

*Research Article*

## **Multiobjective Output Feedback Control of a Class of Stochastic Hybrid Systems with State-Dependent Noise**

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This paper deals with dynamic output feedback control of continuous-time active fault tolerant control systems with Markovian parameters (AFTCSMP) and state-dependent noise. The main contribution is to formulate conditions for multiperformance design, related to this class of stochastic hybrid systems, that take into account the problematic resulting from the fact that the controller only depends on the fault detection and isolation (FDI) process. The specifications and objectives under consideration include stochastic stability,  $\mathcal{H}_2$  and  $\mathcal{H}_\infty$  (or more generally, *stochastic integral quadratic constraints*) performances. Results are formulated as matrix inequalities. The theoretical results are illustrated using a classical example from literature.

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### **1. Introduction**

Control engineers are faced with increasingly complex systems where dependability considerations are sometimes more important than performance. Sensor, actuator, or process (plant) failures may drastically change the system behavior, resulting in performance degradation or even instability. Thus, fault tolerance is essential for modern, highly complex control systems. Fault tolerant control systems (FTCS) are needed in order to preserve or maintain the performance objectives, or if that turns out to be impossible, to assign new (achievable) objectives so as to avoid catastrophic failures. FTCS have been a subject of great practical importance, which has attracted a lot of interest for the last three decades. A bibliographical review on reconfigurable fault tolerant control systems can be found in [1].

Active fault tolerant control systems are feedback control systems that reconfigure the control law in real time based on the response from an automatic fault detection and

identification (FDI) scheme. The dynamic behavior of active fault tolerant control systems (AFTCS) is governed by stochastic differential equations (because the failures and failure detection occur randomly) and can be viewed as a general hybrid system [2]. A major class of hybrid systems is jump linear systems (JLS). In JLS, a single jump process is used to describe the random variations affecting the system parameters. This process is represented by a finite state Markov chain and is called the plant regime mode. The theory of stability, optimal control and  $\mathcal{H}_2/\mathcal{H}_\infty$  control, as well as important applications of such systems, can be found in several papers in the current literature, for instance in [3–12].

To deal with AFTCS, another class of hybrid systems was defined, denoted as active fault tolerant control systems with Markovian parameters (AFTCSMP). In this class of hybrid systems, two random processes are defined: the first random process represents system components failures and the second random process represents the FDI process used to reconfigure the control law. This model was proposed by Srichander and Walker [2]. Necessary and sufficient conditions for stochastic stability of AFTCSMP were developed for a single component failure (actuator failures). In [13], the authors proposed a dynamical model that takes into account multiple failures occurring at different locations in the system, such as in control actuators and plant components. The authors derived necessary and sufficient conditions for the stochastic stability in the mean square sense. The problem of stochastic stability of AFTCSMP in the presence of noise, parameter uncertainties, detection errors, detection delays and actuator saturation limits has also been investigated in [13–16]. Another issue related to the synthesis of fault tolerant control laws was also addressed by [17–19]. In [17], the authors designed an optimal control law for AFTCSMP using the matrix minimum principle to minimize an equivalent deterministic cost function. The problem of  $\mathcal{H}_\infty$  and robust  $\mathcal{H}_\infty$  control (in the presence of norm bounded parameter uncertainties) was treated in [18, 19] for both continuous and discrete time AFTCSMP. The authors defined a single failure process to characterize random failures affecting the system, and they showed that the state feedback control problem can be solved in terms of the solutions of a set of coupled Riccati inequalities. The dynamic/static output feedback counterpart was treated by [20–23] in a convex programming framework. Indeed, the authors provide an LMI characterization of dynamical/static output feedback compensators that stochastically stabilize (robustly stabilize) the AFTCSMP and ensures  $\mathcal{H}_\infty$  (robust  $\mathcal{H}_\infty$ ) constraints. In addition, it is important to mention that the design problem in the framework of AFTCSMP remains an open and challenging problematic. This is due, particularly, to the fact that the controller only depends on the FDI process, that is, the number of controllers to be designed is less than the total number of the closed loop systems modes by combining both failure and FDI processes. The design problem involves searching feasible solutions of a problem where there are more constraints than variables to be solved. Generally speaking, there lacks tractable design methods for this stochastic FTC problem. Indeed, in [16, 18–21], the authors make the assumption that the controller must access both failures and FDI processes. However, this assumption is too restrictive to be applicable in practical FTC systems. In this paper, and inspired by the work of [24] on *mode-independent*  $\mathcal{H}_\infty$  filtering for Markovian jump-linear systems, the assumption on the availability of failure processes, for the synthesis

purposes, is stressed. The results are based on a version of the well-known Finsler lemma and a special parametrization of the Lyapunov matrices. This paper is concerned with dynamic output feedback control of continuous time AFTCSMP with state-dependent noise. The main contribution is to formulate conditions for multiperformance design, related to this class of stochastic hybrid systems, that take into account the problematic resulting from the fact that the controller only depends on the FDI process. The specifications and objectives under consideration include stochastic stability,  $\mathcal{H}_2$  and  $\mathcal{H}_\infty$  (or more generally, *stochastic integral quadratic constraints*) performances. Results are formulated as matrix inequalities. A coordinate descent-type algorithm is provided and its running is illustrated on a VTOL helicopter example.

This paper is organized as follows. Section 2 describes the dynamical model of the system with appropriately defined random processes. A brief summary of basic stochastic terms, results, and definitions are given in Section 3. Section 4 considers the  $\mathcal{H}_2/\mathcal{H}_\infty$  control problem. In Section 5, a coordinate descent-type algorithm is provided and its running is illustrated on a classical example from literature. Finally, a conclusion is given in Section 6.

*Notations.* The notations in this paper are quite standard.  $\mathbb{R}^{m \times n}$  is the set of  $m$ -by- $n$  real matrices.  $A^T$  is the transpose of the matrix  $A$ . The notation  $X \geq Y$  ( $X > Y$ , resp.), where  $X$  and  $Y$  are symmetric matrices, means that  $X - Y$  is positive semidefinite (positive definite, resp.);  $\mathbb{I}$  and  $\mathbf{0}$  are identity and zero matrices of appropriate dimensions, respectively;  $\mathcal{E}\{\cdot\}$  denotes the expectation operator with respect to some probability measure  $P$ ;  $L^2[0, \infty)$  stands for the space of square-integrable vector functions over the interval  $[0, \infty)$ ;  $\|\cdot\|$  refers to either the Euclidean vector norm or the matrix norm, which is the operator norm induced by the standard vector norm;  $\|\cdot\|_2$  stands for the norm in  $L^2[0, \infty)$ ; while  $\|\cdot\|_{\mathcal{E}_2}$  denotes the norm in  $L^2((\Omega, \mathcal{F}, P), [0, \infty))$ ;  $(\Omega, \mathcal{F}, P)$  is a probability space. In block matrices,  $*$  indicates symmetric terms:  $\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} = \begin{bmatrix} A & * \\ B^T & C \end{bmatrix} = \begin{bmatrix} A & B \\ * & C \end{bmatrix}$ .

## 2. Dynamical model of AFTCSMP with state-dependent noise

To describe the class of linear systems with Markovian jumping parameters that we deal with in this paper, let us fix a complete probability space  $(\Omega, \mathcal{F}, P)$ . This class of systems owns a hybrid state vector. The first component vector is continuous and represents the system states, and the second one is discrete and represents the failure processes affecting the system. The dynamical model of the AFTCSMP with Wiener process, defined in the fundamental probability space  $(\Omega, \mathcal{F}, P)$ , is described by the following differential equations:

$$\varphi : \begin{cases} dx(t) = A(\xi(t))x(t)dt + B(\eta(t))u(y(t), \psi(t), t)dt \\ \quad + E(\xi(t), \eta(t))w(t)dt + \sum_{l=1}^v \mathbb{W}_l(\xi(t), \eta(t))x(t)d\omega_l(t), \\ y(t) = C_2x(t) + D_2(\xi(t), \eta(t))w(t), \\ z(t) = C_1x(t) + D_1(\eta(t))u(y(t), \psi(t), t), \end{cases} \quad (2.1)$$

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where  $x(t) \in \mathbb{R}^n$  is the system state,  $u(y(t), \psi(t), t) \in \mathbb{R}^r$  is the system input,  $y(t) \in \mathbb{R}^q$  is the system measured output,  $z(t) \in \mathbb{R}^p$  is the controlled output,  $w(t) \in \mathbb{R}^m$  is the system external disturbance,  $\xi(t)$ ,  $\eta(t)$ , and  $\psi(t)$  represent the plant component failure process, the actuator failure process, and the FDI process, respectively.  $\xi(t)$ ,  $\eta(t)$  and  $\psi(t)$  are separable and measurable Markov processes with finite state spaces  $Z = \{1, 2, \dots, z\}$ ,  $S = \{1, 2, \dots, s\}$ , and  $R = \{1, 2, \dots, \sigma\}$ , respectively.  $\omega(t) = [\omega_1(t) \cdots \omega_v(t)]^T$  is a  $v$ -dimensional standard Wiener process on a given probability space  $(\Omega, \mathcal{F}, P)$ , that is assumed to be independent of the Markov processes. The matrices  $A(\xi(t))$ ,  $B(\eta(t))$ ,  $E(\xi(t), \eta(t))$ ,  $D_2(\xi(t), \eta(t))$ ,  $D_1(\eta(t))$ , and  $\mathbb{W}_i(\xi(t), \eta(t))$  are properly dimensioned matrices which depend on random parameters.

In AFTCS, we consider that the control law is only a function of the measurable FDI process  $\psi(t)$ . Therefore, we introduce a full-order dynamic output feedback compensator ( $\varphi_d$ ) of the form

$$\varphi_d : \begin{cases} dv(t) = A_c(\psi(t))v(t)dt + B_c(\psi(t))y(t)dt \\ u(t) = C_c(\psi(t))v(t). \end{cases} \quad (2.2)$$

Applying the controller  $\varphi_d$  to the AFTCSMP  $\varphi$ , we obtain the following closed loop system:

$$\varphi_{cl} : \begin{cases} d\chi(t) = \Lambda(\xi(t), \eta(t), \psi(t))\chi(t)dt + \bar{E}(\xi(t), \eta(t), \psi(t))w(t)dt \\ \quad + \sum_{l=1}^v \bar{\mathbb{W}}_l(\xi(t), \eta(t))\chi(t)d\omega_l(t), \\ \bar{y}(t) = \bar{C}_2(\psi(t))\chi(t) + \bar{D}_2(\xi(t), \eta(t))w(t), \\ z(t) = \bar{C}_1(\eta(t), \psi(t))\chi(t), \end{cases} \quad (2.3)$$

where

$$\begin{aligned} \chi(t) &= [x(t)^T, v(t)^T]^T; & \bar{y}(t) &= [y(t)^T, u(t)^T]^T; \\ \Lambda(\xi(t), \eta(t), \psi(t)) &= \begin{bmatrix} A(\xi(t)) & B(\eta(t))C_c(\psi(t)) \\ B_c(\psi(t))C_2 & A_c(\psi(t)) \end{bmatrix}; \\ \bar{E}(\xi(t), \eta(t), \psi(t)) &= \begin{bmatrix} E(\xi(t), \eta(t)) \\ B_c(\psi(t))D_2(\xi(t), \eta(t)) \end{bmatrix}; & \bar{C}_2(\psi(t)) &= \begin{bmatrix} C_2 & 0 \\ 0 & C_c(\psi(t)) \end{bmatrix}; \\ \bar{D}_2(\xi(t), \eta(t)) &= \begin{bmatrix} D_2(\xi(t), \eta(t)) \\ 0 \end{bmatrix}; & \bar{C}_1(\eta(t), \psi(t)) &= [C_1 \quad D_1(\eta(t))C_c(\psi(t))]; \\ \bar{\mathbb{W}}_i(\xi(t), \eta(t)) &= \begin{bmatrix} \mathbb{W}_i(\xi(t), \eta(t)) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}. \end{aligned} \quad (2.4)$$

Our goal is to compute dynamical output feedback controllers  $\varphi_d$  that meet various specifications on the closed loop behavior. The specifications and objectives under consideration include stochastic stability,  $\mathcal{H}_2$  performance, and  $\mathcal{H}_\infty$  performance (or more generally, *stochastic integral quadratic constraints* (SIQC)).

**2.1. The FDI and the failure processes.**  $\xi(t)$ ,  $\eta(t)$ , and  $\psi(t)$  being homogeneous Markov processes with finite state spaces, we can define the transition probability of the plant components failure process as [2, 16]

$$\begin{aligned} p_{ij}(\Delta t) &= \pi_{ij}\Delta t + o(\Delta t) \quad (i \neq j), \\ p_{ii}(\Delta t) &= 1 - \sum_{i \neq j} \pi_{ij}\Delta t + o(\Delta t) \quad (i = j). \end{aligned} \quad (2.5)$$

The transition probability of the actuator failure process is given by

$$\begin{aligned} p_{kl}(\Delta t) &= \nu_{kl}\Delta t + o(\Delta t) \quad (k \neq l), \\ p_{kk}(\Delta t) &= 1 - \sum_{k \neq l} \nu_{kl}\Delta t + o(\Delta t) \quad (k = l), \end{aligned} \quad (2.6)$$

where  $\pi_{ij}$  is the plant components failure rate and  $\nu_{kl}$  is the actuator failure rate.

Given that  $\xi = k$  and  $\eta = l$ , the conditional transition probability of the FDI process  $\psi(t)$  is

$$\begin{aligned} p_{iv}^{kl}(\Delta t) &= \lambda_{iv}^{kl}\Delta t + o(\Delta t) \quad (i \neq v), \\ p_{ii}^{kl}(\Delta t) &= 1 - \sum_{i \neq v} \lambda_{iv}^{kl}\Delta t + o(\Delta t) \quad (i = v). \end{aligned} \quad (2.7)$$

Here,  $\lambda_{iv}^{kl}$  represents the transition rate from  $i$  to  $v$  for the Markov process  $\psi(t)$  conditioned on  $\xi = k \in Z$ , and  $\eta = l \in S$ . Depending on the values of  $i, v \in R, k \in Z$ , and  $l \in S$ , various interpretations, such as rate of false detection and isolation, rate of correct detection and isolation, false alarm recovery rate, can be given to  $\lambda_{iv}^{kl}$  [2, 16].

For notational simplicity, we will denote  $A(\xi(t)) = A_i$  when  $\xi(t) = i \in Z$ ,  $B(\eta(t)) = B_j$  and  $D_1(\eta(t)) = D_{1j}$  when  $\eta(t) = j \in S$ ,  $E(\xi(t), \eta(t)) = E_{ij}$ ,  $D_2(\xi(t), \eta(t)) = D_{2ij}$  and  $\mathbb{W}_l(\xi(t), \eta(t)) = \mathbb{W}_{lij}$  when  $\xi(t) = i \in Z$ ,  $\eta(t) = j \in S$  and  $A_c(\psi(t)) = A_{ck}$ ,  $B_c(\psi(t)) = B_{ck}$ ,  $C_c(\psi(t)) = C_{ck}$  when  $\psi(t) = k \in R$ . We also denote  $x(t) = x_t$ ,  $y(t) = y_t$ ,  $z(t) = z_t$ ,  $w(t) = w_t$ ,  $\xi(t) = \xi_t$ ,  $\eta(t) = \eta_t$ ,  $\psi(t) = \psi_t$  and the initial conditions  $x(t_0) = x_0$ ,  $\xi(t_0) = \xi_0$ ,  $\eta(t_0) = \eta_0$  and  $\psi(t_0) = \psi_0$ .

### 3. Definitions and basic results

In this section, we will first give a basic definition related to stochastic stability notion and then we will summarize some results about exponential stability in the mean square sense of the AFTCSMP.

**3.1. Stochastic stability.** For system (2.1), when  $u_t \equiv 0$  for all  $t \geq 0$ , we have the following definition.

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*Defintion 3.1.* System (2.1) is said to be *internally exponentially stable in the mean square sense (IESS)* if there exist positive constants  $\alpha$  and  $\beta$  such that the solution of

$$dx_t = A(\xi_t)x_t dt + \sum_{l=1}^v W_l(\xi_t, \eta_t)x_t d\omega_{lt} \quad (3.1)$$

satisfies the following inequality:

$$\mathcal{E}\{\|x_t\|^2\} \leq \beta\|x_0\|^2 \exp[-\alpha(t-t_0)] \quad (3.2)$$

for arbitrary initial conditions  $(x_0, \xi_0, \eta_0, \psi_0)$ .

The following theorem gives a sufficient condition for internal exponential stability in the mean square sense for the closed loop system (2.3).

**THEOREM 3.2.** *The closed loop system (2.3) is IESS for  $t \geq t_0$  if there exists a Lyapunov function  $\vartheta(\chi_t, \xi_t, \eta_t, \psi_t, t)$  such that*

$$\begin{aligned} K_1\|\chi_t\|^2 &\leq \vartheta(\chi_t, \xi_t, \eta_t, \psi_t, t) \leq K_2\|\chi_t\|^2, \\ \mathcal{L}\vartheta(\chi_t, \xi_t, \eta_t, \psi_t, t) &\leq -K_3\|\chi_t\|^2 \end{aligned} \quad (3.3)$$

for some positive constants  $K_1, K_2$ , and  $K_3$ , where  $\mathcal{L}$  is the weak infinitesimal operator of the joint Markov process  $\{\chi_t, \xi_t, \eta_t, \psi_t\}$ .

*Remark 3.3.* The proof of the Markovian property of the joint process  $\{\chi_t, \xi_t, \eta_t, \psi_t\}$  is given for example in [2, 16, 25].

A necessary condition for internal exponential stability in the mean square sense for the closed loop system  $\varphi_{cl}$  is given by Theorem 3.4.

**THEOREM 3.4.** *If the system (2.3) is IESS, then for any given quadratic positive definite function  $W(\chi_t, \xi_t, \eta_t, \psi_t, t)$  in the variables  $\chi_t$  which is bounded and continuous for all  $t \geq t_0$ , for all  $\xi_t \in Z$ , for all  $\eta_t \in S$  and for all  $\psi_t \in R$ , there exists a quadratic positive definite function  $\vartheta(\chi_t, \xi_t, \eta_t, \psi_t, t)$  in  $\chi_t$  that satisfies the conditions in Theorem 3.2 and is such that  $\mathcal{L}\vartheta(\chi_t, \xi_t, \eta_t, \psi_t, t) = -W(\chi_t, \xi_t, \eta_t, \psi_t, t)$ .*

*Remark 3.5.* The proofs of these theorems follow the same arguments as in [2, 16] for their proposed stochastic Lyapunov functions, so they are not shown in this paper to avoid repetition.

The following proposition gives a necessary and sufficient condition for internal exponential stability in the mean square sense for the system (2.3).

PROPOSITION 3.6. *A necessary and sufficient condition for IESS of the system (2.3) is that there exist symmetric positive-definite matrices  $\mathcal{P}_{ijk}$ ,  $i \in Z$ ,  $j \in S$ , and  $k \in R$  such that*

$$\tilde{\Lambda}_{ijk}^T \mathcal{P}_{ijk} + \mathcal{P}_{ijk} \tilde{\Lambda}_{ijk} + \sum_{l=1}^v \bar{\mathbb{W}}_{lij}^T \mathcal{P}_{ijk} \bar{\mathbb{W}}_{lij} + \sum_{\substack{h \in Z \\ h \neq i}} \pi_{ih} \mathcal{P}_{hjk} + \sum_{\substack{l \in S \\ l \neq j}} \gamma_{jl} \mathcal{P}_{ilk} + \sum_{\substack{v \in R \\ v \neq k}} \lambda_{kv}^{ij} \mathcal{P}_{ijv} < 0 \quad (3.4)$$

for all  $i \in Z$ ,  $j \in S$ , and  $k \in R$ , where

$$\tilde{\Lambda}_{ijk} = \Lambda_{ijk} - 0.5 \mathbb{I} \left( \sum_{\substack{h \in Z \\ h \neq i}} \pi_{ih} + \sum_{\substack{l \in S \\ l \neq j}} \gamma_{jl} + \sum_{\substack{v \in R \\ v \neq k}} \lambda_{kv}^{ij} \right). \quad (3.5)$$

*Proof.* The proof of this proposition is easily deduced from Theorems 3.2 and 3.4.  $\square$

PROPOSITION 3.7. *If the system (2.3) is IESS, for every  $w = \{w_t; t \geq 0\} \in L_2[0, \infty)$ , one has that  $\chi = \{\chi_t; t \geq 0\} \in L_2((\Omega, \mathcal{F}, P), [0, \infty))$ , that is,  $\mathcal{E} \{ \int_0^\infty \chi_t^T \chi_t dt \} < \infty$ , for any initial conditions  $(\chi_0, \xi_0, \eta_0, \psi_0)$ .*

*Proof.* The proof of this proposition follows the same lines as for the proof of [22, Proposition 4].  $\square$

We conclude this section by recalling a version of the well-known Finsler lemma that will be used in the derivation of the main results of this paper.

LEMMA 3.8 [24]. *Given matrices  $\Psi_{ijk} = \Psi_{ijk}^T \in \mathbb{R}^{n \times n}$  and  $H_{ijk} \in \mathbb{R}^{m \times n}$ , for all  $i \in Z$ ,  $j \in S$ , and  $k \in R$ , then*

$$x_t^T \Psi_{ijk} x_t < 0, \quad \forall x_t \in \mathbb{R}^n : H_{ijk} x_t = 0, x_t \neq 0, \quad (3.6)$$

if and only if there exist matrices  $L_{ijk} \in \mathbb{R}^{n \times m}$  such that

$$\Psi_{ijk} + L_{ijk} H_{ijk} + H_{ijk}^T L_{ijk}^T < 0, \quad \forall i \in Z, j \in S, k \in R. \quad (3.7)$$

Note that conditions (3.7) remain sufficient for (3.6) to hold even when arbitrary constraints are imposed to the scaling matrices  $L_{ijk}$ .

## 4. The control problem

**4.1.  $\mathcal{H}_\infty$  control.** Let us consider the system (2.3) with

$$z_t = z_{\infty t} = C_{\infty 1} x_t + D_{\infty 1}(\eta_t) u(y_t, \psi_t, t), \quad (4.1)$$

where  $z_{\infty t}$  stands for the controlled output related to  $\mathcal{H}_\infty$  performance.

In this section, we deal with the design of controllers that stochastically stabilize the closed-loop system and guarantee the disturbance rejection, with a certain level  $\mu > 0$ . Mathematically, we are concerned with the characterization of compensators  $\varphi_d$  that

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stochastically stabilize the system (2.3) and guarantee the following for all  $w \in L^2[0, \infty)$ :

$$\|z_\infty\|_{\mathcal{E}_2} = \mathcal{E} \left\{ \int_0^\infty z_{\infty t}^T z_{\infty t} dt \right\}^{1/2} < \mu \|w\|_2, \quad (4.2)$$

where  $\mu > 0$  is a prescribed level of disturbance attenuation to be achieved. To this end, we need the auxiliary result given by the following proposition.

**PROPOSITION 4.1.** *If there exist symmetric positive-definite matrices  $\mathcal{P}_{\infty ijk}$ ,  $i \in Z$ ,  $j \in S$ , and  $k \in R$  such that*

$$\begin{aligned} & \tilde{\Lambda}_{ijk}^T \mathcal{P}_{\infty ijk} + \mathcal{P}_{\infty ijk} \tilde{\Lambda}_{ijk} \\ & + \sum_{l=1}^v \overline{\mathbb{W}}_{lij}^T \mathcal{P}_{\infty ijk} \overline{\mathbb{W}}_{lij} + \overline{C}_{\infty 1jk}^T \overline{C}_{\infty 1jk} + \mu^{-2} \mathcal{P}_{\infty ijk} \overline{E}_{ijk} \overline{E}_{ijk}^T \mathcal{P}_{\infty ijk} \\ & + \sum_{\substack{h \in Z \\ h \neq i}} \pi_{ih} \mathcal{P}_{\infty hjk} + \sum_{\substack{l \in S \\ l \neq j}} \nu_{jl} \mathcal{P}_{\infty ilk} + \sum_{\substack{v \in R \\ v \neq k}} \lambda_{kv}^{ij} \mathcal{P}_{\infty ijv} < 0 \end{aligned} \quad (4.3)$$

for all  $i \in Z$ ,  $j \in S$ , and  $k \in R$ , then the system (2.3) is IESS and satisfies

$$\|z_\infty\|_{\mathcal{E}_2} = \mathcal{E} \left\{ \int_0^\infty z_{\infty t}^T z_{\infty t} dt \right\}^{1/2} < \mu \|w\|_2. \quad (4.4)$$

*Proof.* The proof of this proposition follows the same arguments as in [21].  $\square$

**Remark 4.2.** Given fixed matrices  $\mathbb{U} \geq 0$ ,  $\mathbb{V} = \mathbb{V}^T$ , and  $\mathbb{Q}$ , the previous characterization extends to more general quadratic constraints on  $w_t$  and  $z_t$  of the form

$$J_{\text{SIQC}} = \mathcal{E} \left\{ \int_{t_0}^{t_f} \begin{pmatrix} z_t \\ w_t \end{pmatrix}^T \begin{pmatrix} \mathbb{U} & \mathbb{Q} \\ \mathbb{Q}^T & \mathbb{V} \end{pmatrix} \begin{pmatrix} z_t \\ w_t \end{pmatrix} dt \right\} < 0. \quad (4.5)$$

These constraints are known as *stochastic quadratic integral constraints* [26].

The following proposition gives a sufficient condition to (4.5) to be hold.

**PROPOSITION 4.3.** *If there exist symmetric positive-definite matrices  $\mathcal{P}_{ijk}$ ,  $i \in Z$ ,  $j \in S$ , and  $k \in R$  such that*

$$\begin{bmatrix} \Theta_{ijk} & \mathcal{P}_{ijk} \overline{E}_{ijk} + \overline{C}_{1jk}^T \mathbb{Q} & \overline{C}_{1jk}^T \Delta \\ * & \mathbb{V} & \mathbf{0} \\ * & * & -\Sigma \end{bmatrix} < 0, \quad (4.6)$$

where

$$\begin{aligned} \mathbb{U} &= \Delta \Sigma^{-1} \Delta^T, \\ \Theta_{ijk} &= \tilde{\Lambda}_{ijk}^T \mathcal{P}_{ijk} + \mathcal{P}_{ijk} \tilde{\Lambda}_{ijk} + \sum_{l=1}^v \overline{\mathbb{W}}_{lij}^T \mathcal{P}_{ijk} \overline{\mathbb{W}}_{lij} + \sum_{\substack{h \in Z \\ h \neq i}} \pi_{ih} \mathcal{P}_{hjk} + \sum_{\substack{l \in S \\ l \neq j}} \nu_{jl} \mathcal{P}_{ilk} + \sum_{\substack{v \in R \\ v \neq k}} \lambda_{kv}^{ij} \mathcal{P}_{ijv}, \end{aligned} \quad (4.7)$$



then

$$J_{SIQC} < 0. \quad (4.8)$$

*Proof.* Let us consider the following quadratic Lyapunov function:

$$\vartheta(\chi_t, \xi_t, \eta_t, \psi_t) = \chi_t^T \mathcal{P}(\xi_t, \eta_t, \psi_t) \chi_t, \quad (4.9)$$

then

$$\mathcal{L}\vartheta(\chi_t, \xi_t, \eta_t, \psi_t) = \begin{pmatrix} \chi_t \\ w_t \end{pmatrix}^T \begin{pmatrix} \Theta(\xi_t, \eta_t, \psi_t) & \mathcal{P}(\xi_t, \eta_t, \psi_t) \bar{E}(\xi_t, \eta_t, \psi_t) \\ * & \mathbf{0} \end{pmatrix} \begin{pmatrix} \chi_t \\ w_t \end{pmatrix} \quad (4.10)$$

adding and subtracting  $\mathcal{E}\{\int_{t_0}^{t_f} \mathcal{L}\vartheta(\chi_t, \xi_t, \eta_t, \psi_t) dt\}$  to  $J_{SIQC}$ , we get

$$\begin{aligned} J_{SIQC} = & \mathcal{E} \left\{ \int_{t_0}^{t_f} \begin{pmatrix} \chi_t \\ w_t \end{pmatrix}^T \left( \Phi(\xi_t, \eta_t, \psi_t) \right. \right. \\ & \left. \left. + \begin{pmatrix} \bar{C}_1(\eta_t, \psi_t) & \mathbf{0} \\ \mathbf{0} & \mathbb{I} \end{pmatrix}^T \begin{pmatrix} \mathbb{U} & \mathbb{Q} \\ \mathbb{Q}^T & \mathbb{V} \end{pmatrix} \begin{pmatrix} \bar{C}_1(\eta_t, \psi_t) & \mathbf{0} \\ \mathbf{0} & \mathbb{I} \end{pmatrix} \right) \begin{pmatrix} \chi_t \\ w_t \end{pmatrix} dt \right\} \\ & - \mathcal{E} \left\{ \int_{t_0}^{t_f} \mathcal{L}\vartheta(\chi_t, \xi_t, \eta_t, \psi_t) dt \right\}, \end{aligned} \quad (4.11)$$

where

$$\Phi(\xi_t, \eta_t, \psi_t) = \begin{pmatrix} \Theta(\xi_t, \eta_t, \psi_t) & \mathcal{P}(\xi_t, \eta_t, \psi_t) \bar{E}(\xi_t, \eta_t, \psi_t) \\ * & \mathbf{0} \end{pmatrix}. \quad (4.12)$$

From Dynkin's formula, we have

$$\mathcal{E}\{\vartheta(\chi_{t_f}, \xi_{t_f}, \eta_{t_f}, \psi_{t_f})\} - \vartheta(\chi_{t_0}, \xi_{t_0}, \eta_{t_0}, \psi_{t_0}) = \mathcal{E}\left\{\int_{t_0}^{t_f} \mathcal{L}\vartheta(\xi_t, \eta_t, \psi_t) dt\right\}. \quad (4.13)$$

Assuming, without loss of generality, that  $\vartheta(\chi_{t_0}, \xi_{t_0}, \eta_{t_0}, \psi_{t_0}) = 0$ , it follows from (4.11) and (4.13) that if

$$\Phi(\xi_t, \eta_t, \psi_t) + \begin{pmatrix} \bar{C}_1(\eta_t, \psi_t) & \mathbf{0} \\ \mathbf{0} & \mathbb{I} \end{pmatrix}^T \begin{pmatrix} \mathbb{U} & \mathbb{Q} \\ \mathbb{Q}^T & \mathbb{V} \end{pmatrix} \begin{pmatrix} \bar{C}_1(\eta_t, \psi_t) & \mathbf{0} \\ \mathbf{0} & \mathbb{I} \end{pmatrix} < 0, \quad (4.14)$$

then (4.5) holds.

Finally, by factorizing  $\mathbb{U} \geq 0$  as

$$\mathbb{U} = \Delta \Sigma^{-1} \Delta^T \quad (4.15)$$

and by Schur complement property, (4.14) is equivalent to (4.6). Hence, the proof is complete.  $\square$

The  $\mathcal{H}_\infty$  output feedback fault tolerant control method to be developed in this paper is based on Lemma 3.8 and with the following parametrization of the Lyapunov matrices  $\mathcal{P}_{\infty ijk}$ :

$$\mathcal{P}_{\infty ijk} = \mathcal{M}_k^T \mathcal{N}_{\infty ijk}^{-1} \mathcal{M}_k > 0, \quad (4.16)$$

where  $\mathcal{N}_{\infty ijk}$  are symmetric positive definite matrices and  $\mathcal{M}_k$  are nonsingular matrices. This parametrization is inspired by the work of [24] on *mode-independent*  $\mathcal{H}_\infty$  filtering for Markovian jumping linear systems.

The next proposition presents the main result of this section, which is derived from Proposition 4.1 with  $\mathcal{P}_{\infty ijk}$  as in (4.16) and using Finsler's lemma together with appropriate parametrization of matrices  $\mathcal{M}_k$  and the dynamical controller matrices.

**PROPOSITION 4.4.** *If there exist matrices  $\mathbb{R}_k, \mathbb{S}_k, \mathbb{D}_k, \mathbb{X}_k, \mathbb{Y}_k, \mathbb{Z}_k$  and symmetric matrices  $\mathbb{N}_{\infty ijk}, i \in Z, j \in S$ , and  $k \in R$  such that*

$$\begin{bmatrix} \Omega_k + \Omega_k^T & * & * & * & * \\ \mathbb{A}_{ijk} - \mathbb{N}_{\infty ijk} & \delta_{ijk} \mathbb{N}_{\infty ijk} & * & * & * \\ \mathbf{0} & \mathbb{E}_{ijk}^T & -\mu^2 \mathbb{I} & * & * \\ \mathbb{C}_{ijk} & \mathbf{0} & \mathbf{0} & -\mathbb{I} & * \\ \mathbb{E}_{ijk} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\Gamma_{ijk} \end{bmatrix} < 0, \quad (4.17)$$

where

$$\begin{aligned} \Omega_k &= \begin{bmatrix} \mathbb{R}_k & \mathbf{0} \\ \mathbb{S}_k + \mathbb{D}_k & \mathbb{D}_k \end{bmatrix}, \\ \mathbb{A}_{ijk} &= \begin{bmatrix} \mathbb{R}_k A_i + \mathbb{R}_k B_j Z_k & \mathbb{R}_k B_j Z_k \\ \mathbb{S}_k A_i + \mathbb{Y}_k C_2 + \mathbb{X}_k + \mathbb{S}_k B_j Z_k & \mathbb{X}_k + \mathbb{S}_k B_j Z_k \end{bmatrix}, \\ \mathbb{E}_{ijk} &= \begin{bmatrix} \mathbb{R}_k E_{ij} \\ \mathbb{S}_k E_{ij} + \mathbb{Y}_k D_{2ij} \end{bmatrix}, \\ \mathbb{C}_{ijk} &= [C_{\infty 1} + D_{\infty 1} Z_k \quad D_{\infty 1} Z_k], \\ \Gamma_{ijk} &= \text{diag} \{ \bar{\Gamma}_{1ijk}, \bar{\Gamma}_{2ijk}, \bar{\Gamma}_{3ijk}, \bar{\Gamma}_{4ijk} \}, \\ \bar{\Gamma}_{1ijk} &= [\mathbb{N}_{\infty 1jk}, \dots, \mathbb{N}_{\infty (i-1)jk}, \mathbb{N}_{\infty (i+1)jk}, \dots, \mathbb{N}_{\infty zjk}], \\ \bar{\Gamma}_{2ijk} &= [\mathbb{N}_{\infty ik}, \dots, \mathbb{N}_{\infty i(j-1)k}, \mathbb{N}_{\infty i(j+1)k}, \dots, \mathbb{N}_{\infty isk}], \\ \bar{\Gamma}_{3ijk} &= [\mathbb{N}_{\infty ij1}, \dots, \mathbb{N}_{\infty ij(k-1)}, \mathbb{N}_{\infty ij(k+1)}, \dots, \mathbb{N}_{\infty ijr}], \\ \bar{\Gamma}_{4ijk} &= [\mathbb{N}_{\infty ijk}, \dots, \mathbb{N}_{\infty ijk}], \end{aligned}$$

$$\begin{aligned}
\Xi_{ijk} &= [F_{1ijk}, F_{2ijk}, F_{3ijk}, F_{4ijk}]^T, \\
F_{1ijk} &= [\sqrt{\pi_{i1}}\Omega_k^T, \dots, \sqrt{\pi_{i(i-1)}}\Omega_k^T, \sqrt{\pi_{i(i+1)}}\Omega_k^T, \dots, \sqrt{\pi_{iz}}\Omega_k^T], \\
F_{2ijk} &= [\sqrt{\nu_{j1}}\Omega_k^T, \dots, \sqrt{\nu_{j(j-1)}}\Omega_k^T, \sqrt{\nu_{j(j+1)}}\Omega_k^T, \dots, \sqrt{\nu_{js}}\Omega_k^T], \\
F_{3ijk} &= [\sqrt{\lambda_{k1}^{ij}}\Omega_k^T, \dots, \sqrt{\lambda_{k(k-1)}^{ij}}\Omega_k^T, \sqrt{\lambda_{k(k+1)}^{ij}}\Omega_k^T, \dots, \sqrt{\lambda_{kr}^{ij}}\Omega_k^T], \\
F_{4ijk} &= [\Phi_{1ijk}^T, \dots, \Phi_{vijk}^T], \\
\Phi_{lijk} &= \begin{bmatrix} \mathbb{R}_k \mathbb{W}_{lij} & \mathbf{0} \\ \mathbb{S}_k \mathbb{W}_{lij} & \mathbf{0} \end{bmatrix}, \\
\delta_{ijk} &= - \left( \sum_{\substack{h \in Z \\ h \neq i}} \pi_{ih} + \sum_{\substack{l \in S \\ l \neq j}} \nu_{jl} + \sum_{\substack{v \in R \\ v \neq k}} \lambda_{kv}^{ij} \right),
\end{aligned} \tag{4.18}$$

and  $\mathbb{T}_{4ijk}, F_{4ijk}$  contain  $v$  elements, then, the system (2.3) is IESS and satisfies (4.4).

Moreover, the transfer functions matrices of suitable controllers are given by

$$H_k(s) = \mathbb{Z}_k(s\mathbb{I} - \mathbb{D}_k^{-1}\mathbb{X}_k)^{-1}\mathbb{D}_k^{-1}\mathbb{Y}_k, \quad \forall k \in \mathbb{R}. \tag{4.19}$$

*Proof.* It will be shown that if the inequalities (4.17) hold, then the controllers (4.19) ensure that conditions (4.3) of Proposition 4.1 are satisfied with a matrix  $\mathcal{P}_{\infty ijk} > 0$  as in (4.16).

First, note that with a matrix  $\mathcal{P}_{\infty ijk} > 0$  as in (4.16), the inequalities (4.3) are equivalent to

$$\Pi_{ijk}^T \mathbb{Y}_{ijk} \Pi_{ijk} < 0, \tag{4.20}$$

where

$$\begin{aligned}
\Pi_{ijk} &= \begin{bmatrix} \mathbb{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathcal{N}_{\infty ijk}^{-1} \mathcal{M}_k & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbb{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbb{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbb{I} \end{bmatrix}, \\
\mathbb{Y}_{ijk} &= \begin{bmatrix} \mathbf{0} & * & \mathbf{0} & \overline{\mathcal{C}}_{\infty 1jk}^T & \mathcal{R}_{\infty ijk} \\ \mathcal{M}_k \Lambda_{ijk} & \delta_{ijk} \mathcal{N}_{\infty ijk} & \mathcal{M}_k \overline{\mathbb{E}}_{ijk} & \mathbf{0} & \mathbf{0} \\ * & * & -\mu^2 \mathbb{I} & \mathbf{0} & \mathbf{0} \\ * & * & * & -\mathbb{I} & \mathbf{0} \\ * & * & * & * & -\mathcal{S}_{\infty ijk} \end{bmatrix},
\end{aligned}$$

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$$\begin{aligned}
\mathcal{R}_{\infty ijk} &= [\mathcal{R}_{\infty 1ijk}, \mathcal{R}_{\infty 2ijk}, \mathcal{R}_{\infty 3ijk}, \mathcal{R}_{\infty 4ijk}], \\
\mathcal{R}_{\infty 1ijk} &= [\sqrt{\pi_{i1}} \mathcal{M}_k^T, \dots, \sqrt{\pi_{i(i-1)}} \mathcal{M}_k^T, \sqrt{\pi_{i(i+1)}} \mathcal{M}_k^T, \dots, \sqrt{\pi_{iz}} \mathcal{M}_k^T], \\
\mathcal{R}_{\infty 2ijk} &= [\sqrt{\nu_{j1}} \mathcal{M}_k^T, \dots, \sqrt{\nu_{j(j-1)}} \mathcal{M}_k^T, \sqrt{\nu_{j(j+1)}} \mathcal{M}_k^T, \dots, \sqrt{\nu_{js}} \mathcal{M}_k^T], \\
\mathcal{R}_{\infty 3ijk} &= [\sqrt{\lambda_{k1}^{ij}} \mathcal{M}_k^T, \dots, \sqrt{\lambda_{k(k-1)}^{ij}} \mathcal{M}_k^T, \sqrt{\lambda_{k(k+1)}^{ij}} \mathcal{M}_k^T, \dots, \sqrt{\lambda_{kr}^{ij}} \mathcal{M}_k^T], \\
\mathcal{R}_{\infty 4ijk} &= [\overline{\mathbb{W}}_{1ij}^T \mathcal{M}_k^T, \dots, \overline{\mathbb{W}}_{vij}^T \mathcal{M}_k^T], \\
\mathcal{S}_{\infty ijk} &= \text{diag} [\mathcal{S}_{\infty 1ijk}, \mathcal{S}_{\infty 2ijk}, \mathcal{S}_{\infty 3ijk}, \mathcal{S}_{\infty 4ijk}], \\
\mathcal{S}_{\infty 1ijk} &= [\mathcal{N}_{\infty 1jk}, \dots, \mathcal{N}_{\infty (i-1)jk}, \mathcal{N}_{\infty (i+1)jk}, \dots, \mathcal{N}_{\infty zjk}], \\
\mathcal{S}_{\infty 2ijk} &= [\mathcal{N}_{\infty ik}, \dots, \mathcal{N}_{\infty i(j-1)k}, \mathcal{N}_{\infty i(j+1)k}, \dots, \mathcal{N}_{\infty isk}], \\
\mathcal{S}_{\infty 3ijk} &= [\mathcal{N}_{\infty ij1}, \dots, \mathcal{N}_{\infty ij(k-1)}, \mathcal{N}_{\infty ij(k+1)}, \dots, \mathcal{N}_{\infty ijr}], \\
\mathcal{S}_{\infty 4ijk} &= [\mathcal{N}_{\infty ijk}, \dots, \mathcal{N}_{\infty ijk}],
\end{aligned} \tag{4.21}$$

where  $\mathcal{S}_{\infty 4ijk}$  contains  $\nu$  elements.

Define  $n \times n$  nonsingular matrices  $\mathcal{U}_k$  and  $\mathcal{V}_k$  such that  $\mathcal{U}_k \mathcal{V}_k = \mathbb{D}_k$  and let the matrix  $\mathcal{M}_k$  be parameterized as follows:

$$\mathcal{M}_k = \begin{bmatrix} \mathbb{R}_k^{-1} & \mathbb{R}_k^{-1} \mathcal{V}_k^T \\ -\mathcal{U}_k^{-1} \mathbb{S}_k \mathbb{R}_k^{-1} & \mathcal{U}_k^{-1} \mathcal{V}_k^T - \mathcal{U}_k^{-1} \mathbb{S}_k \mathbb{R}_k^{-1} \mathcal{V}_k^T \end{bmatrix}^{-1}. \tag{4.22}$$

Moreover, let the transformation matrix

$$\mathcal{T}_k = \begin{bmatrix} \mathbb{R}_k^T & \mathbb{S}_k^T \\ \mathbf{0} & \mathcal{U}_k^T \end{bmatrix}. \tag{4.23}$$

Note that

$$\mathcal{M}_k^{-T} \mathcal{T}_k = \mathfrak{J}_k = \begin{bmatrix} \mathbb{I} & \mathbf{0} \\ \mathcal{V}_k & \mathcal{V}_k \end{bmatrix} \tag{4.24}$$

and  $\mathcal{M}_k$  and  $\mathcal{T}_k$  as defined above are nonsingular. Indeed, from (4.17) it follows that  $\Omega_k + \Omega_k^T < 0$ , which implies that the matrices  $\mathbb{R}_k$  and  $\mathbb{D}_k$  are nonsingular. Therefore, in view of the definition of the matrices  $\mathcal{U}_k$  and  $\mathcal{V}_k$ , the matrices  $\mathcal{T}_k$  and  $\mathcal{M}_k^{-T} \mathcal{T}_k$  are nonsingular and thus  $\mathcal{M}_k^{-T}$  are nonsingular matrices as well.

Next, introduce the matrices

$$\begin{aligned}
\mathbb{F}_k &= \text{diag} \{ \mathfrak{J}_k, \mathfrak{J}_k, \mathbb{I}, \mathbb{I}, \overline{\mathfrak{J}}_k \}, \\
\overline{\mathfrak{J}}_k &= \{ \mathfrak{J}_k, \dots, \mathfrak{J}_k \},
\end{aligned} \tag{4.25}$$

where  $\overline{\mathfrak{J}}_k$  contains  $((z-1) + (s-1) + (r-1) + \nu)$  blocks  $\mathfrak{J}_k$ .

Performing the congruence transformation  $\mathbb{F}_k^T(\cdot)\mathbb{F}_k$  on  $Y_{ijk}$ , inequality (4.20) is equivalent to

$$\kappa^T \mathbb{F}_k^T Y_{ijk} \mathbb{F}_k \kappa < 0, \quad \kappa = \mathbb{F}_k^{-1} \Pi_{ijk} \zeta, \quad \zeta \neq 0. \quad (4.26)$$

Considering that

$$\{\kappa : \kappa = \mathbb{F}_k^{-1} \Pi_{ijk} \zeta, \zeta \neq 0\} = \{\kappa : \mathbb{H}_{ijk} \mathbb{F}_k \kappa = 0, \kappa \neq 0\}, \quad (4.27)$$

where

$$\mathbb{H}_{ijk} = \begin{bmatrix} \mathcal{M}_k & -\mathcal{N}_{\infty ij k} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad (4.28)$$

we have that (4.26) is equivalent to

$$\kappa^T \mathbb{F}_k^T Y_{ijk} \mathbb{F}_k \kappa < 0, \quad \forall \kappa \neq 0 : \mathbb{H}_{ijk} \mathbb{F}_k \kappa = 0. \quad (4.29)$$

By Lemma 3.8, (4.29) holds if and only if the following inequalities are feasible for some matrices  $\bar{\mathbb{L}}_{ijk}$  of appropriate dimensions

$$\mathbb{F}_k^T Y_{ijk} \mathbb{F}_k + \bar{\mathbb{L}}_{ijk} \mathbb{H}_{ijk} \mathbb{F}_k + \mathbb{F}_k^T \mathbb{H}_{ijk}^T \bar{\mathbb{L}}_{ijk}^T < 0. \quad (4.30)$$

Without loss of generality, let the matrices  $\bar{\mathbb{L}}_{ijk}$  be rewritten as  $\bar{\mathbb{L}}_{ijk} = \mathbb{F}_k^T \mathbb{L}_{ijk}$ , then (4.30) is equivalent to

$$\mathbb{F}_k^T (Y_{ijk} + \mathbb{L}_{ijk} \mathbb{H}_{ijk} + \mathbb{H}_{ijk}^T \mathbb{L}_{ijk}^T) \mathbb{F}_k < 0. \quad (4.31)$$

Setting

$$\mathbb{L}_{ijk} = \begin{bmatrix} \mathbb{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad (4.32)$$

inequalities (4.31) become

$$\mathbb{F}_k^T \bar{Y}_{ijk} \mathbb{F}_k < 0, \quad (4.33)$$

where

$$\bar{Y}_{ijk} = \begin{bmatrix} \mathcal{M}_k + \mathcal{M}_k^T & * & * & * & * \\ \mathcal{M}_k \Lambda_{ijk} - \mathcal{N}_{\infty ij k}^T & \delta_{ijk} \mathcal{N}_{\infty ij k} & * & * & * \\ \mathbf{0} & \bar{E}_{ijk}^T \mathcal{M}_k^T & -\mu^2 \mathbb{1} & * & * \\ \bar{C}_{\infty 1jk} & \mathbf{0} & \mathbf{0} & -\mathbb{1} & * \\ \mathcal{R}_{\infty ij k}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathcal{G}_{\infty ij k} \end{bmatrix}. \quad (4.34)$$

Consider the following state-space realization for the controllers (2.2)

$$A_{ck} = \mathcal{V}_k \mathbb{D}_k^{-1} \mathbb{X}_k \mathcal{V}_k^{-1}, \quad B_{ck} = \mathcal{V}_k \mathbb{D}_k^{-1} \mathbb{Y}_k, \quad C_{ck} = \mathbb{Z}_k \mathcal{V}_k^{-1}, \quad (4.35)$$

and let the matrix  $\mathbb{N}_{\infty ij k}$  be defined as

$$\mathbb{N}_{\infty ij k} = \mathcal{T}_k^T \mathcal{M}_k^{-1} \mathcal{N}_{\infty ij k} \mathcal{M}_k^{-T} \mathcal{T}_k. \quad (4.36)$$

By performing straightforward matrix manipulations, it can be easily shown that

$$\begin{aligned} \mathcal{T}_k^T \mathbb{A}_{ijk} \mathcal{M}_k^{-T} \mathcal{T}_k &= \mathbb{A}_{ijk}, & \mathcal{T}_k^T \bar{\mathbb{E}}_{ijk} &= \mathbb{E}_{ijk}, & \bar{\mathbb{C}}_{\infty 1 j k} \mathcal{M}_k^{-T} \mathcal{T}_k &= \mathbb{C}_{ijk}, \\ \mathcal{T}_k^T \mathcal{M}_k^{-T} \mathcal{T}_k &= \Omega_k, & \bar{\mathbb{J}}_k^T \mathcal{R}_{\infty ij k} \bar{\mathbb{J}}_k &= \Xi_{ijk}^T, & & \\ \bar{\mathbb{J}}_k^T \mathcal{S}_{\infty ij k} \bar{\mathbb{J}}_k &= \Gamma_{ijk}. \end{aligned} \quad (4.37)$$

Next, taking into account (4.25) and (4.36)-(4.37), it can be readily verified that (4.33) is identical to (4.17). Thus (4.3) is satisfied with  $\mathcal{P}_{\infty ij k} = \mathcal{M}_k^T \mathcal{N}_{\infty ij k}^{-1} \mathcal{M}_k$ . Finally, the controller transfer matrix of (4.19) is readily obtained from (4.35).  $\square$

From practical point of view, the controller that stochastically stabilizes the AFTCMP and at the same time guarantees the minimum disturbance rejection is of great interest. This controller can be obtained by solving the following optimization problem:

$$\mathbb{C}_{\infty} : \begin{cases} \min & \tau \\ \tau > 0, \mathbb{N}_{\infty ij k} = \mathbb{N}_{\infty ij k}^T > 0, \mathbb{R}_k, \mathbb{S}_k, \mathbb{D}_k, \mathbb{X}_k, \mathbb{Y}_k, \mathbb{Z}_k \\ \text{s.t.} & \\ \left[ \begin{array}{ccccc} \Omega_k + \Omega_k^T & * & * & * & * \\ \mathbb{A}_{ijk} - \mathbb{N}_{\infty ij k} & \delta_{ijk} \mathbb{N}_{\infty ij k} & * & * & * \\ \mathbf{0} & \mathbb{E}_{ijk}^T & -\tau \mathbb{I} & * & * \\ \mathbb{C}_{ijk} & \mathbf{0} & \mathbf{0} & -\mathbb{I} & * \\ \Xi_{ijk} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\Gamma_{ijk} \end{array} \right] < 0, \end{cases} \quad (4.38)$$

where the matrices inequalities in the constraints are obtained from (4.17) by replacing  $\mu^2$  by  $\tau$ . This leads to the following corollary.

**COROLLARY 4.5.** *Let  $\tau > 0$ ,  $\mathbb{N}_{\infty ij k} = \mathbb{N}_{\infty ij k}^T > 0$ ,  $\mathbb{R}_k$ ,  $\mathbb{S}_k$ ,  $\mathbb{D}_k$ ,  $\mathbb{X}_k$ ,  $\mathbb{Y}_k$ ,  $\mathbb{Z}_k$  be the solution of the optimization problem  $\mathbb{C}_{\infty}$ . Then, the controller (2.2) stochastically stabilizes the AFTCSMP considered and moreover the closed loop system satisfies the disturbance rejection of level  $\sqrt{\tau}$ .*

*Remark 4.6.* The above result can be easily extended to general SIQ constraints. This is illustrated by the following corollary.

COROLLARY 4.7. *If there exist matrices  $\mathbb{R}_k, \mathbb{S}_k, \mathbb{D}_k, \mathbb{X}_k, \mathbb{Y}_k, \mathbb{Z}_k$  and symmetric matrices  $\mathbb{N}_{ijk}$ ,  $i \in Z, j \in S$ , and  $k \in R$  such that*

$$\begin{bmatrix} \Omega_k + \Omega_k^T & * & * & * & * \\ \mathbb{A}_{ijk} - \mathbb{N}_{ijk} & \delta_{ijk} \mathbb{N}_{ijk} & * & * & * \\ \mathcal{C}_{ijk} & \mathbb{E}_{ijk}^T & \mathbb{V} & * & * \\ \bar{\mathbb{C}}_{ijk} & \mathbf{0} & \mathbf{0} & -\Sigma & * \\ \Xi_{ijk} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\Gamma_{ijk} \end{bmatrix} < 0, \quad (4.39)$$

where

$$\begin{aligned} \mathcal{C}_{ijk} &= \mathbb{Q}^T \mathbb{C}_{ijk}, \\ \bar{\mathbb{C}}_{ijk} &= \Delta^T \mathbb{C}_{ijk}, \end{aligned} \quad (4.40)$$

then, the SIQ constraints (4.5) are verified.

Moreover, the transfer functions matrices of suitable controllers are given by

$$H_k(s) = \mathbb{Z}_k (s\mathbb{I} - \mathbb{D}_k^{-1} \mathbb{X}_k)^{-1} \mathbb{D}_k^{-1} \mathbb{Y}_k, \quad \forall k \in R. \quad (4.41)$$

**4.2.  $\mathcal{H}_2$  control.** Before introducing the main results of this section, let us consider the following definition which represents a generalization of the  $\mathcal{H}_2$ -norm from Markovian jump linear systems [8, 27, 28] to AFTCSMP.

Let us consider the system (2.3) with

$$z_t = z_{2t} = C_{21}x_t + D_{21}(\eta_t)u(y_t, \psi_t, t), \quad (4.42)$$

where  $z_{2t}$  stands for the controlled output related to  $\mathcal{H}_2$  performance.

*Definition 4.8.* Define the  $\mathcal{H}_2$ -norm of the IESS system  $(\varphi_{cl})$  as

$$\|\varphi_{cl}\|_2^2 = \sum_{d=1}^m \sum_{i,j,k} \mu_{ijk} \|z_{d,i,j,k}\|_{\mathcal{E}_2}^2, \quad (4.43)$$

where  $z_{d,i,j,k}$  represents the output  $\{z_t; t \geq 0\}$  when

- (a) the input is given by  $w_t = \{w_t; t \geq 0\}$ ,  $w_t = e_d \delta_t$ ,  $\delta_t$  the unitary impulse, and  $e_d$  the  $m$ -dimensional unitary vector formed by 1 at the  $d$ th position and zero elsewhere;
- (b)  $\chi_0 = 0$ ,  $\xi_0 = i$ ,  $\eta_0 = j$ ,  $\psi_0 = k$ , and  $\mu = (\mu_{111}, \dots, \mu_{szz})$  is the initial distribution of the joint Markov process.

From the definition above and using the same arguments as in [8, 27, 28], we can state the following corollary.

COROLLARY 4.9. Assume that  $(\varphi_{cl})$  is IESS, then the following hold.

- (i)  $\|\varphi_{cl}\|_2^2 = \sum_{i,j,k} \mu_{ijk} \text{tr}(\bar{E}_{ij}^T \mathcal{P}_{oijk} \bar{E}_{ij})$ , where  $\mathcal{P}_o = \{\mathcal{P}_{o11}, \dots, \mathcal{P}_{oszr}\}$  denotes the observability Gramian, that is,  $\mathcal{P}_{oijk}$  are the unique positive semidefinite solutions of the following equations:

$$\begin{aligned} & \tilde{\Lambda}_{ijk}^T \mathcal{P}_{oijk} + \mathcal{P}_{oijk} \tilde{\Lambda}_{ijk} + \sum_{l=1}^v \bar{\mathbb{W}}_{lij}^T \mathcal{P}_{oijk} \bar{\mathbb{W}}_{lij} + \sum_{\substack{h \in Z \\ h \neq i}} \pi_{ih} \mathcal{P}_{hjk} \\ & + \sum_{\substack{l \in S \\ l \neq j}} \nu_{jl} \mathcal{P}_{ilk} + \sum_{\substack{v \in R \\ v \neq k}} \lambda_{kv}^{ij} \mathcal{P}_{ijv} + \bar{C}_{21}^T \bar{C}_{21} = 0 \end{aligned} \quad (4.44)$$

for all  $i \in Z$ ,  $j \in S$ , and  $k \in R$ .

- (ii)  $\|\varphi_{cl}\|_2^2 < \sum_{i,j,k} \mu_{ijk} \text{tr}(\bar{E}_{ij}^T \mathcal{P}_{2ijk} \bar{E}_{ij})$ , where  $\mathcal{P}_{2ijk}$  is a positive definite solution of the following matrix inequality:

$$\begin{aligned} & \tilde{\Lambda}_{ijk}^T \mathcal{P}_{2ijk} + \mathcal{P}_{2ijk} \tilde{\Lambda}_{ijk} + \sum_{l=1}^v \bar{\mathbb{W}}_{lij}^T \mathcal{P}_{2ijk} \bar{\mathbb{W}}_{lij} + \sum_{\substack{h \in Z \\ h \neq i}} \pi_{ih} \mathcal{P}_{2hjk} \\ & + \sum_{\substack{l \in S \\ l \neq j}} \nu_{jl} \mathcal{P}_{2ilk} + \sum_{\substack{v \in R \\ v \neq k}} \lambda_{kv}^{ij} \mathcal{P}_{2ijv} + \bar{C}_{21}^T \bar{C}_{21} < 0 \end{aligned} \quad (4.45)$$

for all  $i \in Z$ ,  $j \in S$ , and  $k \in R$ .

- (iii) If there exist positive definite matrices  $\mathcal{P}_{2ijk}$ , and matrices  $A_{ck}$ ,  $B_{ck}$ , and  $C_{ck}$  such that

$$\begin{aligned} & \sum_{i,j,k} \mu_{ijk} \text{tr}(\bar{E}_{ij}^T \mathcal{P}_{2ijk} \bar{E}_{ij}) < \gamma^2, \\ & \tilde{\Lambda}_{ijk}^T \mathcal{P}_{2ijk} + \mathcal{P}_{2ijk} \tilde{\Lambda}_{ijk} + \sum_{l=1}^v \bar{\mathbb{W}}_{lij}^T \mathcal{P}_{2ijk} \bar{\mathbb{W}}_{lij} + \sum_{\substack{h \in Z \\ h \neq i}} \pi_{ih} \mathcal{P}_{2hjk} \\ & + \sum_{\substack{l \in S \\ l \neq j}} \nu_{jl} \mathcal{P}_{2ilk} + \sum_{\substack{v \in R \\ v \neq k}} \lambda_{kv}^{ij} \mathcal{P}_{2ijv} + \bar{C}_{21}^T \bar{C}_{21} < 0 \end{aligned} \quad (4.46)$$

for all  $i \in Z$ ,  $j \in S$ , and  $k \in R$ , then  $\varphi_{dk}$  are stabilizing controllers such that  $\|\varphi_{cl}\|_2 < \gamma$ .



(iv) The  $\mathcal{H}_2$  output feedback control problem is solved by the following optimization problem:

$$\begin{aligned}
& \min \sum_{i,j,k} \mu_{ijk} \operatorname{tr}(\mathcal{L}_{ijk}) \\
& \text{s. t.} \\
& \bar{E}_{ij}^T \mathcal{P}_{2ijk} \bar{E}_{ij} < \mathcal{L}_{ijk} \\
& \tilde{\Lambda}_{ijk}^T \mathcal{P}_{2ijk} + \mathcal{P}_{2ijk} \tilde{\Lambda}_{ijk} + \sum_{l=1}^v \bar{\mathbb{W}}_{lij}^T \mathcal{P}_{2ijk} \bar{\mathbb{W}}_{lij} + \sum_{\substack{h \in Z \\ h \neq i}} \pi_{ih} \mathcal{P}_{2hjk} \\
& + \sum_{\substack{l \in S \\ l \neq j}} \nu_{jl} \mathcal{P}_{2ilk} + \sum_{\substack{v \in R \\ v \neq k}} \lambda_{kv}^{ij} \mathcal{P}_{2ijv} + \bar{C}_{21}^T \bar{C}_{21} < 0.
\end{aligned} \tag{4.47}$$

From Definition 4.8 and Corollary 4.9, we can state the following result which solves the  $\mathcal{H}_2$  output feedback control problem.

**PROPOSITION 4.10.** *The  $\mathcal{H}_2$  output feedback control problem is solved by the following optimization problem:*

$$\mathbb{C}_2 : \begin{cases} \min_{\mathcal{L}_{ijk}, \mathbb{N}_{2ijk}, \mathbb{R}_k, \mathbb{S}_k, \mathbb{D}_k, \mathbb{X}_k, \mathbb{Y}_k, \mathbb{Z}_k} \sum_{i,j,k} \mu_{ijk} \operatorname{tr}(\mathcal{L}_{ijk}) \\ \text{s. t.} \\ \begin{bmatrix} \mathcal{L}_{ijk} & * \\ \mathbb{E}_{ijk} & \mathbb{N}_{2ijk} \end{bmatrix} > 0, \\ \begin{bmatrix} \Omega_k + \Omega_k^T & * & * & * \\ \mathbb{A}_{ijk} - \mathbb{N}_{2ijk} & \delta_{ijk} \mathbb{N}_{2ijk} & \mathbf{0} & \mathbf{0} \\ \mathbb{C}_{ijk} & * & -\mathbb{I} & \mathbf{0} \\ \Xi_{ijk} & * & * & -\Gamma_{ijk} \end{bmatrix} < 0 \end{cases} \tag{4.48}$$

for all  $i \in Z$ ,  $j \in S$ , and  $k \in R$ .

If (4.48) holds, then  $\|\varphi_{cl}\|_2^2 < \sum_{i,j,k} \mu_{ijk} \operatorname{tr}(\bar{E}_{ij}^T \mathcal{P}_{2ijk} \bar{E}_{ij})$ . Moreover, the transfer functions matrices of suitable controllers are given by

$$H_k(s) = \mathbb{Z}_k (s\mathbb{I} - \mathbb{D}_k^{-1} \mathbb{X}_k)^{-1} \mathbb{D}_k^{-1} \mathbb{Y}_k, \quad \forall k \in R. \tag{4.49}$$

*Proof.* The proof of this proposition follows the same arguments as for the proof of Proposition 4.4.  $\square$

*Remark 4.11.* We can see from Definition 4.8 that the considered  $\mathcal{H}_2$ -norm depends on the initial distribution of the joint Markov process. However, these distributions are, in general, unknown. This inconvenient could be avoided by replacing, for example, the performance criteria by the one as in [28, equation (19)] (when the input is the unit variance white noise). Proposition 4.10 could be then easily applied for this  $\mathcal{H}_2$ -norm definition.

The only difference is that, in the optimization problem (4.48), the initial distribution  $\mu_{ijk}$  is replaced by  $\varepsilon_{ijk} = \sum_{l,h,v} \tilde{P}_{(lhv)(ijk)}$ , where

$$\tilde{P} = [\tilde{P}_{(lhv)(ijk)}] = \lim_{t \rightarrow \infty} P(t) \quad (4.50)$$

and  $P(t)$  is the probability transition matrix of the joint Markov process.

Indeed, the  $\mathcal{H}_2$ -norm definition considered in this case is given by

$$\|\varphi_{cl}\|_2^2 = \lim_{T \rightarrow \infty} \sum_{i,j,k} \mathcal{E} \left[ \int_0^T \|z_s\|^2 ds \mid \xi_0 = i, \eta_0 = j, \psi_0 = k \right]. \quad (4.51)$$

Using the same arguments as in [28], the following characterization of the  $\mathcal{H}_2$ -norm in terms of observability Gramian can be given:

$$\|\varphi_{cl}\|_2^2 = \sum_{i,j,k} \varepsilon_{ijk} \operatorname{tr} (\bar{E}_{ij}^T \mathcal{P}_{oijk} \bar{E}_{ij}). \quad (4.52)$$

**4.3. Multiobjective synthesis.** The multiobjective synthesis problem amounts to find common controllers that stochastically stabilize the system and ensure  $\mathcal{H}_2/\mathcal{H}_\infty$  performances. This multiperformance synthesis problem can be stated as follows.

*Given positive scalars  $\alpha_2$  and  $\alpha_\infty$ , find stabilizing dynamic output feedback controllers  $\varphi_d$  that solve the following constrained optimization problem:*

$$\begin{aligned} \min_{\gamma_2, \gamma_\infty, A_{ck}, B_{ck}, C_{ck}} \quad & \alpha_2 \gamma_2 + \alpha_\infty \gamma_\infty \\ \text{s.t.} \quad & \\ & \|z_\infty\|_{\mathcal{L}_2} < \gamma_\infty \|w\|_2, \quad \|\varphi_{cl}\|_2 < \gamma_2. \end{aligned} \quad (4.53)$$

The result is straightforward. It amounts to the collection of all related matrix inequality constraints.

*Remark 4.12.* Notice that the developed synthesis conditions are only sufficient. This is due to the fact that the controllers only depend on the FDI process, that is, the number of controllers to be designed is less than the total number of the closed loop system modes by combining both failures and FDI processes.

## 5. Computational issues and example

The inequality conditions in Propositions 4.4 and 4.10 are not linear in the variables and it is difficult to verify these conditions directly. However, the characterizations given in these propositions have the following nice properties:

- (i) the given parametrization enables us to express these norm minimization problems in closed form, that is, all variables explicitly appear in constraints (4.17) and (4.48), which is not possible with projection-like conditions;
- (ii) the  $\mathcal{H}_2/\mathcal{H}_\infty$  synthesis problem with fixed matrices  $C_c(\psi_t)$  (synthesis problem similar to the one considered by [29] in the case of linear time invariant systems) is a LMI problem;

- (iii) with  $\mathbb{Z}_k$  fixed, conditions (4.17) and (4.48) are LMIs in  $\mathcal{L}_{ijk}, \mathbb{N}_{2ijk}, \mathbb{N}_{\infty ijk}, \mathbb{R}_k, \mathbb{S}_k, \mathbb{D}_k, \mathbb{X}_k, \mathbb{Y}_k$ ;
- (iv) with  $\mathbb{R}_k$  and  $\mathbb{S}_k$  fixed, conditions (4.17) and (4.48) are LMIs in  $\mathcal{L}_{ijk}, \mathbb{N}_{2ijk}, \mathbb{N}_{\infty ijk}, \mathbb{D}_k, \mathbb{X}_k, \mathbb{Y}_k, \mathbb{Z}_k$ .

From the two last properties, one can see that the synthesis problem is expressed as a bilinear matrix inequalities (BMI) problem. BMI problems are known to be generally nonconvex and NP-hard [30]. This means that any algorithm which is guaranteed to find a global optimum cannot be expected to have a polynomial time complexity. There exist different approaches to the solution of this problem, which can be classified into global [31–33] and local [34–36]. Most of the global algorithms to the BMI problem are variations of the Branch and Bound algorithm [31, 32]. Although the major focus of global search algorithms is the computational complexity, none of them is polynomial time due to the NP-hardness of the problem. As a result, these approaches can currently be applied only to problems with modest size.

Most of the existing local approaches, on the other hand, are computationally fast but, depending on the initial condition, may not converge to the global optimum. The simplest local approach makes use of the fact that by fixing some of the variables,  $x$ , the BMI problem becomes convex in the remaining variables  $y$ , and vice versa, and iterates between them [36]. The algorithm (coordinate descent-type algorithm) used in this paper belongs to this class of methods. Nevertheless, these types of algorithms, called coordinate descent methods in [36], alternating SDP method in [31], and the dual iteration in [36], are not guaranteed to converge to a local solution [31]. Such an algorithm is given by Algorithm 5.1. For the purposes of its initialization, we adopt the same methodology as in [37]. These can be summarized as follows.

#### ALGORITHM 5.1

*Step 1.* Initialization: set  $q = 0$ . Design state feedback matrices  $\mathbb{Z}_{kq}$  that solves the following optimization problem:

$$\begin{aligned} & \min_{\gamma_{2q}, \gamma_{\infty q}, \mathbb{Z}_{kq}} \alpha_2 \gamma_{2q} + \alpha_{\infty} \gamma_{\infty q} \\ & \text{s.t.} \\ & \|z_{\infty}\|_{\mathcal{E}_2} < \gamma_{\infty q} \|w\|_2, \quad \|\varphi_{\text{cls}}\|_2 < \gamma_{2q}, \end{aligned} \quad (5.1)$$

where  $\varphi_{\text{cls}}$  is the closed loop system obtained by applying the controller  $\varphi_s$  given by

$$\varphi_s : u_t = \mathbb{Z}_{kq} x_t \quad (5.2)$$

to the system (2.1).

*Step 2.* Fix the matrices  $\mathbb{Z}_{kq}$  and search for a solution to the multiobjective control problem, defined in (4.52), in terms of the remaining unknown matrices  $\mathcal{L}_{ijkq}, \mathbb{N}_{2ijkq}, \mathbb{N}_{\infty ijkq}, \mathbb{R}_{kq}, \mathbb{S}_{kq}, \mathbb{D}_{kq}, \mathbb{X}_{kq}, \mathbb{Y}_{kq}$ . Set the current cost value  $v_q = \alpha_2 \gamma_{2q} + \alpha_{\infty} \gamma_{\infty q}$ .

*Step 3.* Fix the matrices  $\mathbb{R}_{kq}$  and  $\mathbb{S}_{kq}$  and search for a solution to the multiobjective control problem, defined in (4.52), in terms of the remaining unknown matrices  $\mathcal{L}_{ijkq}, \mathbb{N}_{2ijkq}, \mathbb{N}_{\infty ijkq}, \mathbb{D}_{kq}, \mathbb{X}_{kq}, \mathbb{Y}_{kq}, \mathbb{Z}_{kq}$ . Set the current cost value  $\omega_q = \alpha_2 \gamma_{2q} + \alpha_{\infty} \gamma_{\infty q}$ .

*Step 4.* If  $v_q - \omega_q < \epsilon$ ,  $\epsilon > 0$ , stop. Otherwise, set  $q \leftarrow q + 1$  and go back to Step 1.

As in the usual coordinate descent methods, the above algorithm generates a nonincreasing sequence of the objective function values, and thus the convergence is guaranteed. Note, however, that the limit of the sequence may not be optimal.

**5.1. A VTOL example.** In this section, the proposed multiobjective dynamic output feedback control of AFTCSMP is illustrated using a VTOL helicopter model adapted from [9]. Consider the nominal system with

$$\begin{aligned}
 A &= \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.01 & 0.0024 & -4.0208 \\ 0.1002 & 0.3681 & -0.707 & 1.4200 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \\
 B &= \begin{bmatrix} 0.4422 & 0.1761 \\ 3.5446 & -7.5922 \\ -5.52 & 4.49 \\ 0 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0.0468 & 0 \\ 0.0457 & 0.0099 \\ 0.0437 & 0.0011 \\ -0.0218 & 0 \end{bmatrix}, \\
 W_1 &= \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0.1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \\
 C_{\infty 1} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad D_{\infty 1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad D_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.
 \end{aligned} \tag{5.3}$$

The state vector  $x_t \in \mathbb{R}^4$  is composed by the following:

- $x_1$ : longitudinal velocity;
- $x_2$ : vertical velocity;
- $x_3$ : rate of pitch;
- $x_4$ : pitch angle.

And the components of command vector are

- $u_1$ : general cyclic command;
- $u_2$ : longitudinal cyclic command.

For illustration purposes, we will consider two faulty modes

- (i) Mode 2: A 50% power loss on the first actuator;
- (ii) Mode 3: A 50% power loss on both actuators.

From above, we have that  $S = \{1, 2, 3\}$ , where Mode 1 represents the nominal case. The failure process is assumed to have Markovian transition characteristics. The FDI process is also Markovian with three states  $R = \{1, 2, 3\}$ .

The actuator failure rates are assumed to be

$$[\pi_{ij}] = \begin{bmatrix} -0.002 & 0.0010 & 0.0010 \\ 0.0010 & -0.002 & 0.0010 \\ 0.0010 & 0.0010 & -0.002 \end{bmatrix}. \quad (5.4)$$

The FDI conditional transition rates are

$$\begin{aligned} [\lambda_{ij}^1] &= \begin{bmatrix} -0.02 & 0.01 & 0.01 \\ 1.00 & -1.01 & 0.01 \\ 1.00 & 0.01 & -1.01 \end{bmatrix}, & [\lambda_{ij}^2] &= \begin{bmatrix} -1.01 & 1.00 & 0.01 \\ 0.01 & -0.02 & 0.01 \\ 0.01 & 1.00 & -1.01 \end{bmatrix}, \\ [\lambda_{ij}^3] &= \begin{bmatrix} -1.01 & 0.01 & 1.00 \\ 0.01 & -1.01 & 1.00 \\ 0.01 & 0.01 & -0.02 \end{bmatrix}. \end{aligned} \quad (5.5)$$

For the above AFTCSMP, and using Algorithm 5.1 with  $\alpha_2 = \alpha_\infty = 1$ , we obtain the following  $\mathcal{H}_2/\mathcal{H}_\infty$  performances from  $w_t$  to  $z_2$  and  $z_\infty$ , respectively:  $\gamma_2 = 1.3813$ ,  $\gamma_\infty = 7.0430$ .

The corresponding dynamical controllers are given as follows:

$$\begin{aligned} &\left[ \begin{array}{cccc|cc} -4.5605 & 6.1141 & -0.7661 & -2.0089 & 4.6913 & -4.9146 \\ 1.2505 & -13.949 & 2.1057 & 2.8241 & -2.0707 & 10.8691 \\ -0.8586 & 9.0361 & -3.6110 & -6.1882 & 4.1034 & -6.6348 \\ 1.7167 & 0.8173 & 0.8173 & 0.8173 & -1.8753 & -1.2901 \\ \hline -0.5951 & -0.1871 & 0.3333 & 0.4416 & \mathbf{0} & \\ 0.0519 & 0.8206 & -0.6220 & -1.5290 & & \end{array} \right]_{\psi_t=1}, \\ &\left[ \begin{array}{cccc|cc} -1.8554 & -0.8810 & -0.0395 & -0.5837 & 1.6635 & 0.9851 \\ -2.7850 & -8.6549 & 4.1078 & 3.8934 & -1.7314 & 5.2828 \\ 0.2180 & 2.2321 & -2.8481 & -4.2791 & 3.0771 & -1.2354 \\ 1.3433 & 3.3313 & 1.1581 & 0.1551 & -1.4002 & -3.4934 \\ \hline -0.6729 & 0.0591 & 0.1496 & 0.2176 & \mathbf{0} & \\ 2.4528 & 1.4950 & -2.1517 & -3.1308 & & \end{array} \right]_{\psi_t=2}, \quad (5.6) \\ &\left[ \begin{array}{cccc|cc} -2.0190 & -0.9464 & 0.0742 & -0.4221 & 1.7047 & 0.9938 \\ -0.7222 & -6.3567 & 2.2273 & 1.2006 & -1.3945 & 4.5383 \\ -0.5881 & 0.9657 & -2.1838 & -3.2807 & 2.9401 & -0.5600 \\ 1.3238 & 3.2111 & 1.2362 & 0.1852 & -1.3741 & -3.4574 \\ \hline -0.7787 & 0.0063 & 0.2350 & 0.3908 & \mathbf{0} & \\ 1.8829 & 1.2999 & -1.9038 & -3.0831 & & \end{array} \right]_{\psi_t=3}, \end{aligned}$$

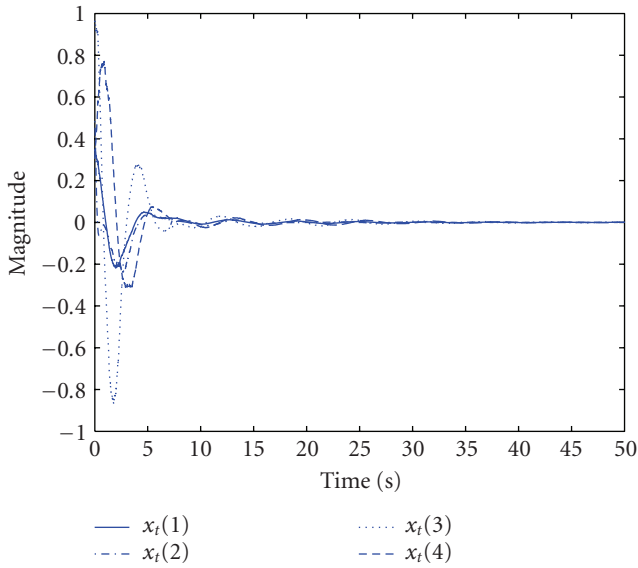


Figure 5.1. State variables evolution: single sample path simulation.

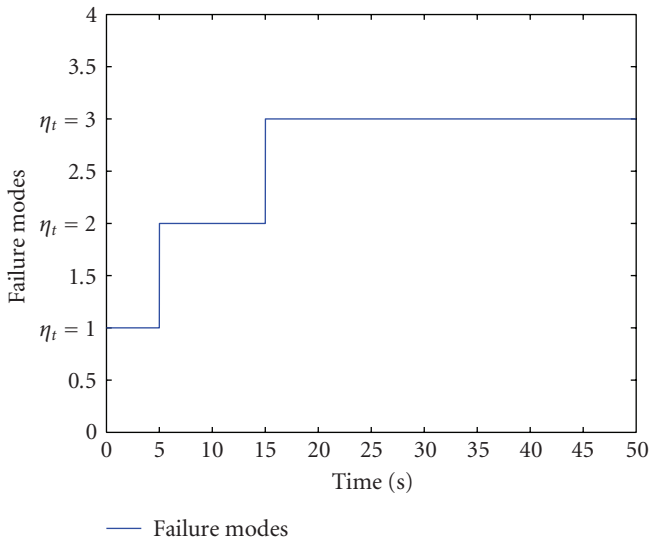


Figure 5.2. Failure modes.

where

$$\left[ \begin{array}{c|c} * & * \\ \hline * & \mathbf{0} \end{array} \right]_{\psi_t=i} \tag{5.7}$$

is a realization of the controller ( $\varphi_d$ ) for  $\psi_t = i$ .

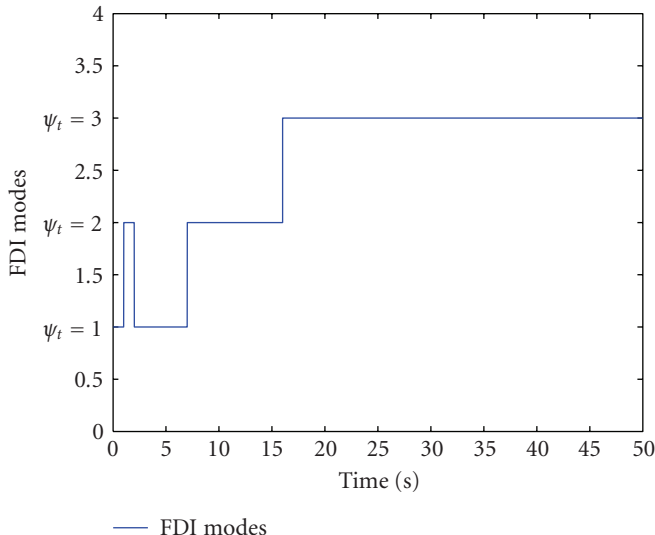
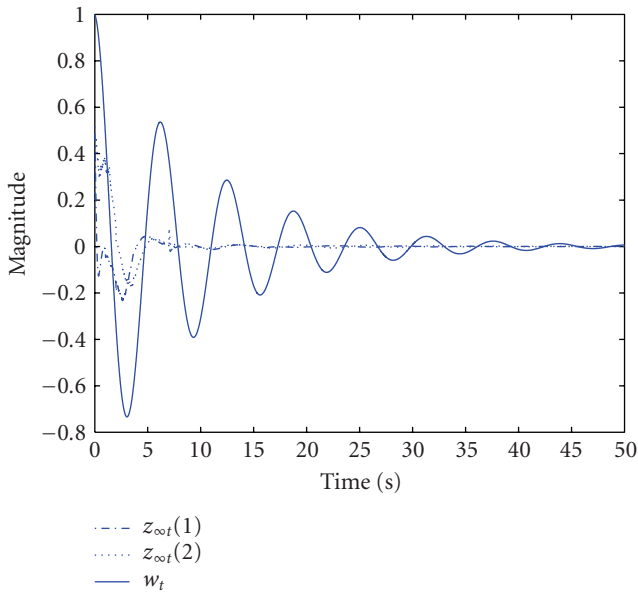


Figure 5.3. FDI modes.

Figure 5.4. Evolution of the variables  $z_{\infty t}$ : single sample path simulation.

The state trajectories of the closed loop system resulting from the obtained controllers are shown in Figure 5.1. These trajectories represent a single sample path simulation corresponding to a realization of the failure process  $\eta_t$  and the FDI process  $\psi_t$  given by Figures 5.2 and 5.3, respectively. Figure 5.4 represents the evolution of the controlled outputs

$z_{\infty t}$ . It can be seen that the closed loop system is stochastically stable and that the disturbance attenuation is achieved.

## 6. Conclusion

In this paper, the dynamic output feedback multiobjective control of continuous time AFTCSMP was considered within a framework that allows to take into account the problematic resulting from the fact that the controller only depends on the FDI process. The specifications and objectives considered include stochastic stability,  $\mathcal{H}_2$  and  $\mathcal{H}_\infty$  performances. The main results were derived using a version of the well-known Finsler's lemma and a parametrization of the Lyapunov matrices. The numerical resolution of the obtained results was done using a coordinate descent-type algorithm. The effectiveness of the developed method was illustrated on a VTOL helicopter example.

## References

- [1] Y. Zhang and J. Jiang, "Bibliographical review on reconfigurable fault-tolerant control systems," in *Proceedings of the 5th IFAC Symposium on Fault Detection, Supervision and Safety of Technical Processes (SAFEPROCESS '03)*, pp. 265–276, Washington, DC, USA, June 2003.
- [2] R. Srichander and B. K. Walker, "Stochastic stability analysis for continuous-time fault tolerant control systems," *International Journal of Control*, vol. 57, no. 2, pp. 433–452, 1993.
- [3] E. K. Boukas, P. Shi, and K. Benjelloun, "On stabilization of uncertain linear systems with jump parameters," *International Journal of Control*, vol. 72, no. 9, pp. 842–850, 1999.
- [4] E. K. Boukas, P. Shi, M. Karan, and C. Y. Kaya, "Linear discrete-time systems with Markovian jumps and mode dependent time-delay: stability and stabilizability," *Mathematical Problems in Engineering*, vol. 8, no. 2, pp. 123–133, 2002.
- [5] E. K. Boukas and N. F. Al-Muthairi, " $\mathcal{H}_\infty$  constant gain state feedback stabilization of stochastic hybrid systems with Wiener process," *Mathematical Problems in Engineering*, vol. 2004, no. 4, pp. 333–345, 2004.
- [6] E. K. Boukas, "Stabilization of stochastic nonlinear hybrid systems," *International Journal of Innovative Computing, Information and Control*, vol. 1, no. 1, pp. 131–141, 2005.
- [7] E. K. Boukas, "Static output feedback control for stochastic hybrid systems: LMI approach," *Automatica*, vol. 42, no. 1, pp. 183–188, 2006.
- [8] O. L. V. Costa, J. B. R. do Val, and J. C. Geromel, "Continuous-time state-feedback  $\mathcal{H}_2$ -control of Markovian jump linear systems via convex analysis," *Automatica*, vol. 35, no. 2, pp. 259–268, 1999.
- [9] D. P. de Farias, J. C. Geromel, J. B. R. do Val, and O. L. V. Costa, "Output feedback control of Markov jump linear systems in continuous-time," *IEEE Transactions on Automatic Control*, vol. 45, no. 5, pp. 944–949, 2000.
- [10] C. E. de Souza and M. D. Fragoso, " $\mathcal{H}_\infty$  control for linear systems with Markovian jumping parameters," *Control Theory and Advanced Technology*, vol. 9, no. 2, pp. 457–466, 1993.
- [11] Y. Ji and H. J. Chizeck, "Controllability, stabilizability, and continuous-time Markovian jump linear quadratic control," *IEEE Transactions on Automatic Control*, vol. 35, no. 7, pp. 777–788, 1990.
- [12] Y. Ji and H. J. Chizeck, "Jump linear quadratic Gaussian control in continuous time," *IEEE Transactions on Automatic Control*, vol. 37, no. 12, pp. 1884–1892, 1992.
- [13] M. Mahmoud, J. Jiang, and Y. Zhang, "Analysis of the stochastic stability for fault tolerant control systems," in *Proceedings of the 38th IEEE Conference on Decision & Control (CDC '99)*, vol. 4, pp. 3188–3193, Phoenix, Ariz, USA, December 1999.



- [14] M. Mahmoud, J. Jiang, and Y. Zhang, “Stochastic stability analysis of fault-tolerant control systems in the presence of noise,” *IEEE Transactions on Automatic Control*, vol. 46, no. 11, pp. 1810–1815, 2001.
- [15] M. Mahmoud, J. Jiang, and Y. Zhang, “Stability of fault tolerant control systems driven by actuators with saturation,” in *Proceedings of the 15th IFAC Triennial World Congress*, Barcelona, Spain, July 2002.
- [16] M. Mahmoud, J. Jiang, and Y. Zhang, *Active Fault Tolerant Control Systems: Stochastic Analysis and Synthesis*, vol. 287 of *Lecture Notes in Control and Information Sciences*, Springer, Berlin, Germany, 2003.
- [17] M. Mahmoud, J. Jiang, and Y. Zhang, “Optimal control law for fault tolerant control systems,” in *Proceedings of the 39th IEEE Conference on Decision & Control (CDC '00)*, vol. 4, pp. 4126–4131, Sydney, Australia, December 2000.
- [18] P. Shi and E. K. Boukas, “ $\mathcal{H}_\infty$ -control for Markovian jumping linear systems with parametric uncertainty,” *Journal of Optimization Theory and Applications*, vol. 95, no. 1, pp. 75–99, 1997.
- [19] P. Shi, E. K. Boukas, S. K. Nguang, and X. Guo, “Robust disturbance attenuation for discrete-time active fault tolerant control systems with uncertainties,” *Optimal Control Applications and Methods*, vol. 24, no. 2, pp. 85–101, 2003.
- [20] S. Aberkane, J. C. Ponsart, and D. Sauter, “Output feedback stochastic stabilization of active fault tolerant control systems: LMI formulation,” in *Proceedings of 16th IFAC World Congress*, Prague, Czech Republic, July 2005.
- [21] S. Aberkane, D. Sauter, J. C. Ponsart, and D. Theilliol, “ $\mathcal{H}_\infty$  stochastic stabilization of active fault tolerant control systems: convex approach,” in *Proceedings of the 44th IEEE Conference on Decision & Control, and the European Control Conference (CDC-ECC '05)*, pp. 3783–3788, Seville, Spain, December 2005.
- [22] S. Aberkane, J. C. Ponsart, and D. Sauter, “Output feedback  $\mathcal{H}_\infty$  control of a class of stochastic hybrid systems with Wiener process via convex analysis,” *International Journal of Innovative Computing, Information and Control*, vol. 2, no. 6, pp. 1179–1196, 2006.
- [23] S. Aberkane, J. C. Ponsart, and D. Sauter, “Ellipsoidal output-feedback sets for  $\mathcal{H}_\infty$  control of a class of stochastic hybrid systems with state-dependant noise,” in *Taming Heterogeneity and Complexity of Embedded Control*, F. Lamnabhi-Lagarriague, A. Loria, E. Panteley, and S. Laghrouche, Eds., ISTE, London, UK, 2007.
- [24] C. E. de Souza, A. Trofino, and K. A. Barbosa, “Mode-independent  $\mathcal{H}_\infty$  filters for hybrid Markov linear systems,” in *Proceedings of the 43rd IEEE Conference on Decision & Control (CDC '04)*, vol. 1, pp. 947–952, Atlantis, Paradise Island, Bahamas, December 2004.
- [25] R. Z. Has'minskii, *Stochastic Stability of Differential Equations*, vol. 7 of *Monographs and Textbooks on Mechanics of Solids and Fluids: Mechanics and Analysis*, Sijthoff & Noordhoff, Alphen aan den Rijn, The Netherlands, 1980.
- [26] C. Cheng, Q. Zhao, and F. Tao, “Stability and performance of the stochastic fault tolerant control systems,” in *Proceedings of the 42nd IEEE Conference on Decision & Control (CDC '03)*, vol. 3, pp. 2484–2489, Maui, Hawaii, USA, December 2003.
- [27] V. Dragan and T. Morozan, “Stability and robust stabilization to linear stochastic systems described by differential equations with Markovian jumping and multiplicative white noise,” *Stochastic Analysis and Applications*, vol. 20, no. 1, pp. 33–92, 2002.
- [28] V. Dragan, T. Morozan, and A. Stoica, “ $\mathcal{H}_2$  optimal control for linear stochastic systems,” *Automatica*, vol. 40, no. 7, pp. 1103–1113, 2004.
- [29] H. Tokunaga, T. Iwasaki, and S. Hara, “Multi-objective robust control with transient specifications,” in *Proceedings of the 35th IEEE Conference on Decision & Control (CDC '96)*, vol. 3, pp. 3482–3483, Kobe, Japan, December 1996.

- [30] O. Toker and H. Özbay, "On the NP-hardness of solving bilinear matrix inequalities and simultaneous stabilization with static output feedback," in *Proceedings of the American Control Conference*, vol. 4, pp. 2525–2526, Seattle, Wash, USA, June 1995.
- [31] M. Fukuda and M. Kojima, "Branch-and-cut algorithms for the bilinear matrix inequality eigenvalue problem," *Computational Optimization and Applications*, vol. 19, no. 1, pp. 79–105, 2001.
- [32] H. D. Tuan and P. Apkarian, "Low nonconvexity-rank bilinear matrix inequalities: algorithms and applications in robust controller and structure designs," *IEEE Transactions on Automatic Control*, vol. 45, no. 11, pp. 2111–2117, 2000.
- [33] H. D. Tuan, P. Apkarian, and Y. Nakashima, "A new Lagrangian dual global optimization algorithm for solving bilinear matrix inequalities," *International Journal of Robust and Nonlinear Control*, vol. 10, no. 7, pp. 561–578, 2000.
- [34] A. Hassibi, J. How, and S. Boyd, "A path-following method for solving BMI problems in control," in *Proceedings of the American Control Conference*, vol. 2, pp. 1385–1389, San Diego, Calif, USA, June 1999.
- [35] S. Ibaraki and M. Tomizuka, "Rank minimization approach for solving BMI problems with random search," in *Proceedings of the American Control Conference*, vol. 3, pp. 1870–1875, Arlington, Va, USA, June 2001.
- [36] T. Iwasaki, "The dual iteration for fixed-order control," *IEEE Transactions on Automatic Control*, vol. 44, no. 4, pp. 783–788, 1999.
- [37] S. Kanev, C. Scherer, M. Verhaegen, and B. De Schutter, "Robust output-feedback controller design via local BMI optimization," *Automatica*, vol. 40, no. 7, pp. 1115–1127, 2004.

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