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Research Article

Well-Posedness of the Boundary Value Problem for Parabolic Equations in Difference Analogues of Spaces of Smooth Functions

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The first and second orders of accuracy difference schemes for the approximate solutions of the nonlocal boundary value problem $v'(t)+Av(t)=f(t)$ $(0\le t\le 1)$, $v(0)=v(\lambda)+\mu$, $0<\lambda\le 1$, for differential equation in an arbitrary Banach space E with the strongly positive operator A are considered. The well-posedness of these difference schemes in difference analogues of spaces of smooth functions is established. In applications, the coercive stability estimates for the solutions of difference schemes for the approximate solutions of the nonlocal boundary value problem for parabolic equation are obtained.

1. Introduction: difference schemes

It is known that (see, e.g., [1–5] and the references given therein) many applied problems in fluid mechanics and other areas of physics and mathematical biology were formulated into nonlocal mathematical models. However, such problems were not well investigated in general.

In [6], the well-posedness in the spaces of smooth functions of the nonlocal boundary value problem

$$v'(t)+Av(t)=f(t)\quad(0\le t\le 1),\qquad v(0)=v(\lambda)+\mu\quad(0<\lambda\le 1)\tag{1.1}$$

for differential equation in an arbitrary Banach space E with the strongly positive operator A was established. The importance of coercive (well-posedness) inequalities is well known [7, 8].

For the construction of difference schemes, we consider the uniform grid space

$$[0,1]_\tau = \{t_k = k\tau,\ 0 \le k \le N,\ N\tau = 1\}. \tag{1.2}$$

Assume that $\tau \le \lambda$. We consider the first order of accuracy implicit Rothe difference scheme

$$\begin{gathered} \frac{u_k - u_{k-1}}{\tau} + Au_k = \varphi_k, \quad \varphi_k = f(t_k),\ t_k = k\tau,\ 1 \le k \le N, \\ u_0 = u_{[\lambda/\tau]} + \mu, \end{gathered} \tag{1.3}$$

and the second order of accuracy implicit difference scheme

$$\begin{gathered} \frac{u_k - u_{k-1}}{\tau} + A\left(I + \frac{\tau A}{2}\right)u_k = \left(I + \frac{\tau A}{2}\right)\varphi_k, \quad \varphi_k = f\left(t_k - \frac{\tau}{2}\right),\ t_k = k\tau,\ 1 \le k \le N, \\ u_0 = \left(I - \left(\lambda - \left[\frac{\lambda}{\tau}\right]\tau\right)A\right)u_{[\lambda/\tau]} + \mu + \left(\lambda - \left[\frac{\lambda}{\tau}\right]\tau\right)\varphi_{[\lambda/\tau]}, \end{gathered} \tag{1.4}$$

approximately solving the boundary value problem (1.1).

Let $F_\tau(E)$ be the linear space of mesh functions $\varphi^\tau = \{\varphi_k\}_1^N$ with values in the Banach space E. Next on $F_\tau(E)$, we introduce the Banach spaces $C_\tau(E) = C([0,1]_\tau, E)$, $C_\tau^{\beta,\gamma}(E) = C^{\beta,\gamma}([0,1]_\tau, E)$ $(0 \le \gamma \le \beta < 1)$ with the norms

$$\begin{gathered} \|\varphi^\tau\|_{C_\tau(E)} = \max_{1\le k\le N} \|\varphi_k\|_E, \\ \|\varphi^\tau\|_{C_\tau^{\beta,\gamma}(E)} = \|\varphi^\tau\|_{C_\tau(E)} + \sup_{1\le k<k+r\le N} \|\varphi_{k+r} - \varphi_k\|_E \frac{((k+r)\tau)^\gamma}{(r\tau)^\beta}. \end{gathered} \tag{1.5}$$

We introduce the fractional space $E_\alpha = E_\alpha(E,A)$ $(0 < \alpha < 1)$, consisting of all $v \in E$ for which the following norm is finite:

$$\|v\|_{E_\alpha} = \sup_{\lambda>0} \lambda^\alpha \|A(\lambda + A)^{-1}v\|_E. \tag{1.6}$$

The difference scheme (1.3) or (1.4) is said to be coercively stable (well-posed) in $F_\tau(E)$ if we have the coercive inequality

$$\left\|\{\tau^{-1}(u_k - u_{k-1})\}_1^N\right\|_{F_\tau(E)} \le M\left[\|A\mu\|_{E'} + \|\varphi^\tau\|_{F_\tau(E)}\right], \quad E' \subset E, \tag{1.7}$$

where M is independent not only of φ^τ, μ but also of τ.

In [9, 10], the stability and coercive stability of the difference schemes (1.3) and (1.4) in $C_\tau^{\alpha,\alpha}(E)$ and $C_\tau(E_\alpha)$ $(0 < \alpha < 1)$ spaces and almost coercive stability (with multiplier $\min\{\ln 1/\tau, 1 + |\ln\|A\|_{E\to E}|\}$) of the difference schemes (1.3) and (1.4) in $C_\tau(E)$ spaces are established.

In the present paper, the coercive stability of difference schemes (1.3) and (1.4) in $C_\tau^{\beta,\gamma}(E)$ $(0\le\gamma\le\beta<1)$ and $C_\tau^{\beta,\gamma}(E_{\alpha-\beta})$ $(0\le\gamma\le\beta\le\alpha<1)$ spaces under the assumption that the operator $-A$ generates an analytic semigroup $\exp\{-tA\}$ $(t\ge 0)$ with exponentially decreasing norm,when $t\to+\infty$,

$$\|\exp\{-tA\}\|_{E\to E}\le Me^{-\delta t},\quad \|A\exp\{-tA\}\|_{E\to E}\le\frac{M}{t},\quad t>0,\ \delta,M>0,\tag{1.8}$$

is established. In applications, this abstract result permits us to obtain the almost coercivity inequality and the coercive stability estimates for the solutions of difference schemes of the first and second orders of accuracy over time and of an arbitrary order of accuracy over space variables in the case of the nonlocal boundary value problem for the $2m$-order multidimensional parabolic equation.

Finally, methods for numerical solutions of the evolution differential equations have been studied extensively by many researchers (see [8, 11–32] and the references therein).

2. Well-posedness of (1.3) and (1.4)

Theorem 2.1. *Let τ be a sufficiently small number. Then the solutions of the difference schemes (1.3) and (1.4) in $C_\tau^{\beta,\gamma}(E)$ $(0\le\gamma\le\beta,\ 0<\beta<1)$ obey the coercivity inequality*

$$\begin{aligned}&\left\|\{\tau^{-1}(u_k-u_{k-1})\}_1^N\right\|_{C_\tau^{\beta,\gamma}(E)}+\left\|\{\tau^{-1}(u_k-u_{k-1})\}_1^N\right\|_{C_\tau(E_1^{\beta,\gamma})}\\&\quad\le\frac{M_1}{\beta(1-\beta)}\|\varphi^\tau\|_{C_\tau^{\beta,\gamma}(E)}+M_1\left|\mu+A^{-1}(\varphi_{[\lambda/\tau]}-\varphi_1)\right|_1^{\beta,\gamma},\end{aligned}\tag{2.1}$$

where M_1 is independent not only of φ^τ, μ, β, γ, but also of τ.

Here, the space of traces $E_1^{\beta,\gamma}=E^{\beta,\gamma}(E)$ which consist of the elements $w\in E$ for which the norm

$$\begin{aligned}|w|_1^{\beta,\gamma}=\sup_{0<\tau\le\tau_0}\Bigg[&\max_{1\le i\le N}\|\tau^{-1}(I-R)R^{i-1}w\|_E\\&+\sup_{1\le i<i+r\le N}(r\tau)^{-\beta}((i+r)\tau)^{\gamma}\|\tau^{-1}(I-R)(R^{i+r-1}-R^{i-1})w\|_E\Bigg]\end{aligned}\tag{2.2}$$

is finite, where $R=(I+\tau A)^{-1}$ for (1.3) and $R=(I+\tau A+(\tau A)^2/2)^{-1}$ for (1.4).

Proof. Let us prove (2.1) for difference scheme (1.3). By [7, formula (0.2) in Chapter 2],

$$u_k=R^ku_0+\sum_{j=1}^{k}R^{k-j+1}\varphi_j\tau,\quad k=1,\dots,N,\tag{2.3}$$

for the solution of the first order of accuracy implicit difference scheme for the approximate solutions of Cauchy problem

$$u'(t)+Au(t)=f(t) \quad (0\le t\le 1) \quad u(0)=u_0. \tag{2.4}$$

From this formula and the condition $u_0=u_{[\lambda/\tau]}+\mu$, it follows that

$$u_0=R^{[\lambda/\tau]}u_0+\sum_{j=1}^{[\lambda/\tau]}R^{[\lambda/\tau]-j+1}\varphi_j\tau+\mu. \tag{2.5}$$

Since the semigroup $\exp\{-tA\}$ obeys the exponential decay estimate (1.8), we have that

$$||R^k||_{E\to E}\le M(1+\delta\tau)^{-k}, \quad ||k\tau AR^k||_{E\to E}\le M, \quad k\ge 1. \tag{2.6}$$

From this estimate, it follows that the operator $I-R^{[\lambda/\tau]}$ has a bounded inverse $T_\tau=(I-R^{[\lambda/\tau]})^{-1}$ and

$$||T_\tau||_{E\to E}\le M(\lambda,\delta). \tag{2.7}$$

Actually, we have that

$$T_\tau-(I-\exp\{-\lambda A\})^{-1}=T_\tau(I-\exp\{-\lambda A\})^{-1}(R^{[\lambda/\tau]}-\exp\{-\lambda A\}),$$
$$R^{[\lambda/\tau]}-\exp\{-\lambda A\}=\int_0^1(I+s\tau A)^{-([\lambda/\tau]+1)}\left(\left(\lambda-\left[\frac{\lambda}{\tau}\right]\tau\right)A+s\tau\lambda A^2\right)\exp\{-\lambda(1-s)A\}\,ds. \tag{2.8}$$

Then, using the triangle inequality and the estimates

$$||(I-\exp\{-\lambda A\})^{-1}||_{E\to E}\le M(\lambda,\delta), \tag{2.9}$$

$$||R^{[\lambda/\tau]}-\exp\{-\lambda A\}||_{E\to E}\le M(\lambda,\delta)\tau, \tag{2.10}$$

we obtain estimate (2.7). The proof of (2.9) is based on the estimate (1.8) and it was proved in [33]. The proof of (2.10) is based on the estimates (1.8) and (2.6) and it was proved in [34].

So, we have the formula

$$u_k=R^ku_0+\sum_{j=1}^{k}R^{k-j+1}\varphi_j\tau, \quad k=1,\ldots,N,$$
$$u_0=T_\tau\left\{\sum_{j=1}^{[\lambda/\tau]}R^{[\lambda/\tau]-j+1}\varphi_j\tau+\mu\right\} \tag{2.11}$$

for the solution of problem (1.3). By [7, Theorems 5.1 and 5.2 in Chapter 2],

$$
\begin{aligned}
&\left\|\{\tau^{-1}(u_k-u_{k-1})\}_1^N\right\|_{C_\tau^{\beta,\gamma}(E)}\\
&\quad\le M\Big[\max_{1\le i\le N}\|\tau^{-1}(I-R)R^{i-1}(u_0-A^{-1}\varphi_1)\|_E\\
&\qquad+\sup_{1\le i<i+r\le N}(r\tau)^{-\beta}((i+r)\tau)^{\gamma}\|\tau^{-1}(I-R)(R^{i+r-1}-R^{i-1})(u_0-A^{-1}\varphi_1)\|_E\\
&\qquad+\frac{M}{\beta(1-\beta)}\|\varphi^\tau\|_{C_\tau^{\beta,\gamma}(E)}\Big]
\end{aligned}
\tag{2.12}
$$

for the solution of the first order of accuracy implicit difference scheme for the approximate solutions of Cauchy problem (2.4). The proof of estimate (2.1) for difference scheme (1.3) is based on the estimate (2.12) and the following estimate:

$$
\begin{aligned}
&\max_{1\le i\le N}\|\tau^{-1}(I-R)R^{i-1}(u_0-A^{-1}\varphi_1)\|_E\\
&\qquad+\sup_{1\le i<i+r\le N}(r\tau)^{-\beta}((i+r)\tau)^{\gamma}\|\tau^{-1}(I-R)(R^{i+r-1}-R^{i-1})(u_0-A^{-1}\varphi_1)\|_E\\
&\quad\le M_1\Big[\,|\mu+A^{-1}(\varphi_{[\lambda/\tau]}-\varphi_1)|_1^{\beta,\gamma}+\frac{M}{\beta(1-\beta)}\|\varphi^\tau\|_{C_\tau^{\beta,\gamma}(E)}\Big]
\end{aligned}
\tag{2.13}
$$

for the solution of problem (1.3). Using formula (2.11) and estimate (2.7), we obtain

$$
\begin{aligned}
&\max_{1\le i\le N}\|\tau^{-1}(I-R)R^{i-1}(u_0-A^{-1}\varphi_1)\|_E\\
&\quad+\sup_{1\le i<i+r\le N}(r\tau)^{-\beta}((i+r)\tau)^{\gamma}\|\tau^{-1}(I-R)(R^{i+r-1}-R^{i-1})(u_0-A^{-1}\varphi_1)\|_E\\
&\le M(\lambda,\delta)\Bigg[\max_{1\le i\le N}\Bigg\|\tau^{-1}(I-R)R^{i-1}\Bigg(\sum_{j=1}^{[\lambda/\tau]}R^{[\lambda/\tau]-j+1}(\varphi_j-\varphi_{[\lambda/\tau]})\tau\\
&\qquad\qquad+\mu-(I-R^{[\lambda/\tau]})A^{-1}(\varphi_1-\varphi_{[\lambda/\tau]})\Bigg)\Bigg\|_E\\
&\qquad+\sup_{1\le i<i+r\le N}(r\tau)^{-\beta}((i+r)\tau)^{\gamma}\Bigg\|\tau^{-1}(I-R)(R^{i+r-1}-R^{i-1})\\
&\qquad\qquad\times\Bigg(\sum_{j=1}^{[\lambda/\tau]}R^{[\lambda/\tau]-j+1}(\varphi_j-\varphi_{[\lambda/\tau]})\tau+\mu-(I-R^{[\lambda/\tau]})A^{-1}(\varphi_1-\varphi_{[\lambda/\tau]})\Bigg)\Bigg\|_E\Bigg].
\end{aligned}
\tag{2.14}
$$

By [7, Theorem 5.2 in Chapter 2],

$$
\begin{aligned}
&\max_{1\le i\le N}\|\tau^{-1}(I-R)R^{i-1}(\mu+A^{-1}(-\varphi_1+\varphi_{[\lambda/\tau]}))\|_E \\
&\quad+\sup_{1\le i<i+r\le N}(r\tau)^{-\beta}((i+r)\tau)^{\gamma}\|\tau^{-1}(I-R)(R^{i+r-1}-R^{i-1})(\mu+A^{-1}(-\varphi_1+\varphi_{[\lambda/\tau]}))\|_E \\
&\le \sup_{0<\tau\le\tau_0}\Big[\max_{1\le i\le N}\|\tau^{-1}(I-R)R^{i-1}(\mu+A^{-1}(-\varphi_1+\varphi_{[\lambda/\tau]}))\|_E \\
&\qquad+\sup_{1\le i<i+r\le N}(r\tau)^{-\beta}((i+r)\tau)^{\gamma}\|\tau^{-1}(I-R)(R^{i+r-1}-R^{i-1})(\mu+A^{-1}(-\varphi_1+\varphi_{[\lambda/\tau]}))\|_E\Big] \\
&= |\mu+A^{-1}(-\varphi_1+\varphi_{[\lambda/\tau]})|_1^{\beta,\gamma}.
\end{aligned} \tag{2.15}
$$

Similar to estimate (2.12) and using estimates (2.6), we can show that

$$
\begin{aligned}
&\max_{1\le i\le N}\left\|\tau^{-1}(I-R)R^{i-1}\left(\sum_{j=1}^{[\lambda/\tau]}R^{[\lambda/\tau]-j+1}(\varphi_j-\varphi_{[\lambda/\tau]})\tau+R^{[\lambda/\tau]}A^{-1}(\varphi_1-\varphi_{[\lambda/\tau]})\right)\right\|_E \\
&\quad+\sup_{1\le i<i+r\le N}(r\tau)^{-\beta}((i+r)\tau)^{\gamma} \\
&\quad\times\left\|\tau^{-1}(I-R)(R^{i+r-1}-R^{i-1})\left(\sum_{j=1}^{[\lambda/\tau]}R^{[\lambda/\tau]-j+1}(\varphi_j-\varphi_{[\lambda/\tau]})\tau+R^{[\lambda/\tau]}A^{-1}(\varphi_1-\varphi_{[\lambda/\tau]})\right)\right\|_E \\
&\le\frac{M}{\beta(1-\beta)}\|\varphi^{\tau}\|_{C_\tau^{\beta,\gamma}(E)}.
\end{aligned} \tag{2.16}
$$

From this estimate and (2.15), estimate (2.13) follows. Now let us consider the difference scheme (1.4). In a similar manner with the difference scheme (1.3), we can obtain the formula

$$
\begin{aligned}
u_k&=D^k u_0+\sum_{j=1}^{k}\left(I+\frac{\tau}{2}A\right)D^{k-j+1}\varphi_j\tau,\quad k=1,\dots,N,\\
u_0&=T_\tau\bigg\{\left(I-\left(\lambda-\left[\frac{\lambda}{\tau}\right]\tau\right)A\right)\sum_{j=1}^{[\lambda/\tau]-1}\left(I+\frac{\tau}{2}A\right)D^{[\lambda/\tau]-j+1}\varphi_j\tau\\
&\qquad+\mu+D\left(\left(1+\frac{\lambda}{\tau}-\left[\frac{\lambda}{\tau}\right]\right)I+\frac{\tau}{2}A\right)\tau\varphi_{[\lambda/\tau]}\bigg\}
\end{aligned} \tag{2.17}
$$

for the solution of problem (1.4). Here

$$
T_\tau=\left(I-\left(I-\left(\lambda-\left[\frac{\lambda}{\tau}\right]\tau\right)A\right)D^{[\lambda/\tau]}\right)^{-1},\quad D=\left(I+\tau A+\frac{(\tau A)^2}{2}\right)^{-1}. \tag{2.18}
$$

By [7, Theorem 5.1 in Chapter 3],

$$
\begin{aligned}
&\|\{\tau^{-1}(u_k - u_{k-1})\}_1^N\|_{C_\tau^{\beta,\gamma}(E)} \\
&\quad\le M\Big[\max_{1\le i\le N}\Big\|A\Big(I+\frac{\tau}{2}A\Big)D^i(u_0 - A^{-1}\varphi_1)\Big\|_E \\
&\qquad + \sup_{1\le i<i+r\le N}(r\tau)^{-\beta}((i+r)\tau)^{\gamma}\Big\|A\Big(I+\frac{\tau}{2}A\Big)(D^{i+r}-D^i)(u_0 - A^{-1}\varphi_1)\Big\|_E \\
&\qquad + \frac{M}{\beta(1-\beta)}\|\varphi^\tau\|_{C_\tau^{\beta,\gamma}(E)}\Big]
\end{aligned}
\tag{2.19}
$$

for the solution of the second order of accuracy implicit difference scheme for the approximate solutions of Cauchy problem (2.4). The proof of estimate (2.1) for difference scheme (1.4) is based on the estimate (2.19) and the following estimate:

$$
\begin{aligned}
&\max_{1\le i\le N}\|\tau^{-1}(I-D)D^{i-1}(u_0 - A^{-1}\varphi_1)\|_E \\
&\qquad + \sup_{1\le i<i+r\le N}(r\tau)^{-\beta}((i+r)\tau)^{\gamma}\|\tau^{-1}(I-D)(D^{i+r-1}-D^{i-1})(u_0 - A^{-1}\varphi_1)\|_E \\
&\quad\le M_1\Big[\,|\mu + A^{-1}(\varphi_{[\lambda/\tau]} - \varphi_1)|_1^{\beta,\gamma} + \frac{M}{\beta(1-\beta)}\|\varphi^\tau\|_{C_\tau^{\beta,\gamma}(E)}\Big]
\end{aligned}
\tag{2.20}
$$

for the solution of problem (1.4). We have that

$$\|T_\tau\|_{E\to E} \le M(\lambda,\delta). \tag{2.21}$$

Actually, we can write

$$
\begin{aligned}
T_\tau - (I - \exp\{-\lambda A\})^{-1} &= T_\tau(I - \exp\{-\lambda A\})^{-1} \\
&\quad\times\Big(\Big(I - \Big(\lambda - \Big[\frac{\lambda}{\tau}\Big]\tau\Big)A\Big)D^{[\lambda/\tau]} - \exp\{-\lambda A\}\Big).
\end{aligned}
\tag{2.22}
$$

Then, using the triangle inequality and estimates (2.9),

$$\Big\|\Big(\lambda - \Big[\frac{\lambda}{\tau}\Big]\tau\Big)AD^{[\lambda/\tau]}\Big\|_{E\to E} \le M(\lambda,\delta)\tau, \tag{2.23}$$

$$\|D^{[\lambda/\tau]} - \exp\{-\lambda A\}\|_{E\to E} \le M(\lambda,\delta)\tau, \tag{2.24}$$

we obtain estimate (2.7). Estimate (2.23) follows from

$$\|D^k\|_{E\to E} \le M, \quad \|k\tau AD^k\|_{E\to E} \le M, \quad k \ge 1. \tag{2.25}$$

The proof of (2.24) is based on estimates (1.8) and (2.25) and it was proved in [34].

Using formula (2.17) and estimate (2.21), we obtain

$$\begin{aligned}
&\max_{1\le i\le N}\|\tau^{-1}(I-D)D^{i-1}(u_0-A^{-1}\varphi_1)\|_E\\
&\quad+\sup_{1\le i<i+r\le N}(r\tau)^{-\beta}((i+r)\tau)^{\gamma}\|\tau^{-1}(I-D)(D^{i+r-1}-D^{i-1})(u_0-A^{-1}\varphi_1)\|_E\\
&\le M(\lambda)\Bigg[\max_{1\le i\le N}\Bigg\|\tau^{-1}(I-D)D^{i-1}\Bigg(\sum_{j=1}^{[\lambda/\tau]}D^{[\lambda/\tau]-j+1}(\varphi_j-\varphi_{[\lambda/\tau]})\tau\\
&\qquad\qquad+\mu-(I-D^{[\lambda/\tau]})A^{-1}(\varphi_1-\varphi_{[\lambda/\tau]})\Bigg)\Bigg\|_E\\
&\qquad+\sup_{1\le i<i+r\le N}(r\tau)^{-\beta}((i+r)\tau)^{\gamma}\Bigg\|\tau^{-1}(I-D)(D^{i+r-1}-D^{i-1})\\
&\qquad\qquad\times\Bigg(\sum_{j=1}^{[\lambda/\tau]}D^{[\lambda/\tau]-j+1}(\varphi_j-\varphi_{[\lambda/\tau]})\tau+\mu-(I-D^{[\lambda/\tau]})A^{-1}(\varphi_1-\varphi_{[\lambda/\tau]})\Bigg)\Bigg\|_E\Bigg].
\end{aligned}\tag{2.26}$$

By [7, Theorem 5.2 in Chapter 3],

$$\begin{aligned}
&\max_{1\le i\le N}\|\tau^{-1}(I-D)d^{i-1}(\mu+A^{-1}(-\varphi_1+\varphi_{[\lambda/\tau]}))\|_E\\
&\quad+\sup_{1\le i<i+r\le N}(r\tau)^{-\beta}((i+r)\tau)^{\gamma}\|\tau^{-1}(I-D)(D^{i+r-1}-D^{i-1})(\mu+A^{-1}(-\varphi_1+\varphi_{[\lambda/\tau]}))\|_E\\
&\le\sup_{0<\tau\le\tau_0}\Bigg[\max_{1\le i\le N}\|\tau^{-1}(I-D)D^{i-1}(\mu+A^{-1}(-\varphi_1+\varphi_{[\lambda/\tau]}))\|_E\\
&\qquad+\sup_{1\le i<i+r\le N}(r\tau)^{-\beta}((i+r)\tau)^{\gamma}\|\tau^{-1}(I-D)(D^{i+r-1}-D^{i-1})(\mu+A^{-1}(-\varphi_1+\varphi_{[\lambda/\tau]}))\|_E\Bigg]\\
&=|\mu+A^{-1}(-\varphi_1+\varphi_{[\lambda/\tau]})|_1^{\beta,\gamma}.
\end{aligned}\tag{2.27}$$

Similar to estimate (2.19) and using estimates (2.25), we can show that

$$\begin{aligned}
&\max_{1\le i\le N}\Bigg\|\tau^{-1}(I-D)D^{i-1}\Bigg(\sum_{j=1}^{[\lambda/\tau]}D^{[\lambda/\tau]-j+1}(\varphi_j-\varphi_{[\lambda/\tau]})\tau+D^{[\lambda/\tau]}A^{-1}(\varphi_1-\varphi_{[\lambda/\tau]})\Bigg)\Bigg\|_E\\
&\quad+\sup_{1\le i<i+r\le N}(r\tau)^{-\beta}((i+r)\tau)^{\gamma}\\
&\quad\times\Bigg\|\tau^{-1}(I-D)(D^{i+r-1}-D^{i-1})\Bigg(\sum_{j=1}^{[\lambda/\tau]}D^{[\lambda/\tau]-j+1}(\varphi_j-\varphi_{[\lambda/\tau]})\tau\\
&\qquad\qquad+D^{[\lambda/\tau]}A^{-1}(\varphi_1-\varphi_{[\lambda/\tau]})\Bigg)\Bigg\|_E\le\frac{M}{\beta(1-\beta)}\|\varphi^{\tau}\|_{C_\tau^{\beta,\gamma}(E)}.
\end{aligned}\tag{2.28}$$

From these estimates, estimate (2.20) follows. □

Remark 2.2. The parameter γ can be chosen freely in $[0,\beta)$, which increases the number of spaces $C_\tau^{\beta,\gamma}(E)$ $(0 \leq \gamma \leq \beta,\ 0 < \beta < 1)$ of grid functions in which difference schemes (1.3) and (1.4) are well-posed.

THEOREM 2.3. *Let τ be a sufficiently small number. Then the solutions of the difference schemes (1.3) and (1.4) in $C_\tau^{\beta,\gamma}(E_{\alpha-\beta})$ $(0 \leq \gamma \leq \beta \leq \alpha < 1)$ satisfy the following coercivity inequalities:*

$$\left\| \{\tau^{-1}(u_k - u_{k-1})\}_1^N \right\|_{C_\tau^{\beta,\gamma}(E_{\alpha-\beta})} \leq \frac{M_1}{\alpha(1-\alpha)} \|\varphi^\tau\|_{C_\tau^{\beta,\gamma}(E_{\alpha-\beta})} + M_1 \left| \mu + A^{-1}(\varphi_{[\lambda/\tau]} - \varphi_1) \right|_{1+\alpha-\beta}^{\beta,\gamma}, \quad E_{\alpha-\beta}^{\beta,\gamma} = E^{\beta,\gamma}(E_{\alpha-\beta}), \tag{2.29}$$

where M_1 is independent not only of φ^τ, φ, α, β, γ, but also of τ.

Proof. Let us prove (2.29) for difference scheme (1.3). By [7, Theorem 5.3 in Chapter 2],

$$\begin{aligned}
&\left\| \{\tau^{-1}(u_k - u_{k-1})\}_1^N \right\|_{C_\tau^{\beta,\gamma}(E_{\alpha-\beta})} \\
&\quad \leq M\Big[\max_{1\leq i\leq N} \|\tau^{-1}(I-R)R^{i-1}(u_0 - A^{-1}\varphi_1)\|_{E_{\alpha-\beta}} \\
&\qquad + \sup_{1\leq i<i+r\leq N} (r\tau)^{-\beta}((i+r)\tau)^{\gamma} \|\tau^{-1}(I-R)(R^{i+r-1} - R^{i-1})(u_0 - A^{-1}\varphi_1)\|_{E_{\alpha-\beta}} \\
&\qquad + \frac{M}{\alpha(1-\alpha)} \|\varphi^\tau\|_{C_\tau^{\beta,\gamma}(E_{\alpha-\beta})} \Big]
\end{aligned} \tag{2.30}$$

for the solution of the first order of accuracy implicit difference scheme for the approximate solutions of Cauchy problem (2.4). The proof of estimate (2.29) for difference scheme (1.3) is based on estimate (2.30) and the following estimate:

$$\begin{aligned}
&\max_{1\leq i\leq N} \|\tau^{-1}(I-R)R^{i-1}(u_0 - A^{-1}\varphi_1)\|_{E_{\alpha-\beta}} \\
&\qquad + \sup_{1\leq i<i+r\leq N} (r\tau)^{-\beta}((i+r)\tau)^{\gamma} \|\tau^{-1}(I-R)(R^{i+r-1} - R^{i-1})(u_0 - A^{-1}\varphi_1)\|_{E_{\alpha-\beta}} \\
&\quad \leq M_1\Big[\left| \mu + A^{-1}(\varphi_{[\lambda/\tau]} - \varphi_1) \right|_{1+\alpha-\beta}^{\beta,\gamma} + \frac{M}{\alpha(1-\alpha)} \|\varphi^\tau\|_{C_\tau^{\beta,\gamma}(E_{\alpha-\beta})} \Big]
\end{aligned} \tag{2.31}$$

for the solution of problem (1.3). Using formula (2.11) and estimate (2.7), we obtain

$$\begin{aligned}
&\max_{1\le i\le N}\|\tau^{-1}(I-R)R^{i-1}(u_0-A^{-1}\varphi_1)\|_{E_{\alpha-\beta}}\\
&\quad+\sup_{1\le i<i+r\le N}(r\tau)^{-\beta}((i+r)\tau)^{\gamma}\|\tau^{-1}(I-R)(R^{i+r-1}-R^{i-1})(u_0-A^{-1}\varphi_1)\|_{E_{\alpha-\beta}}\\
&\le M(\lambda)\Bigg[\max_{1\le i\le N}\Bigg\|\tau^{-1}(I-R)R^{i-1}\Bigg(\sum_{j=1}^{[\lambda/\tau]}R^{[\lambda/\tau]-j+1}(\varphi_j-\varphi_{[\lambda/\tau]})\tau\\
&\qquad\qquad+\mu-(I-R^{[\lambda/\tau]})A^{-1}(\varphi_1-\varphi_{[\lambda/\tau]})\Bigg)\Bigg\|_{E_{\alpha-\beta}}\\
&\qquad+\sup_{1\le i<i+r\le N}(r\tau)^{-\beta}((i+r)\tau)^{\gamma}\Bigg\|\tau^{-1}(I-R)(R^{i+r-1}-R^{i-1})\\
&\qquad\qquad\times\Bigg(\sum_{j=1}^{[\lambda/\tau]}R^{[\lambda/\tau]-j+1}(\varphi_j-\varphi_{[\lambda/\tau]})\tau+\mu-(I-R^{[\lambda/\tau]})A^{-1}(\varphi_1-\varphi_{[\lambda/\tau]})\Bigg)\Bigg\|_{E_{\alpha-\beta}}\Bigg].
\end{aligned}\tag{2.32}$$

By [7, Theorem 5.2 in Chapter 2],

$$\begin{aligned}
&\max_{1\le i\le N}\|\tau^{-1}(I-R)R^{i-1}(\mu+A^{-1}(-\varphi_1+\varphi_{[\lambda/\tau]}))\|_{E_{\alpha-\beta}}\\
&\quad+\sup_{1\le i<i+r\le N}(r\tau)^{-\beta}((i+r)\tau)^{\gamma}\|\tau^{-1}(I-R)(R^{i+r-1}-R^{i-1})(\mu+A^{-1}(-\varphi_1+\varphi_{[\lambda/\tau]}))\|_{E_{\alpha-\beta}}\\
&\le\sup_{0<\tau\le\tau_0}\Bigg[\max_{1\le i\le N}\|\tau^{-1}(I-R)R^{i-1}(\mu+A^{-1}(-\varphi_1+\varphi_{[\lambda/\tau]}))\|_{E_{\alpha-\beta}}\\
&\qquad+\sup_{1\le i<i+r\le N}(r\tau)^{-\beta}((i+r)\tau)^{\gamma}\|\tau^{-1}(I-R)(R^{i+r-1}-R^{i-1})(\mu+A^{-1}(-\varphi_1+\varphi_{[\lambda/\tau]}))\|_{E_{\alpha-\beta}}\Bigg]\\
&=|\mu+A^{-1}(-\varphi_1+\varphi_{[\lambda/\tau]})|^{\beta,\gamma}_{1+\alpha-\beta}.
\end{aligned}\tag{2.33}$$

Similar to estimate (2.30) and using estimates (2.6), we can show that

$$\begin{aligned}
&\max_{1\le i\le N}\Bigg\|\tau^{-1}(I-R)R^{i-1}\Bigg(\sum_{j=1}^{[\lambda/\tau]}R^{[\lambda/\tau]-j+1}(\varphi_j-\varphi_{[\lambda/\tau]})\tau+R^{[\lambda/\tau]}A^{-1}(\varphi_1-\varphi_{[\lambda/\tau]})\Bigg)\Bigg\|_{E_{\alpha-\beta}}\\
&\quad+\sup_{1\le i<i+r\le N}(r\tau)^{-\beta}((i+r)\tau)^{\gamma}\Bigg\|\tau^{-1}(I-R)(R^{i+r-1}-R^{i-1})\\
&\qquad\qquad\times\Bigg(\sum_{j=1}^{[\lambda/\tau]}R^{[\lambda/\tau]-j+1}(\varphi_j-\varphi_{[\lambda/\tau]})\tau+R^{[\lambda/\tau]}A^{-1}(\varphi_1-\varphi_{[\lambda/\tau]})\Bigg)\Bigg\|_{E_{\alpha-\beta}}\\
&\le\frac{M}{\alpha(1-\alpha)}\|\varphi^{\tau}\|_{C_{\tau}^{\beta,\gamma}(E_{\alpha-\beta})}.
\end{aligned}\tag{2.34}$$

From this estimate and (2.33), estimate (2.31) follows. Now let us consider the difference scheme (1.4). By [7, Theorem 5.3 in Chapter 3],

$$\begin{aligned}
&\left\|\left\{\tau^{-1}(u_k-u_{k-1})\right\}_1^N\right\|_{C_\tau^{\beta,\gamma}(E_{\alpha-\beta})}\\
&\quad\leq M\bigg[\max_{1\leq i\leq N}\left\|A\left(I+\frac{\tau}{2}A\right)D^i(u_0-A^{-1}\varphi_1)\right\|_{E_{\alpha-\beta}}\\
&\qquad+\sup_{1\leq i<i+r\leq N}(r\tau)^{-\beta}((i+r)\tau)^{\gamma}\left\|A\left(I+\frac{\tau}{2}A\right)(D^{i+r}-D^i)(u_0-A^{-1}\varphi_1)\right\|_{E_{\alpha-\beta}}\\
&\qquad+\frac{M}{\alpha(1-\alpha)}\|\varphi^\tau\|_{C_\tau^{\beta,\gamma}(E_{\alpha-\beta})}\bigg]
\end{aligned}\tag{2.35}$$

for the solution of the second order of accuracy implicit difference scheme for the approximate solutions of Cauchy problem (2.4). The proof of estimate (2.29) for difference scheme (1.4) is based on estimate (2.35) and the following estimate:

$$\begin{aligned}
&\max_{1\leq i\leq N}\|\tau^{-1}(I-D)D^{i-1}(u_0-A^{-1}\varphi_1)\|_{E_{\alpha-\beta}}\\
&\qquad+\sup_{1\leq i<i+r\leq N}(r\tau)^{-\beta}((i+r)\tau)^{\gamma}\|\tau^{-1}(I-D)(D^{i+r-1}-D^{i-1})(u_0-A^{-1}\varphi_1)\|_{E_{\alpha-\beta}}\\
&\quad\leq M_1\bigg[\left|\mu+A^{-1}(\varphi_{[\lambda/\tau]}-\varphi_1)\right|^{\beta,\gamma}_{1+\alpha-\beta}+\frac{M}{\beta(1-\beta)}\|\varphi^\tau\|_{C_\tau^{\beta,\gamma}(E_{\alpha-\beta})}\bigg]
\end{aligned}\tag{2.36}$$

for the solution of problem (1.4). Using formula (2.17) and estimate (2.21), we obtain

$$\begin{aligned}
&\max_{1\leq i\leq N}\|\tau^{-1}(I-D)D^{i-1}(u_0-A^{-1}\varphi_1)\|_{E_{\alpha-\beta}}\\
&\quad+\sup_{1\leq i<i+r\leq N}(r\tau)^{-\beta}((i+r)\tau)^{\gamma}\|\tau^{-1}(I-D)(D^{i+r-1}-D^{i-1})(u_0-A^{-1}\varphi_1)\|_{E_{\alpha-\beta}}\\
&\leq M(\lambda)\bigg[\max_{1\leq i\leq N}\bigg\|\tau^{-1}(I-D)D^{i-1}\bigg(\sum_{j=1}^{[\lambda/\tau]}D^{[\lambda/\tau]-j+1}(\varphi_j-\varphi_{[\lambda/\tau]})\tau\\
&\qquad\qquad+\mu-(I-D^{[\lambda/\tau]})A^{-1}(\varphi_1-\varphi_{[\lambda/\tau]})\bigg)\bigg\|_{E_{\alpha-\beta}}\\
&\qquad+\sup_{1\leq i<i+r\leq N}(r\tau)^{-\beta}((i+r)\tau)^{\gamma}\bigg\|\tau^{-1}(I-D)(D^{i+r-1}-D^{i-1})\\
&\qquad\qquad\times\bigg(\sum_{j=1}^{[\lambda/\tau]}D^{[\lambda/\tau]-j+1}(\varphi_j-\varphi_{[\lambda/\tau]})\tau+\mu-(I-D^{[\lambda/\tau]})A^{-1}(\varphi_1-\varphi_{[\lambda/\tau]})\bigg)\bigg\|_{E_{\alpha-\beta}}\bigg].
\end{aligned}\tag{2.37}$$

By [7, Theorem 5.3 in Chapter 3],

$$\begin{aligned}
&\max_{1\le i\le N}\|\tau^{-1}(I-D)d^{i-1}(\mu+A^{-1}(-\varphi_1+\varphi_{[\lambda/\tau]}))\|_E\\
&\quad+\sup_{1\le i<i+r\le N}(r\tau)^{-\beta}((i+r)\tau)^{\gamma}\|\tau^{-1}(I-D)(D^{i+r-1}-D^{i-1})(\mu+A^{-1}(-\varphi_1+\varphi_{[\lambda/\tau]}))\|_{E_{\alpha-\beta}}\\
&\le\sup_{0<\tau\le\tau_0}\Big[\max_{1\le i\le N}\|\tau^{-1}(I-D)D^{i-1}(\mu+A^{-1}(-\varphi_1+\varphi_{[\lambda/\tau]}))\|_{E_{\alpha-\beta}}\\
&\qquad+\sup_{1\le i<i+r\le N}(r\tau)^{-\beta}((i+r)\tau)^{\gamma}\|\tau^{-1}(I-D)(D^{i+r-1}-D^{i-1})(\mu+A^{-1}(-\varphi_1+\varphi_{[\lambda/\tau]}))\|_{E_{\alpha-\beta}}\Big]\\
&=|\mu+A^{-1}(-\varphi_1+\varphi_{[\lambda/\tau]})|^{\beta,\gamma}_{1+\alpha-\beta}.
\end{aligned}\tag{2.38}$$

Similar to estimate (2.35) and using estimates (2.25), we can show that

$$\begin{aligned}
&\max_{1\le i\le N}\left\|\tau^{-1}(I-D)D^{i-1}\left(\sum_{j=1}^{[\lambda/\tau]}D^{[\lambda/\tau]-j+1}(\varphi_j-\varphi_{[\lambda/\tau]})\tau+D^{[\lambda/\tau]}A^{-1}(\varphi_1-\varphi_{[\lambda/\tau]})\right)\right\|_{E_{\alpha-\beta}}\\
&\quad+\sup_{1\le i<i+r\le N}(r\tau)^{-\beta}((i+r)\tau)^{\gamma}\bigg\|\tau^{-1}(I-D)(D^{i+r-1}-D^{i-1})\\
&\qquad\times\left(\sum_{j=1}^{[\lambda/\tau]}D^{[\lambda/\tau]-j+1}(\varphi_j-\varphi_{[\lambda/\tau]})\tau+D^{[\lambda/\tau]}A^{-1}(\varphi_1-\varphi_{[\lambda/\tau]})\right)\bigg\|_{E_{\alpha-\beta}}\\
&\le\frac{M}{\alpha(1-\alpha)}\|\varphi^{\tau}\|_{C_\tau^{\beta,\gamma}(E_{\alpha-\beta})}.
\end{aligned}\tag{2.39}$$

From these estimates, estimate (2.36) follows. □

Remark 2.4. The spaces $C_\tau^{\beta,\gamma}(E_{\alpha-\beta})$ of grid functions, in which coercive solvability has been established, depend on the parameters α, β, and γ. However, the constants in the coercive inequalities depend only on α. Hence, we can be choose the parameters β and γ freely, which increases the number of spaces of grid functions in which difference schemes (1.3) and (1.4) are well-posed.

Remark 2.5. Using the coercive stability estimates of Theorems 2.1–2.3 and by passing to the limit for $\tau\to 0$, one can recover theorems on coercive solvability of the nonlocal-boundary value problem (1.1) [6].

3. Applications

We consider the boundary value problem on the range $\{0 \le t \le 1,\ x \in \mathbb{R}^n\}$ for $2m$-order multidimensional parabolic equation

$$\frac{\partial v(t,x)}{\partial t} + \sum_{|\tau|=2m} a_\tau(x) \frac{\partial^{|\tau|} v(t,x)}{\partial x_1^{\tau_1} \cdots \partial x_n^{\tau_n}} + \delta v(t,x) = f(t,x), \quad 0 \le t \le 1,$$
$$v(0,x) = v(\lambda,x) + \mu(x), \quad 0 < \lambda \le 1,\ x \in R^n, \quad |\tau| = \tau_1 + \cdots + \tau_n, \tag{3.1}$$

where $a_r(x)$, $f(t,x)$ and $\mu(x)$ are given sufficiently smooth functions and $\delta > 0$ is a sufficiently large positive constant.

Now, the abstract theorems given above are applied in the investigation of difference schemes for approximate solution of (3.1). The discretization of problem (3.1) is carried out in two steps. Let us define the grid space $\mathbb{R}_h^n$ $(0 < h \le h_0)$ as the set of all points of the Euclidean space $\mathbb{R}^n$ whose coordinates are given by

$$x_k = s_k h, \quad s_k = 0, \pm 1, \pm 2, \ldots, \quad k = 1, \ldots, n. \tag{3.2}$$

In the first step, let us give the difference operator A_h^x by the formula

$$A_h^x u^h = \sum_{2m \le |r| \le S} b_r^x D_h^r u^h + \delta u^h. \tag{3.3}$$

The coefficients are chosen in such a way that the operator A_h^x approximates in a specified way the operator

$$\sum_{|r|=2m} a_r(x) \frac{\partial^{|r|}}{\partial x_1^{r_1} \cdots \partial x_n^{r_n}} + \delta. \tag{3.4}$$

We will assume that for $|\xi_k h| \le \pi$ and fixed x, the symbol $A^x(\xi h, h)$ of the operator $A_h^x - \delta$ satisfies the inequalities

$$(-1)^m A^x(\xi h, h) \ge M_1 |\xi|^{2m}, \qquad |\arg A^x(\xi h, h)| \le \phi < \phi_0 \le \frac{\pi}{2}. \tag{3.5}$$

With the help of A_h^x, we arrive at the nonlocal boundary value problem

$$\frac{dv^h(t,x)}{dt} + A_h^x v^h(t,x) = f^h(t,x), \quad 0 \le t \le 1,$$
$$v^h(0,x) = v^h(\lambda,x) + \mu^h(x), \quad x \in \mathbb{R}_h^n, \tag{3.6}$$

for an infinite system of ordinary differential equations.

In the second step, we replace problem (3.6) by the difference schemes

$$\begin{gathered}
\frac{u_k^h(x)-u_{k-1}^h(x)}{\tau}+A_h^x u_k^h(x)=\varphi_k^h(x), \quad \varphi_k^h(x)=f^h(t_k,x), \quad t_k=k\tau, \\
1\le k\le N, \quad u_0^h(x)=u_{[\lambda/\tau]}^h(x)+\mu^h(x), \quad x\in\mathbb{R}_h^n, \\
\frac{u_k^h(x)-u_{k-1}^h(x)}{\tau}+A_h^x\left(I+\frac{\tau A_h^x}{2}\right)u_k^h(x)=\left(I+\frac{\tau A_h^x}{2}\right)\varphi_k^h(x), \\
\varphi_k^h(x)=f^h\left(t_k-\frac{\tau}{2},x\right), \quad t_k=k\tau,\ 1\le k\le N, \\
u_0^h(x)=\left(I-\left(\lambda-\left[\frac{\lambda}{\tau}\right]\tau\right)A_h^x\right)u_{[\lambda/\tau]}^h(x)+\mu^h(x)+\left(\lambda-\left[\frac{\lambda}{\tau}\right]\tau\right)\varphi_{[\lambda/\tau]}^h(x), \quad x\in\mathbb{R}_h^n.
\end{gathered} \tag{3.7}$$

Let us give a number of corollaries of the abstract theorems given above. To formulate our result, we need to introduce the spaces $C_h=C(\mathbb{R}_h^n)$ and $C_h^\beta=C^\beta(\mathbb{R}_h^n)$ of all bounded grid functions $u^h(x)$ defined on $\mathbb{R}_h^n$, equipped with the norms

$$\begin{gathered}
||u^h||_{C_h}=\sup_{x\varepsilon\mathbb{R}_h^n}|u^h(x)|, \\
||u^h||_{C_h^\beta}=\sup_{x\varepsilon\mathbb{R}_h^n}|u^h(x)|+\sup_{x,y\varepsilon\mathbb{R}_h^n}\frac{|u^h(x)-u^h(x+y)|}{|y|^\beta}.
\end{gathered} \tag{3.8}$$

THEOREM 3.1. *The solutions of the difference schemes (3.7) satisfy the coercivity estimates:*

$$\begin{gathered}
\left|\left|\{\tau^{-1}(u_k^h-u_{k-1}^h)\}_1^{N-1}\right|\right|_{C_\tau^{\beta,\gamma}(C_h^\gamma)} \\
\le M(\alpha,\nu)\left[\sum_{|r|=2m}||D_h^r\mu^h||_{C_h^{\gamma+2m\alpha}}+||\varphi^{\tau,h}||_{C_\tau^{\beta,\gamma}(C_h^\gamma)}\right], \quad 0<2m\alpha+\nu<1,\ \nu>0,
\end{gathered} \tag{3.9}$$

where $M(\alpha,\nu)$ does not depend on $\varphi^{\tau,h}$, μ^h, h, and τ.

The proof of this theorem is based on the abstract theorems (Theorems 2.1 and 2.3) and the positivity of the operator A_h^x in C_h [35] and on the coercivity inequality for an elliptic operator A_h^x in C_h^β [7] and on the following theorem.

THEOREM 3.2 (see [7]). *For any $0<\beta<1/2m$, the norms in the spaces $E_\beta(C_h,A_h^x)$ and $C_h^{2m\beta}$ are equivalent uniformly in h.*

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