p^*)^{1/(M_1+M_2+M_0)} \prod_{i=1}^{M_1} \left(\frac{N_{1i} + x(s)}{N_{1i} + 1} \right)^{2/(M_1+M_2+M_0)}$$

$$\times \prod_{i=1}^{M_2} \left(\frac{N_{2i} + x(s)}{N_{2i} + 1} \right)^{2/(M_1+M_2+M_0)} \prod_{i=1}^{M_0-1} \left(\frac{N_{0i} + x(s)}{N_{0i} + 1} \right)^{2/(M_1+M_2+M_0)}.$$

$$(2.6)$$

THEOREM 2.2 *Assume* $\sum_{i=1}^{M_1} N_{1i}^{-1} + \sum_{i=1}^{M_2} N_{2i}^{-1} + \sum_{i=1}^{M_0-1} N_{0i}^{-1} <$
$(M_1 + M_2 + M_0)/2$. *Then the recursive procedure* (2.6) *is a contraction on* $[0, 1]$. *Moreover,*

$$\lim_{s \to \infty} x(s) = PR^*, \tag{2.7}$$

where PR^* is defined by (2.5). *In addition, the values of p_{ij}, $i = 0, 1, 2$, $j = 1, \ldots, M_i$, that result in PR^* are*

$$p_{11}^* = \left(\frac{N_{11} + 1}{N_{11} + PR^*} \right) PR^*,$$

$$p_{21}^* = \left(\frac{N_{21} + 1}{N_{21} + PR^*} \right) PR^*,$$

$$p_{ij}^* = \left(\frac{N_{i,j-1} + 1}{N_{i,j-1} + PR^*} \right) \left(\frac{N_{ij} + 1}{N_{ij} + PR^*} \right) PR^*, \quad \forall ij \neq 11, 21, 01, 0M_0,$$

$$p_{01}^* = \left(\frac{N_{1M_1} + 1}{N_{1M_1} + PR^*} \right) \left(\frac{N_{2M_2} + 1}{N_{2M_2} + PR^*} \right) \left(\frac{N_{01} + 1}{N_{01} + PR^*} \right) PR^*,$$

$$p_{0M_0}^* = \left(\frac{N_{0,M_0-1} + 1}{N_{0,M_0-1} + PR^*} \right) PR^*.$$

$$\tag{2.8}$$

Proof See the Appendix.

Remark 2.2 If all buffers are of identical capacity, it follows from the above expressions that the following "bowl" phenomenon takes place: Component machines m_{11}, m_{21}, and m_{0M_0} should have the smallest production rate in isolation; assembly machine m_{01} should have the largest production rate in isolation; all other machines should have the isolation production rate between the above two. This conclusion is in agreement with that of [8].

2.2 Improvability with Respect to Workforce and Work-in-Process Simultaneously

2.2.1 Improvability Conditions

THEOREM 2.3 *Assembly system* (i)–(vi) *is unimprovable with respect to WF and WIP re-allocation simultaneously if and only if* (2.1) *takes*

place and, in addition,

$$\mu_i^f = \mu_i^b, \quad i = 2, \ldots, M_1 + M_0 - 1,$$
$$\nu_i^f = \nu_i^b, \quad i = 2, \ldots, M_2 + M_0 - 1, \tag{2.9}$$

where $\mu_i^f, \mu_i^b, \nu_i^f$ and ν_i^b are the steady states of the recursive procedure (1.3).

Proof See the Appendix.

COROLLARY 2.2 *Under conditions* (2.1) *and* (2.9),

(a)

$$\widetilde{cb}_{ij} = \widetilde{cs}_{ij} + \mathcal{O}(\delta), \quad \forall ij \neq 11, 21, 01, 0M_0,$$
$$\widetilde{cb}_{01} = \widetilde{cs}_{01_1} + \mathcal{O}(\delta) = \widetilde{cs}_{01_2} + \mathcal{O}(\delta), \tag{2.10}$$

where $\widetilde{cb}_{ij}, \widetilde{cs}_{ij}, ij \neq 01$, and $\widetilde{cs}_{01_i}, i = 1, 2$, are given in (2.2);

(b)

$$\bar{\bar{h}}_{ij} = \bar{\bar{h}}_{11} + \mathcal{O}(\delta), \quad \forall ij \neq 0M_0, \tag{2.11}$$

where $\bar{\bar{h}}_{ij}, \forall ij \neq 0M_0$, are the average steady state occupancy of buffer b_{ij}.

Proof See the Appendix.

Although relationship (2.11) may be used to formulate an improvability indicator, we postpone this until the next subsubsection, which provides for a more convenient formulation.

2.2.2 Unimprovable Distribution of p^* and N^*

The vectors p_i^*, $i = 0, 1, 2$, and N_j^*, $j = 0, 1, 2$, which render the system unimprovable with respect to WF and WIP can be calculated as follows:

THEOREM 2.4 *Assume that N^* is an integer multiple of $M_1 + M_2 + M_0 - 1$ and denote*

$$PR^{**} = \max_{\substack{p_1, p_2, p_0; \prod_{i=1}^{M_1} p_{1i}^* \prod_{i=1}^{M_2} p_{2i}^* \prod_{i=1}^{M_0} p_{0i}^* = p^* \\ N_1, N_2, N_0; \sum_{i=1}^{M_1} N_{1i} + \sum_{i=1}^{M_2} N_{2i} + \sum_{i=1}^{M_0-1} N_{0i} = N^*}} PR(p_1, p_2, p_0, N_1, N_2, N_0).$$

$$\tag{2.12}$$

Then conditions (2.1) and (2.9) are satisfied if and only if,

$$p_{11}^* = p_{21}^* = p_{0M_0}^* = \left(\frac{N^*/(M_1 + M_2 + M_0 - 1) + 1}{N^*/(M_1 + M_2 + M_0 - 1) + PR^{**}} \right) PR^{**},$$

$$p_{ij}^* = \left(\frac{N^*/(M_1 + M_2 + M_0 - 1) + 1}{N^*/(M_1 + M_2 + M_0 - 1) + PR^{**}} \right)^2 PR^{**}, \quad \forall ij \neq 11, 21, 0M_0, 01,$$

$$p_{01}^* = \left(\frac{N^*/(M_1 + M_2 + M_0 - 1) + 1}{N^*/(M_1 + M_2 + M_0 - 1) + PR^{**}} \right)^3 PR^{**},$$

$$N_{ij}^* = \frac{N^*}{M_1 + M_2 + M_0 - 1}, \quad i = 0, 1, 2, \quad \forall ij \neq 0M_0,$$

(2.13)

where PR^{**} is calculated according to (2.6) with N_{ij}^*'s defined by the last equation in (2.13).

Thus, in an assembly system unimprovable with respect to both WF and WIP, all buffers are of equal capacity and $p_{11} = p_{21} = p_{0M_0} < p_{ij}$, $\forall ij \neq 11, 21, 0M_0, 01$, and $p_{ij} < p_{01}$, $\forall ij \neq 01$, as indicated in (2.13).

From Corollary 2.1 and Theorem 2.4, we derive the following:

IMPROVABILITY INDICATOR 2.2 *An assembly system is unimprovable with respect to WF and WIP re-distribution simultaneously if all buffers are of equal capacity and are, on the average, almost half full.*

2.3 Improvability with Respect to Work-in-Process

Unfortunately, no simple and convenient criterion of improvability for this problem was found. Therefore, we resort to a heuristic formulation and then justify it numerically.

The heuristics are based on expressions (2.10): As it follows from Corollary 2.2, these expressions are due to the WIP constraint. They imply that each machine is starved and blocked (in the communication sense) with almost equal frequency. The rationale behind these conditions can be described as follows: Consider a part of the system consisting of two buffers and a machine between them. The first buffer protects the machine from the starvations and the second from blockages. Thus, if the machine is starved more often than it is blocked, the second buffer is relatively too large, and a unit capacity of the second buffer may be re-allocated to the first one. Only when blockages and

starvations are the same, the re-allocation would not increase the production rate. This is the situation insured by unimprovable distribution of p^* and N^* simultaneously.

When only N^* is being distributed, due to the discrete nature of N_{ij}'s and due to the fixed values of p_{ij}'s, no exact equality of blockages and starvations can be achieved. The only thing possible is the minimization of the largest difference between the blockages and starvations. It turns out that this minimization results in a larger $\widetilde{PR}$ when the differences are normalized by the machine production rate in isolation, i.e., when less productive machines are closer to having blockages equal starvations. Moreover, since the assembly machine may be starved by either of the immediately preceding buffers, it should be "balanced" twice: from the point of view of cs_{01_1} and cs_{01_2} (see formula (2.14) below). This leads to

IMPROVABILITY INDICATOR 2.3   *An assembly system is unimprovable with respect to WIP re-distribution if the quantity*

$$\Delta = \max\left(\frac{1}{p_{01}(1 - \tilde{c}s_{01_2})} |\tilde{cb}_{01} - \tilde{c}s_{01_1}|, \frac{1}{p_{01}(1 - \tilde{c}s_{01_1})} |\tilde{cb}_{01} - \tilde{c}s_{01_2}|, \right.$$

$$\left. \max_{ij \neq 11,21,01,0M_0} \frac{1}{p_{ij}} |\tilde{cb}_{ij} - \tilde{c}s_{ij}| \right) \qquad (2.14)$$

is minimized over all vectors N_1, N_2, *and* N_0 *such that* $\sum_{i=1}^{M_1} N_{1i}^* + \sum_{i=1}^{M_2} N_{2i}^* + \sum_{i=1}^{M_0-1} N_{0i}^* = N^*$.

Numerical Justification   The performance of this indicator was analyzed using a large number of examples. In each example, the direct Markov chain analysis approach has been utilized, i.e., transition matrices have been constructed, stationary probabilities of all states have been calculated, and the communication blockage and starvation of each machine and two isolation production rates of the assembly machine have been evaluated. A typical example is shown in Table I. Obviously, max $\widetilde{PR}$ takes place if Δ is minimized. Similar results have been obtained in the majority cases analyzed. Several counterexamples, however, have been found. One of them is shown in Table II. Although in this example min Δ does not result in the max $\widetilde{PR}$, the difference between max $\widetilde{PR}$ and the $\widetilde{PR}$ corresponding to min Δ is quite small. A similar situation was observed in every case where the

TABLE I Illustration of Improvability Indicator 2.3 ($p_{11} = 0.8$, $p_{12} = 0.8$, $p_{21} = 0.75$, $p_{22} = 0.8$, $p_{01} = 0.8$, $p_{02} = 0.8$, $p_{03} = 0.8$, and $N_{11} + N_{12} + N_{21} + N_{22} + N_{01} + N_{02} = 7$)

N_{11}	N_{12}	N_{21}	N_{22}	N_{01}	N_{02}	Δ	$\widetilde{PR}$
2	1	1	1	1	1	0.3938	0.4661
1	2	1	1	1	1	0.2018	0.482
1	1	2	1	1	1	0.3623	0.4728
1	1	1	2	1	1	0.1935	0.4865
1	1	1	1	2	1	0.2018	0.482
1	1	1	1	1	2	0.3938	0.4661

TABLE II Counterexample for Improvability Indicator 2.3 ($p_{11} = 0.8$, $p_{12} = 0.8$, $p_{21} = 0.63$, $p_{22} = 0.8$, $p_{01} = 0.9$, $p_{02} = 0.8$, $p_{03} = 0.8$, and $N_{11} + N_{12} + N_{21} + N_{22} + N_{01} + N_{02} = 7$)

N_{11}	N_{12}	N_{21}	N_{22}	N_{01}	N_{02}	Δ	$\widetilde{PR}$
2	1	1	1	1	1	0.5072	0.4574
1	2	1	1	1	1	0.2799	0.4691
1	1	2	1	1	1	0.2666	0.4834
1	1	1	2	1	1	0.252	0.4816
1	1	1	1	2	1	0.2799	0.4691
1	1	1	1	1	2	0.5072	0.4574

above rule failed. Therefore, we conclude that Improvability Indicator 2.3 results in either the best or "almost" the best WIP distribution.

Remark 2.3 The unimprovable distribution of N_1^*, N_2^* and N_0^* can be reached not only through the complete search as in Tables I and II but also using the following algorithm:

ALGORITHM 2.1

(1) Consider the assembly system defined by (i)–(vi). Calculate

$$\Delta_{01_1} = \frac{1}{p_{01}(1 - \widetilde{cs}_{01_2})} |\widetilde{cb}_{01} - \widetilde{cs}_{01_1}|,$$

$$\Delta_{01_2} = \frac{1}{p_{01}(1 - \widetilde{cs}_{01_1})} |\widetilde{cb}_{01} - \widetilde{cs}_{01_2}|, \qquad (2.15)$$

$$\Delta_{ij} = \frac{1}{p_{ij}} |\widetilde{cb}_{ij} - \widetilde{cs}_{ij}|, \quad ij \neq 11, 21, 01, 0M_0.$$

Let machine i^*j^* be the machine with the largest value of the quantities (2.15) and the quantity corresponding to i^*j^*, $i^*j^* \neq 01$, be $\Delta_{i^*j^*}$,

i.e., $\Delta_{\max} = \Delta_{i^*j^*}$. If the assembly machine has the largest value, then $\Delta_{\max} = \max(\Delta_{01_1}, \Delta_{01_2})$.

(2) If machine $m_{i^*j^*}$ is blocked in the communication sense with larger probability than it is starved in **the communication sense** and the capacity of buffer b_{i^*,j^*-1} is greater than 1, re-allocate a unit of the buffer capacity from buffer b_{i^*,j^*-1} to buffer $b_{i^*j^*}$. If machine $m_{i^*j^*}$ is starved in the communication sense with larger probability than it is blocked in the communication sense and the capacity of buffer $b_{i^*j^*}$ is greater than 1, transfer a unit of the buffer capacity from buffer $b_{i^*j^*}$ to buffer b_{i^*,j^*-1}. In the case of $i^*j^* = 01$, if the assembly machine m_{01} is blocked in the communication sense with larger probability than it is starved in the communication sense due to the lack of parts produced by $m_{k^*M_{k^*}}$, and the capacity of buffer $b_{k^*M_{k^*}}$ is greater than 1, move a unit of the buffer capacity from $b_{k^*M_{k^*}}$ to buffer b_{01}. The opposite action is taken when the assembly machine is starved due to the lack of parts produced by $m_{k^*M_{k^*}}$ more often than blocked. If the capacity of the buffer from which a unit of the capacity must be transferred is greater than 1, go to Step (3); else let machine i^*j^* be the machine with the second largest value of Δ, and repeat this step.

(3) Go to Step (1) and continue this process until a periodic cycle of length k, $k > 1$, is reached; choose vectors N_1, N_2 and N_0 from the values on the cycle that give the largest $\widetilde{PR}$; stop.

The performance of this algorithm in comparison with the complete search procedure is illustrated in Table III. The parameters of cases in Table III are given in Table IV. In all cases analyzed, it resulted in the same distribution of N_{ij}'s as the complete search.

TABLE III Implementation of Improvability Indicator 2.3 based on the complete search and the algorithm

Cases	Algorithm						Complete search					
	N_{11}	N_{12}	N_{21}	N_{22}	N_{01}	N_{02}	N_{11}	N_{12}	N_{21}	N_{22}	N_{01}	N_{02}
Case I	1	1	1	2	1	1	1	1	1	2	1	1
Case II	1	1	1	2	1	1	1	1	1	2	1	1
Case III	1	1	1	2	1	1	1	1	1	2	1	1
Case IV	1	1	2	1	1	1	1	1	2	1	1	1
Case V	1	1	1	2	1	1	1	1	1	2	1	1
Case VI	1	2	1	2	1	1	1	2	1	2	1	1
Case VII	1	2	1	2	1	1	1	2	1	2	1	1

TABLE IV   Parameters of cases in Table III

Cases	N^*	p_{11}	p_{12}	p_{21}	p_{22}	p_{01}	p_{02}	p_{03}
I	7	0.8	0.8	0.8	0.65	0.8	0.8	0.8
II	7	0.8	0.8	0.75	0.8	0.8	0.8	0.8
III	7	0.8	0.8	0.7	0.8	0.9	0.8	0.8
IV	7	0.8	0.8	0.63	0.8	0.9	0.8	0.8
V	7	0.8	0.77	0.75	0.8	0.8	0.8	0.8
VI	8	0.8	0.77	0.75	0.8	0.8	0.8	0.8
VII	8	0.7	0.9	0.9	0.7	0.7	0.9	0.7

To illustrate the efficacy of Improvability Indicator 2.3 in comparison with the popular in industrial circles Theory of Constraints [9,10], consider the following example. Assume the assembly system shown in Fig. 1 is given with $M_1 = 2$, $M_2 = 2$, $M_0 = 3$, $p_{ij} = 0.8$, $ij \neq 12$, $p_{12} = 0.6$ (i.e. m_{12} is the worst machine) and the total capacity of the buffers $N^* = 12$. The problem is to distribute N^* among the buffer b_{ij}, $\forall ij \neq 0M_0$, so the system production rate is maximized. Following the ideology of the Theory of Constraints, all available inventory should be allocated for the protection of the "bottleneck", i.e., $N_{11} = N_{12} = 4$, $N_{ij} = 1$, $\forall ij \neq 11, 12$ (since at least one part is needed at each machine). This distribution of buffer capacity results in $\widetilde{PR} = 0.4946$. On the other hand, if N^* is distributed according to Improvability Indicator 2.3, using the above algorithm, we arrive at the following buffer structure: $N_{11} = N_{01} = 2$, $N_{12} = N_{22} = 3$, $N_{21} = N_{02} = 1$. The resulting $\widetilde{PR}$ is 0.547, i.e., over 10% improvement in comparison with the Theory of Constraints.

3   UNCONSTRAINED IMPROVABILITY

In terms of the production rate estimate (1.4), the notion of bottleneck, introduced in [6], can be formulated as follows:

– Machine m_{ij} is the *bottleneck machine* (*BN-M*) of the assembly system (i)–(vi) if

$$\frac{\partial PR}{\partial p_{ij}} > \frac{\partial PR}{\partial p_{mn}}, \quad \forall mn \neq ij. \tag{3.1}$$

– Buffer b_{ij} is the *bottleneck buffer* (*BN-B*) of the assembly system (i)–(vi) if

$$PR(p_1, p_2, p_0, [N_{11}, N_{12}, \ldots, N_{ij} + 1, \ldots, N_{0M_0-1}])$$
$$> PR(p_1, p_2, p_0, [N_{11}, N_{12}, \ldots, N_{mn} + 1, \ldots, N_{0M_0-1}]),$$
$$\forall mn \neq ij. \qquad (3.2)$$

A method for identifying these bottlenecks is given below.

3.1 Bottleneck Machine in Workforce Unimprovable Systems

THEOREM 3.1   *If assembly system* (i)–(vi) *is unimprovable with respect to WF, then*

$$p_{ij} \frac{\partial PR}{\partial p_{ij}} = \text{const}, \quad i = 0, 1, 2, \ j = 1, \ldots, M_i. \qquad (3.3)$$

Proof   See the Appendix.

Therefore, the machine with the smallest p_{ij} has the largest $\partial PR/\partial p_{ij}$. Thus, we arrive at

IMPROVABILITY INDICATOR 3.1   *To expose the BN-M, first ensure that the assembly line is WF-unimprovable; then the machine with the smallest production rate in isolation is the bottleneck.*

In WF-improvable systems, the above indicator does not take place, i.e., any machine, including the one with the largest production rate in isolation, may be the bottleneck. An example of this situation is given in Fig. 2, the sensitivities, indicated under each machine, have been evaluated numerically, using the Markov chain analysis. A method for identifying the bottleneck in this case is described in two subsections that follow.

3.2 Bottleneck Machine Identification: Single BN Case

Direct identification of BN-M using definition (3.1) is practically impossible since the partial derivatives involved cannot be either measured on the factory floor or conveniently calculated using the
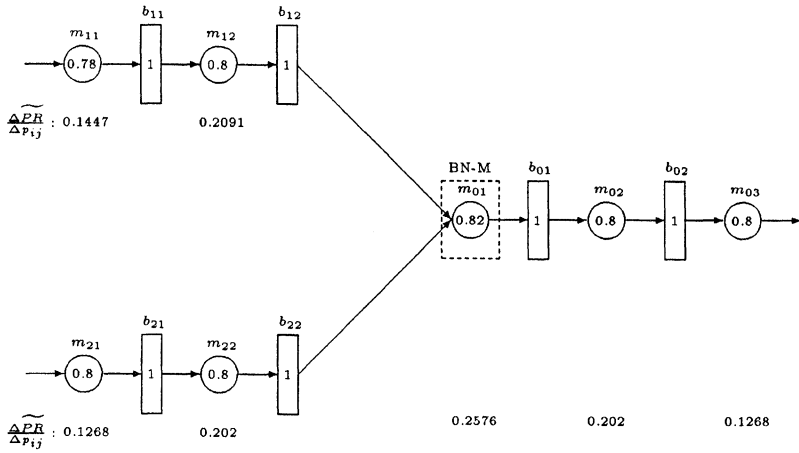
FIGURE 2   Bottleneck machine example.

machine and buffer parameters. Therefore, an indirect approach is necessary. The approach described below is based on probabilities of manufacturing blockages and starvations defined in (1.1) and (1.2) and evaluated in (1.6).

This approach is based on the results of [3], where BNs in serial production lines have been investigated. In the framework of these lines, it was shown, both analytically and numerically, that BN machines could be identified by analyzing relationships between probabilities of manufacturing blockage and starvation for each pair of consecutive machines. It is shown below that an extension of this approach leads to BN-M identification in assembly lines. Although we do not have an analytical proof of this result (due to technical difficulties in extending the proofs of [3] to model (i)–(vi)), we justify it numerically and show that, with a few exceptions, it leads to correct BN-M identification.

Consider the assembly system shown in Fig. 3 and assume that probabilities of manufacturing blockage and starvation are known, either from real time measurements or from calculations. Place these data under each machine as shown in Fig. 3. In addition, place arrows, directed from one machine to another, according to the following rule: If $\widetilde{mb}_{ij} > \widetilde{ms}_{i,j+1}$, $\forall ij \neq 1M_1, 2M_2, 0M_0$, the arrow is directed from the upstream to the downstream machine; if $\widetilde{ms}_{ij} > \widetilde{mb}_{i,j-1}$, $\forall ij \neq 11, 21, 01$, the direction of the arrow is reversed. If $\widetilde{mb}_{iM_i} > \widetilde{ms}_{01_i}$, the arrow
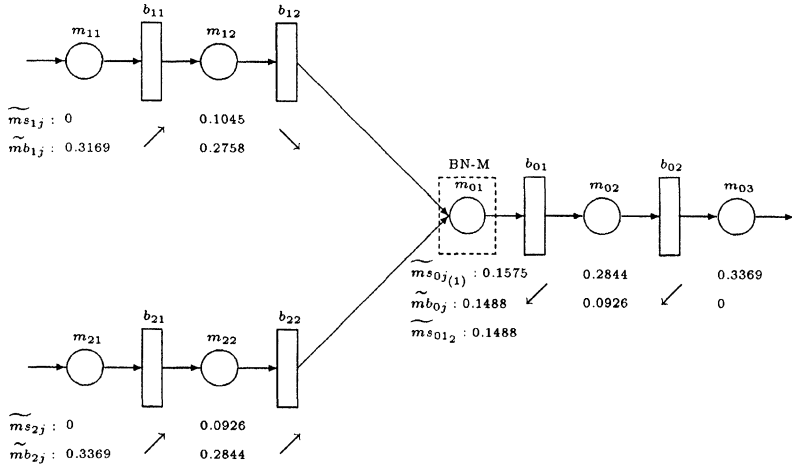
FIGURE 3   Assembly system with the probabilities of manufacturing starvation and blockage.

is directed from the upstream to the downstream machine; If $\widetilde{mb}_{iM_i} < \widetilde{ms}_{01_i}$, the direction of the arrow is reversed.

IMPROVABILITY INDICATOR 3.2   *Assume that there is only one machine with no arrows emanating from it. Then this machine is the bottleneck.*

Numerical Justification   The transition matrix for the Markovian chain defined by assumptions (i)–(vi) has been constructed and iterated upon to obtain stationary probability distributions for p_{ij} and $p_{ij} + \Delta p_{ij}$, with the step $\Delta p_{ij} = 0.01$. Then, the difference between the resulting production rates, $\Delta \widetilde{PR}$, has been calculated and $\Delta \widetilde{PR}/\Delta p_{ij}$ has been evaluated. Two typical examples are shown in Fig. 4. The system in Fig. 4(a) is the same as the ones in Figs. 2 and 3. According to Improvability Indicator 3.2, the bottleneck is the machine without arrows emanating from it. Hence, the bottlenecks in Fig. 4(a) and (b) are assembly machine m_{01} and component machine m_{22}, respectively. This conclusion is supported by the values of $\Delta \widetilde{PR}/\Delta p_{ij}$.

For most systems considered, the bottleneck identified using Improvability Indicator 3.2 and $\Delta \widetilde{PR}/\Delta p_{ij}$ coincides. An example in which the two results differ is shown in Fig. 5. According to Improvability Indicator 3.2, the bottleneck is m_{01}, whereas according to $\Delta \widetilde{PR}/\Delta p_{ij}$
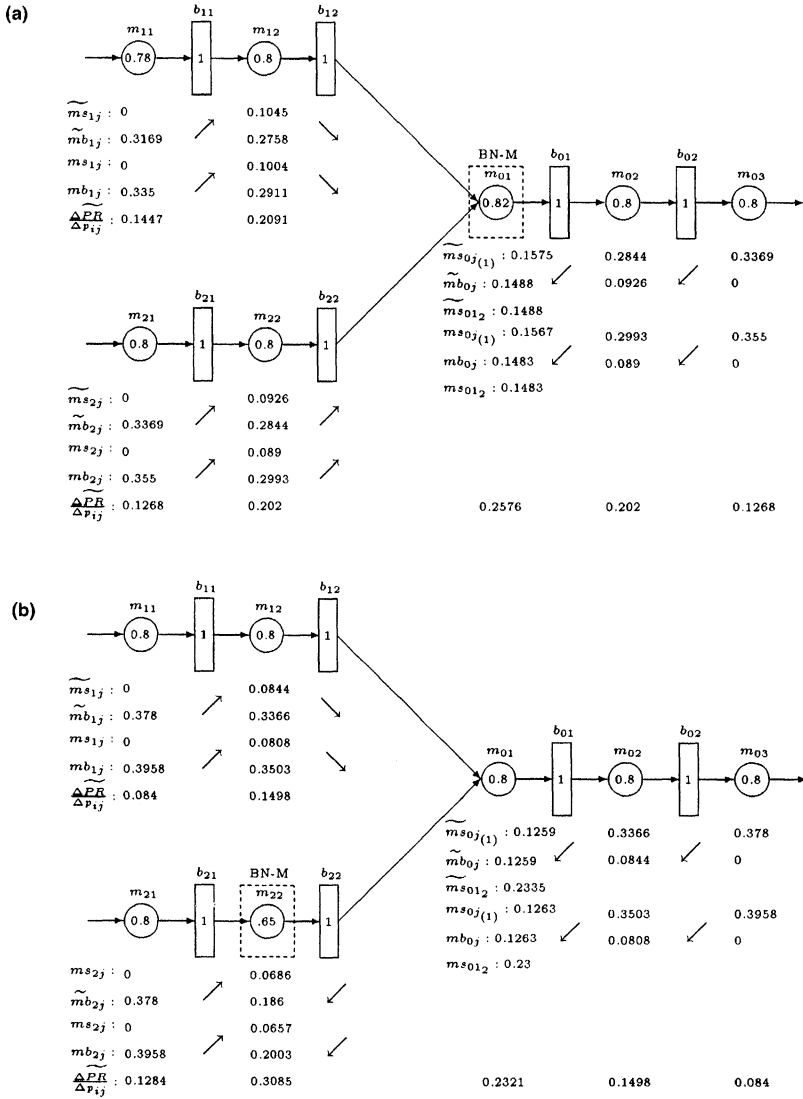
FIGURE 4   Illustration of Improvability Indicator 3.2.

the bottleneck is m_{22}. However, as it can be seen in Fig. 5, the difference between $\Delta \widetilde{PR}/\Delta p_{01}$ and $\Delta \widetilde{PR}/\Delta p_{22}$ is quite small. The same situation has been observed in all counterexamples discovered. Therefore, we conclude that Improvability Indicator 3.2 identifies
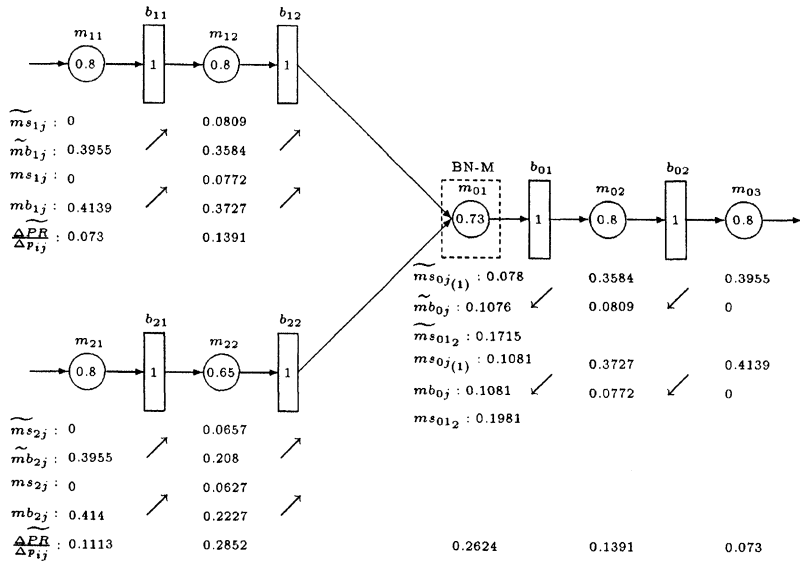
FIGURE 5   Counterexample for Improvability Indicator 3.2.

either the machine with the largest $\partial \widetilde{PR}/\partial p_{ij}$ or a machine with $\partial \widetilde{PR}/\partial p_{ij}$ being close to the largest one.

3.3   Bottleneck Machine Identification: Multiple BNs Case

In some systems there may be multiple machines with no arrows emanating from them. Which one then is the BN-M? To answer this question, introduce the measure of bottleneck severity, S_{ij}, defined as follows:

$$S_{ij} = (mb_{i,j-1} + ms_{i,j+1}) - (mb_{ij} + ms_{ij}),$$
$$\forall ij \neq 11, 21, 1M_1, 2M_2, 01, 0M_0.$$

For the first and the last machine in each line, the severity is defined as

$$S_{i1} = ms_{i2} - mb_{i1}, \quad i = 1, 2,$$
$$S_{iM_i} = (mb_{i,M_i-1} + ms_{01_i}) - (mb_{iM_i} + ms_{iM_i}), \quad i = 1, 2,$$
$$S_{01} = (mb_{1M_1} + mb_{2M_2} + ms_{02}) - (mb_{01} + ms_{01_1} + ms_{01_2}),$$
$$S_{0M_0} = mb_{0,M_0-1} - ms_{0M_0}.$$

In terms of measured probabilities of manufacturing blockages and starvations, $\tilde{S}_{ij}$ are defined analogously.

IMPROVABILITY INDICATOR 3.3 *If there are multiple machines with no arrows emanating from them, the one with the largest severity is the primary bottleneck (PBN-M).*

Numerical Justification Improvability Indicator 3.3 has also been justified using the numerical Markov chain analysis. Two typical examples are shown in Fig. 6, where the last row of numbers is the bottleneck severity. In most cases analyzed, Improvability Indicator 3.3 resulted in the correct identification of the primary bottleneck. A number of counterexamples have also been found. One of them is shown in Fig. 7. Nevertheless, since in the majority of cases analyzed Improvability Indicator 3.3 identified the machine with the best possible effect on the production rate, we conclude that it can be employed as a tool for the primary bottleneck identification.
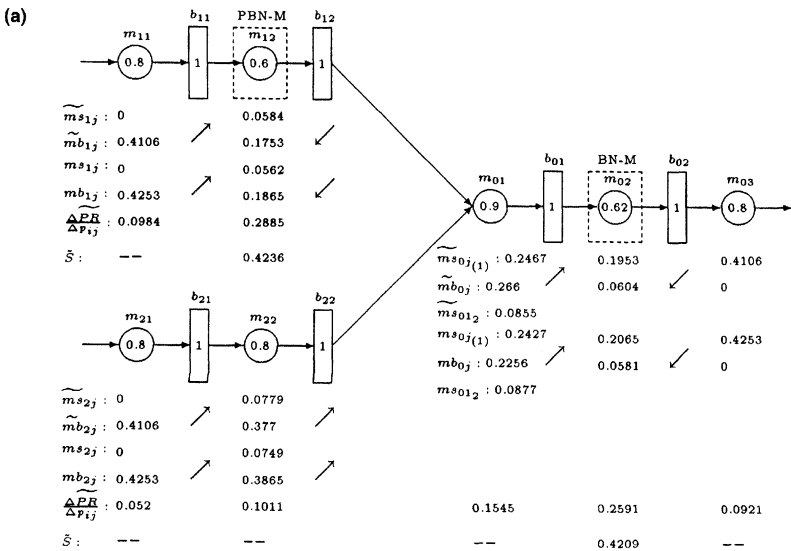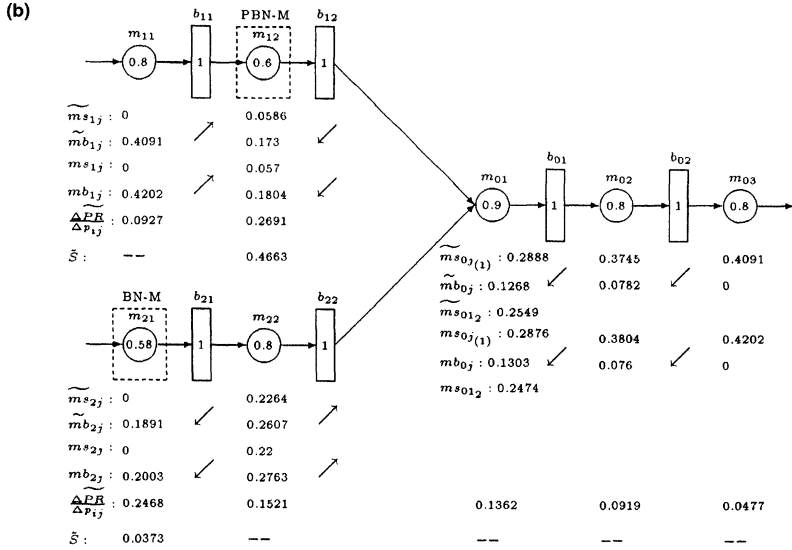

FIGURE 6(a)

(b)

m_{11} (0.8) → b_{11} (1) → PBN-M $\widetilde{m}_{12}$ (0.6) → b_{12} (1)

$\widetilde{ms}_{1j}$: 0	0.0586	↗ ↙
$\widetilde{mb}_{1j}$: 0.4091	0.173	
ms_{1j} : 0	0.057	
mb_{1j} : 0.4202	0.1804	↗ ↙
$\frac{\Delta PR}{\Delta p_{ij}}$: 0.0927	0.2691	
$\tilde{S}$: $--$	0.4663	

BN-M $\widetilde{m}_{21}$ (0.58) → b_{21} (1) → m_{22} (0.8) → b_{22} (1)

$\widetilde{ms}_{2j}$: 0	0.2264	↙ ↗
$\widetilde{mb}_{2j}$: 0.1891	0.2607	
ms_{2j} : 0	0.22	
mb_{2j} : 0.2003	0.2763	↙ ↗
$\frac{\Delta PR}{\Delta p_{ij}}$: 0.2468	0.1521	
$\tilde{S}$: 0.0373	$--$	

m_{01} (0.9) → b_{01} (1) → m_{02} (0.8) → b_{02} (1) → m_{03} (0.8)

	m_{01}	m_{02}	m_{03}
$\widetilde{ms}_{0j(1)}$:	0.2888	0.3745	0.4091
$\widetilde{mb}_{0j}$:	0.1268	0.0782	0
$\widetilde{ms}_{012}$:	0.2549		
$ms_{0j(1)}$:	0.2876	0.3804	0.4202
mb_{0j} :	0.1303	0.076	0
ms_{012} :	0.2474		
$\frac{\Delta PR}{\Delta p_{ij}}$:	0.1362	0.0919	0.0477
$\tilde{S}$:	$--$	$--$	$--$

FIGURE 6(b)

FIGURE 6   Illustration of Improvability Indicator 3.3.

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m_{11} (0.8) → b_{11} (1) → BN-M $\widetilde{m}_{12}$ (0.6) → b_{12} (2)

$\widetilde{ms}_{1j}$: 0	0.0656	↗ ↙
$\widetilde{mb}_{1j}$: 0.3626	0.1122	
ms_{1j} : 0	0.0643	
mb_{1j} : 0.3715	0.1201	↗ ↙
$\frac{\Delta PR}{\Delta p_{ij}}$: 0.1239	0.2866	
$\tilde{S}$: $--$	0.3452	

m_{21} (0.8) → b_{21} (1) → m_{22} (0.8) → b_{22} (1)

$\widetilde{ms}_{2j}$: 0	0.0875	↗ ↗
$\widetilde{mb}_{2j}$: 0.3626	0.3147	
ms_{2j} : 0	0.0857	
mb_{2j} : 0.3716	0.3202	↗ ↗
$\frac{\Delta PR}{\Delta p_{ij}}$: 0.0791	0.1282	
$\tilde{S}$: $--$	$--$	

m_{01} (0.9) → b_{01} (1) → PBN-M $\widetilde{m}_{02}$ (0.66) → b_{02} (1) → m_{03} (0.8)

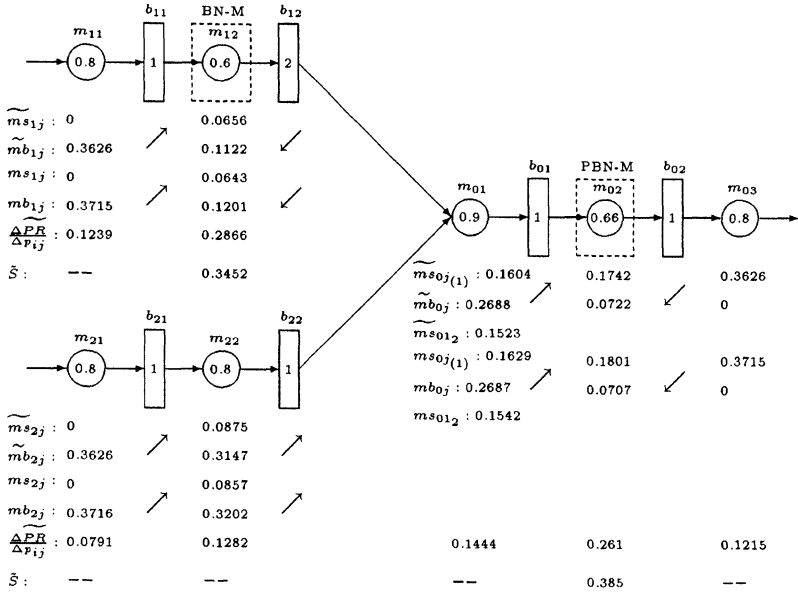|	m_{01}	m_{02}	m_{03}
$\widetilde{ms}_{0j(1)}$:	0.1604	0.1742	0.3626
$\widetilde{mb}_{0j}$:	0.2688	0.0722	0
$\widetilde{ms}_{012}$:	0.1523		
$ms_{0j(1)}$:	0.1629	0.1801	0.3715
mb_{0j} :	0.2687	0.0707	0
ms_{012} :	0.1542		
$\frac{\Delta PR}{\Delta p_{ij}}$:	0.1444	0.261	0.1215
$\tilde{S}$:	$--$	0.385	$--$

FIGURE 7   Counterexample for Improvability Indicator 3.3.

3.4 Identification of Potency of the Material Handling System

According to [6],

- Material handling system (MHS) is *weakly potent* if the machine with the smallest p_{ij} is the BN-M; otherwise it is not potent.
- MHS is *potent* if it is weakly potent and, in addition, PR is sufficiently close to the smallest p_{ij}.

Improvability Indicators 3.2 and 3.3 provide a possibility to evaluate the potency of a MHS. Indeed, if the BN-M, identified using these rules, is the machine with the smallest production rate in isolation, the MHS is weakly potent. Otherwise, the MHS causes a better machine to be most detrimental for the overall system performance, which testifies to the lack of MHS potency. Even if the machine with the smallest p_{ij} is the BN-M but the PR is sufficiently far from this p_{ij}, the lack of the performance again is attributed to the MHS, and the most beneficial continuous improvement project might be directed towards modifying the MHS. In more than a dozen industrial applications that we have carried out in the automotive industry, the identification of potency of MHS proved to be the most efficient tool for designing continuous improvement projects.

3.5 Bottleneck Buffer Identification

A buffer is the bottleneck (BN-B) if a unit increase of its capacity results in the largest increase of the PR, as compared with the similar increase of any other buffer capacity [6]. The smallest capacity buffer is not necessarily the BN-B (see an example in Fig. 8). The identification of the BN-B can be carried out using the following:

IMPROVABILITY INDICATOR 3.4 *BN-B is one of the buffers surrounding the BN-M. It is the buffer in front of the BN-M if the probability of manufacturing starvation of this machine is larger than its probability of manufacturing blockage; otherwise, it is the buffer after the BN-M.*

Numerical Justification Improvability Indicator 3.4 has also been justified using the direct numerical Markov chain analysis. Two typical examples are given in Fig. 9, where the last row of numbers is $(\Delta \widetilde{PR}/\Delta N_{ij})$ with the step $\Delta N_{ij} = 1$. With a very few exception, in all systems considered the BN-B identified using the indicator and

$\Delta \widetilde{PR}/\Delta N_{ij}$ were the same. A few counterexamples have also been found. One of them are shown in Fig. 10. In each counterexample, however, the difference between the sensitivity of the production rate to the buffer size and the one identified by the indicator is very small.
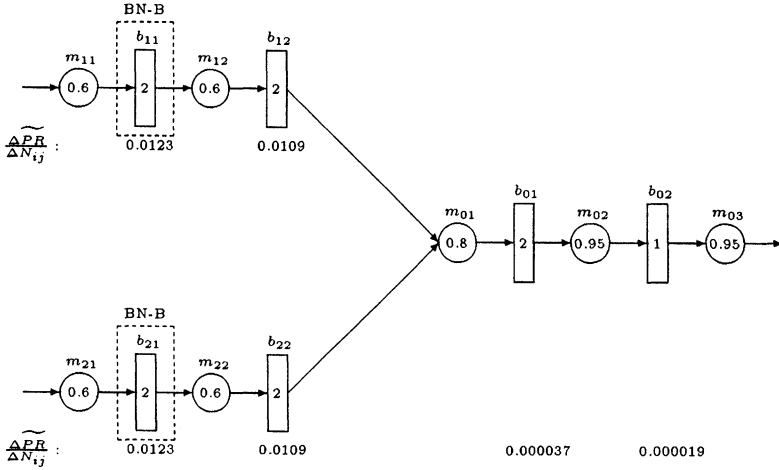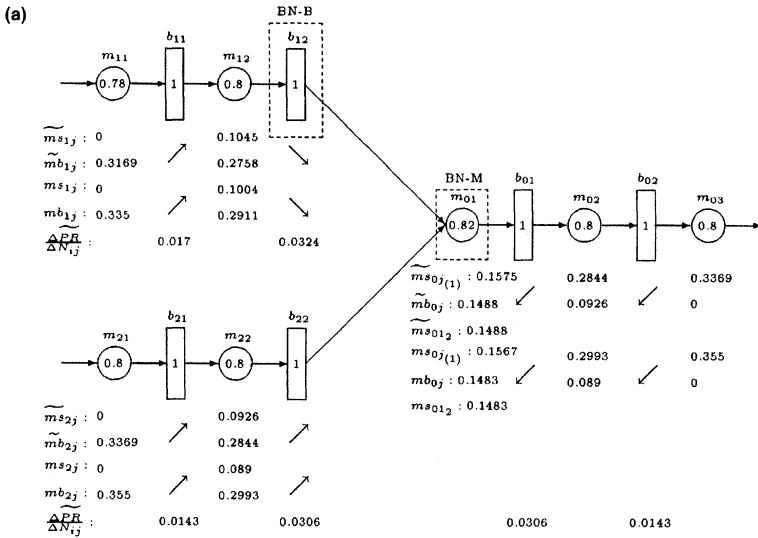

FIGURE 8   Bottleneck buffer example.
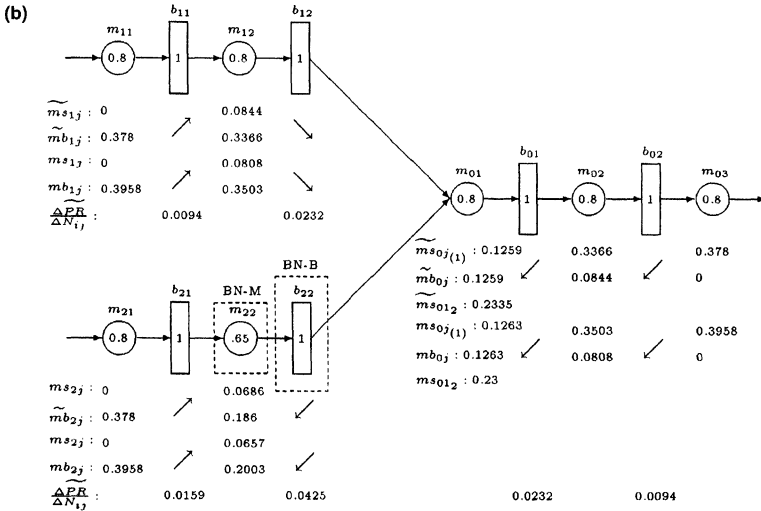

FIGURE 9(a)

FIGURE 9(b)

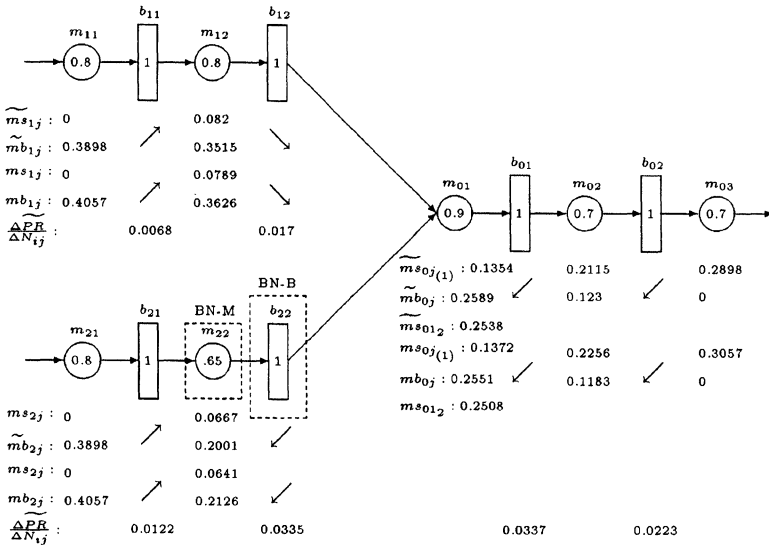FIGURE 9 Illustration for Improvability Indicator 3.4.

FIGURE 10 Counterexample for Improvability Indicator 3.4.

Therefore, we conclude that it can be used as a tool for the BN-B identification in assembly systems.

4   CASE STUDY

4.1   Analysis of MHS Potency

The assembly system of the case study has been described and identified in [6]. Its structure is shown in Fig. 11 along with the probabilities of manufacturing blockages and starvations calculated according to (1.6) (for one month data). Using Improvability Indicator 3.2, we conclude that Operation 10 is the bottleneck. Similar analyses for five other months of the study lead to the conclusions summarized in Table V. In addition, Table V includes the machine with the smallest p_{ij} during the respective month. As it follows from these data, MHS is consistently not potent and an increase of the system PR may be obtained by improving MHS.

The extent of the potential PR improvement was evaluated as follows: PR losses due to the machines can be evaluated as the difference between the nominal production rate (600 parts/h) and the
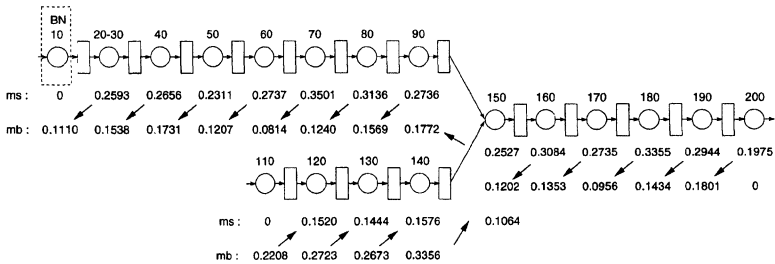

FIGURE 11   Bottleneck identification for Product A production system (based on May data).

TABLE V   Bottleneck and the slowest machine of the system

	Month					
	May	*June*	*July*	*August*	*September*	*October*
Bottleneck	Op. 10	Op. 150	Op. 150	Op. 150	Op. 150	Op. 10
Machine with the smallest isolation production rate	Op. 50	Op. 50	Op. 50	Op. 130	Op. 50	Op. 160

TABLE VI   Losses analysis

	Month					
	May	*June*	*July*	*August*	*September*	*October*
Isolation *PR* of the slowest machine (parts/h)	522	534	468	498	540	492
Losses due to machine (parts/h)	78	66	132	102	60	108
PR of the system (parts/h)	337	347	378	340	384	383
Losses due to MHS (parts/h)	185	187	90	158	156	109

isolation production rate of the slowest machine. *PR* losses due to the material handling system can be evaluated as the difference between the isolation production rate of the slowest machine and the actual production rate obtained in the system. These data for the 6 months of the study are summarized in Table VI.

Thus, as it follows from these data, roughly speaking, out of 240 parts/h lost, 80 parts/h are attributable to the machines and 160 parts/h to MHS. Interestingly, the 1 : 2 ratio has been found to be quite typical in a number of practical applications analyzed. This leads to a conclusion that improvements in the MHS could result in a substantial improvement of the overall system performance.

4.2   Design of a Continuous Improvement Project

Three avenues for potential improvement have been investigated: increasing capacity of all buffers, increasing capacity of the buffer conveyor only, and elimination of starvation of Operations 10 and 110 and blockage of Operation 200. The results obtained are described below.

4.2.1   Increasing Capacity of All Buffers

Assuming that the capacity of each buffer is increased from 1 to 5, and there is no starvation in Operations 10 and 110 and no blockage in Operation 200, the *PR* for each month has been calculated using the recursive procedure (1.3). The results are shown in Table VII. As it follows from these data, the MHS is potent: BN-M is the slowest machine and the *PR* is almost equal to that of the slowest machine. However, due to practical considerations, all buffers cannot be increased without substantial capital investments (re-building the

TABLE VII   Estimated production rates of the system with all buffers increased to 5 parts

	Month					
	May	June	July	August	September	October
PR (parts/h)	501	518	452	490	527	491
Isolation PR of the slowest machine (parts/h)	522	534	468	498	540	492
BN machine	Op. 50	Op. 50	Op. 50	Op. 130	Op. 50	Op. 160
Losses due to MHS (parts/h)	21	16	16	8	13	1

TABLE VIII   Estimated production rates of the system with no blockages and starvations of Op. 10, Op. 110, and Op. 200

	Month					
	May	June	July	August	September	October
PR (parts/h)	372	393	325	359	415	374
Improvement (%)	12.7	17	7.6	7.2	9.2	11
BN machine	Op. 60	Op. 60	Op. 60	Op. 150	Op. 140	Op. 60

conveyor system). Therefore two other avenues of continuous improvement have been investigated.

4.2.2   Increasing Capacity of the Buffer Conveyor

The buffering capacity of this conveyor can be increased by using additional pallets. At the time of the study, this conveyor contained 19 pallets. We have shown that 40 pallets would provide additional buffering capacity without creating congestion due to accumulation of empty pallets. With 40 pallets utilized, the capacity of the buffer b_{18}, i.e. the buffer in front of the bottleneck, becomes 29 parts. This results in the $PR = 368$ parts/h, which is 9.2% improvement in comparison with the average production rate over the months of May–October.

4.2.3   Eliminating Starvation of Operations 10 and 110 and Blockage of Operation 200

This can be accomplished by either manual or robotic material handling. For instance, if Operation 90 is blocked, a part can be removed from the pallet, manually or by a robot, making this pallet available to Operation 10. If these modification are put in place, the system performance is improved as shown in Table VIII, i.e. the production rate increase by 7–17% is possible.

The plant management has accepted the above two recommendations, and they have been put in place.

5 CONCLUSIONS

This work describes improvability theory for assembly systems with unreliable (Bernoulli) machines. The main results are the improvability indicators. They allow factory floor personnel to design continuous improvement projects. The data necessary to implement these indicators consist of machine and buffer parameters and frequencies (probabilities) of machine blockages and starvations. The former are typically measured on the factory floor (often, however, with a very low accuracy); the latter rarely are. This work shows that the knowledge (through real-time measurements or calculations developed in the work) of these probabilities is centrally important for production line management and control: they lead to the determination whether the system is improvable and to identification of bottlenecks. Case study, described in this paper, illustrates that these indicators are relatively easy to apply in industrial environment. At present, application efforts are centered on including these indicators as a part of standard operating procedures at an automotive engine plant.

APPENDIX: PROOFS FOR SECTIONS 2 AND 3

The proofs of Theorems 2.1 and 2.2 and Corollary 2.1 are based on Lemmas A.1–A.5 formulated below. The proofs of these lemmas are similar to those of Lemmas B.3–B.7 of [1] (see [11] for details).

LEMMA A.1   *Let* $c_i' = \sqrt{\mu_i^f \mu_{i+1}^b}, i = 1, \ldots, M_1 + M_0 - 1, c_i'' = \sqrt{\nu_i^f \nu_{i+1}^b},$
$i = 1, \ldots, M_2 + M_0 - 1.$

Then

$$c_i' \geq PR \frac{\Gamma_i + 1}{\Gamma_i + PR}, \quad i = 1, \ldots, M_1 + M_0 - 1,$$

$$c_i'' \geq PR \frac{\Lambda_i + 1}{\Lambda_i + PR}, \quad i = 1, \ldots, M_2 + M_0 - 1.$$

The equality takes place in the upper line if and only if $\mu_i^f = \mu_{i+1}^b$ and in the lower line $\nu_i^f = \nu_{i+1}^b$.

LEMMA A.2   *The total workforce, p^*, necessary to achieve the production rate value PR, is bounded by*

$$p^* \geq \prod_{i=1}^{M_1+M_0-1} \left(\frac{\Gamma_i + 1}{\Gamma_i + PR}\right)^2 \prod_{i=1}^{M_2} \left(\frac{\Lambda_i + 1}{\Lambda_i + PR}\right)^2 PR^{M_1+M_2+M_0}.$$

The equality holds if and only if $\mu_i^f = \mu_{i+1}^b$, $i = 1, \ldots, M_1 + M_0 - 1$, and if $\nu_i^f = \nu_{i+1}^b$, $i = 1, \ldots, M_2 + M_0 - 1$.

LEMMA A.3   *The conditions $\mu_i^f = \mu_{i+1}^b$, $i = 1, \ldots, M_1 + M_0 - 1$, and $\nu_i^f = \nu_{i+1}^b$, $i = 1, \ldots, M_2 + M_0 - 1$, is achieved if and only if the workforce is distributed as*

$$p_{11} = \left(\frac{N_{11} + 1}{N_{11} + PR}\right) PR,$$

$$p_{21} = \left(\frac{N_{21} + 1}{N_{21} + PR}\right) PR,$$

$$p_{ij} = \left(\frac{N_{i,j-1} + 1}{N_{i,j-1}+PR}\right)\left(\frac{N_{ij} + 1}{N_{ij} + PR}\right) PR, \qquad (A.1)$$

$$i = 0, 1, 2, \ j = 2, \ldots, M_i \ (except \ p_{0M_0}),$$

$$p_{01} = \left(\frac{N_{1M_1} + 1}{N_{1M_1} + PR}\right)\left(\frac{N_{2M_2} + 1}{N_{2M_2} + PR}\right)\left(\frac{N_{01} + 1}{N_{01} + PR}\right) PR,$$

$$p_{0M_0} = \left(\frac{N_{0,M_0-1} + 1}{N_{0,M_0-1} + PR}\right) PR,$$

where PR is the production rate of the assembly system.

LEMMA A.4   *The minimum workforce $p_{\min}^*$ required to achieve production rate PR is given by*

$$p_{\min}^* \prod_{i=1}^{M_1+M_0-1} \left(\frac{\Gamma_i + 1}{\Gamma_i + PR}\right)^2 \prod_{i=1}^{M_2} \left(\frac{\Lambda_i + 1}{\Lambda_i + PR}\right)^2 PR^{M_1+M_2+M_0}.$$

Moreover, this production rate is achieved if and only if p^* is distributed among vectors $p_1, p_2, p_0, \prod_{i=1}^{M_1} p_{1i} \prod_{i=1}^{M_2} p_{2i} \prod_{i=1}^{M_0} p_{0i} = p^*$, so that $\mu_i^f = \mu_{i+1}^b$, $i = 1, \ldots, M_1 + M_0 - 1$, and $\nu_i^f = \nu_{i+1}^b$, $i = 1, \ldots, M_2 + M_0 - 1$.

LEMMA A.5 *The minimum workforce $p^*_{\min}$ necessary to achieve the production rate PR is a monotonically increasing function of PR.*

Proof of Theorem 2.1 "Only if": Suppose the assembly system is unimprovable, but that there exists an i such that $\mu_i^f \neq \mu_{i+1}^b$ in the upper line or $\nu_i^f \neq \nu_{i+1}^b$ in the lower line. Then by Lemma A.4, $p^* > p^*_{\min}$. Thus, from Lemma A.5, workforce p^* optimally distributed can achieve a larger production rate, which is a contradiction.

The "if" part follows from Lemma A.4.

Proof of Corollary 2.1 By Lemmas B.3 and B.5 of [6], the distribution of parts in buffer i in the upper line can be approximated with error $\mathcal{O}(\delta)$ by the distribution of parts in the buffer of the two machine line $L_i' = \{\mu_i^f, \Gamma_i, \mu_{i+1}^b\}$. Applying Lemma A.5 of [1] to line L_i', Prob$\{m_i'$ is starved in the communication sense$\} = Q(\mu_i^f, \mu_{i+1}^b, \Gamma_i) + \mathcal{O}(\delta)$ and Prob$\{m_i'$ is blocked in the communication sense$\} = Q(\mu_{i+1}^b, \mu_i^f, \Gamma_i) + \mathcal{O}(\delta)$. Since $\mu_i^f = \mu_{i+1}^b$, using (1.3), the result of part (a) for the upper line follows. The result of part (b) for the lower line can be proved analogously. From Eq. (A.3) of Lemma A.5 in [1], when applied to line L_i',

$$X_{i'}(j) = \frac{X_{i'}(0)}{1 - \mu_i^f}, \quad 1 \leq j \leq \Gamma_i, \ i = 1, \ldots, M_1 + M_0 - 1, \qquad (A.2)$$

where

$$X_{i'}(0) = \frac{1 - \mu_i^f}{\Gamma_i + 1 - \mu_i^f}, \quad i = 1, \ldots, M_1 + M_0 - 1. \qquad (A.3)$$

Thus,

$$\begin{aligned}
\bar{h}_i' E[\tilde{h}_i'] &= \sum_{j=0}^{\Gamma_i} j X_{i'}(j) + \mathcal{O}(\delta) \\
&= \sum_{j=1}^{\Gamma_i} j \left(\frac{1}{1 - \mu_i^f}\right) \left(\frac{1 - \mu_i^f}{\Gamma_i + 1 - \mu_i^f}\right) + \mathcal{O}(\delta) \\
&= \frac{\Gamma_i(\Gamma_i + 1)}{2(\Gamma_i + 1 - \mu_i^f)} + \mathcal{O}(\delta) \\
&\approx \frac{\Gamma_i}{2}, \quad i = 1, \ldots, M_1 + M_0 - 1. \qquad (A.4)
\end{aligned}$$

Using (1.3), this proves statement (b) for the upper line of the corollary. Statement (b) for the lower line can be proved similarly.

Proof of Theorem 2.2 Follows from Lemma A.3, and the proof of Theorem 3.2 of [12] (see [11] for details).

Proof of Theorem 2.3 Similar to the proof of Theorem 2.7 of [1] (see [11] for details).

Proof of Corollary 2.2 Similar to that of Corollary 2.1.

Proof of Theorem 2.4 Similar to that of Theorem 2.2.

Proof of Theorem 3.1 Consider an assembly system (i)–(vi) with (2.1) taking place. Assume that the workforce distribution is p_{ij}^*, $i = 0, 1, 2$, $j = 1, \ldots, M_i$, where p_{ij}^* are defined by (2.8). Modify the workforce distribution to $p_{ij} = g p_{ij}^*$, $i = 0, 1, 2$, $j = 1, \ldots, M_i$, and $p_{mn} = (1/g) p_{mn}^*$, $m = 0, 1, 2$, $n = 1, \ldots, M_m$, where $g \neq 0$. Then, we find that the total workforce p^* is preserved and is not depending on g, but that the line is unimprovable when $g = 1$. That is, the production rate achieves its maximum value at $g = 1$. Let $PR = PR(g)$. Consequently, we have

$$\left. \frac{\partial PR(g)}{\partial g} \right|_{g=1} = 0. \tag{A.5}$$

By the chain rule,

$$\left. \frac{\partial PR(g)}{\partial g} \right|_{g=1} = \left(\frac{\partial PR(g)}{\partial (g p_{ij}^*)} \frac{\partial (g p_{ij}^*)}{\partial g} + \frac{\partial PR(g)}{\partial ((p_{mn}^*)/g)} \frac{\partial ((p_{mn}^*)/g)}{\partial g} \right) \Bigg|_{g=1}$$

$$= \left(p_{ij}^* \frac{\partial PR(g)}{\partial p_{ij}} - p_{mn}^* \left(\frac{1}{g} \right)^2 \frac{\partial PR(g)}{\partial p_{mn}} \right) \Bigg|_{g=1},$$

$$i = 0, 1, 2, \; j = 1, \ldots, M_i, \; m = 0, 1, 2, \; n = 1, \ldots, M_m. \tag{A.6}$$

From (A.5),

$$p_{ij}^* \left. \frac{\partial PR(g)}{\partial p_{ij}} \right|_{g=1} - p_{mn}^* \left. \frac{\partial PR(g)}{\partial p_{mn}} \right|_{g=1} = 0.$$

Since i, j, m and n are chosen arbitrarily, we therefore conclude that

$$p_{ij}^* \frac{\partial PR(g)}{\partial p_{ij}}\bigg|_{g=1} = p_{mn}^* \frac{\partial PR(g)}{\partial p_{mn}}\bigg|_{g=1}, \quad i = 0, 1, 2, \ j = 1, \ldots, M_i,$$
$$m = 0, 1, 2, \ n = 1, \ldots, M_m.$$

Therefore, Eq. (3.3) follows.

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