B-SPLINE COLLOCATION METHODS FOR NUMERICAL SOLUTIONS OF THE BURGERS' EQUATION

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Both time- and space-splitted Burgers' equations are solved numerically. Cubic B-spline collocation method is applied to the time-splitted Burgers' equation. Quadratic B-spline collocation method is used to get numerical solution of the space-splitted Burgers' equation. The results of both schemes are compared for some test problems.

1. Introduction

The Burgers' equation first appeared in the paper by Bateman [3], who mentioned two of the essentially steady solutions. Due to extensive works of Burgers [4] involving the Burgers' equation especially as a mathematical model for the turbulence, it is known as Burgers' equation. The equation is used as a model in fields as wide as heat conduction [5], gas dynamics [13], shock waves [4], longitudinal elastic waves in an isotropic solid [15], number theory [18], continues stochastic processes [5], and so forth. Hopf [8] and Cole [5] solved the Burgers' equation analytically and independently for arbitrary initial conditions. In many cases, these solutions involve infinite series which may converge very slowly for small values of viscosity coefficients ν, which correspond to steep wave fronts in the propagation of the dynamic wave forms. Burgers' equation shows a similar features with Navier-Stokes equation due to the form of the nonlinear convection term and the occurrence of the viscosity term. Before concentrating on the numerical solution of the Navier-Stokes equation, it seems reasonable to first study a simple model of the Burgers' equation. Therefore, the Burgers' equation has been used as a model equation to test the numerical methods in terms of accuracy and stability for the Navier-Stokes equation. Many authors have used a variety of numerical techniques for getting the numerical solution of the Burgers' equation. Numerical difficulties have been come across in the numerical solution of the Burgers' equation with a very small viscosity. Various numerical techniques accompanied with spline functions have been set up for computing the solutions of the Burgers' equation. Rubin and Graves have used the cubic spline function technique and quasilinearisation for the numerical solutions of the Burgers' equation in one space variable at low Reynolds numbers [16]. A cubic spline collocation procedure has been developed for the numerical solution of the Burgers' equation [17]. A combination

of the time-splitted scheme and cubic spline functions was used to set up implicit finite difference schemes for obtaining the numerical solution of the Burgers' equation in the papers [9, 10, 11, 14]. The equation is solved numerically by the collocation method with cubic B-spline interpolation functions over uniform elements by Ali et al. [2]. A finite element solution of the Burgers' equation based on Galerkin method using B-splines as both element shape and test functions is developed in the studies [1, 6, 7]. Least-squares formulation using quadratic B-splines as trial function is given over the finite intervals by Kutluay et al. [12]. Numerical solutions of the partial differential equations have been found by splitting the equation both to make the numerical method applicable and to increase the accuracy of the method. We have written two algorithms for the splitted Burgers' equation. First, Burgers' equation is splitted in time and then cubic B-spline collocation method is applied. To be able to use the quadratic B-splines as trial functions in the collocation method, setting $V = -U_x$ in the Burgers' equation gives a first-order coupled system. This system of equations involving the first-order derivatives can be computed by employing the quadratic B-spline collocation method. Numerical results for some known initial and boundary conditions are illustrated for both methods.

Briefly, outline is as follows. In Section 2, numerical methods are described. Numerical experiments are carried out for two test problems and results of those methods are compared with each other and with theoretical results in Section 3.

2. B-spline collocation methods

The form of one-dimensional Burgers' equation is

$$U_t + UU_x - \nu U_{xx} = 0, \tag{2.1}$$

where $\nu > 0$ is the coefficient of the kinematic viscosity and subscripts x and t denote differentiation. Initial and boundary conditions are chosen as follows:

$$U(x,0) = f(x), \quad a \le x \le b, \tag{2.2}$$

$$U(a,t) = \alpha_1, \quad U(b,t) = \alpha_2, \quad t \in [0,T]. \tag{2.3}$$

We consider a mesh $a = x_0 < x_1 \cdots < x_N = b$ as a uniform partition of the solution domain $a \le x \le b$ by the knots x_m and $h = x_m - x_{m-1}$, $m = 1,\ldots,N$, throughout paper.

2.1. Quadratic B-spline collocation method (QBCM). A direct application of the quadratic B-spline collocation method requires the first-order derivatives in the equation to obtain smooth solutions. To start with formulation of the quadratic B-spline collocation method for the Burgers' equation, the space splitting is done with setting $V = -U_x$. So that Burgers' equation turns into the system of equations which involves the first-order derivatives. In the system of equations, the unknown functions U, V and their space derivatives U_x, V_x are discretized by the quadratic B-splines.

Let $Q_m(x)$, $m = -1,\dots,N$,

$$Q_m(x) = \frac{1}{h^2} \begin{cases} (x_{m+2} - x)^2 - 3(x_{m+1} - x)^2 + 3(x_m - x)^2, & [x_{m-1}, x_m], \\ (x_{m+2} - x)^2 - 3(x_{m+1} - x)^2, & [x_m, x_{m+1}], \\ (x_{m+2} - x)^2, & [x_{m+1}, x_{m+2}], \\ 0, & \text{otherwise,} \end{cases} \tag{2.4}$$

be quadratic B-splines with the knots x_m, $m = 0,\dots,N$. A basis is formed with quadratic B-splines $Q_m(x)$ over the domain region $a \le x \le b$.

The first-order space splitting of the Burgers' equation, with $V(x,t) = -U_x(x,t)$, gives a coupled system for U and V:

$$U_t - UV + \nu V_x = 0,$$
$$V + U_x = 0 \tag{2.5}$$

and boundary and initial conditions are

$$U(a,t) = \alpha_1, \quad U(b,t) = \alpha_2, \quad V(a,t) = V(b,t) = 0, \quad t \in [0,T],$$
$$U(x,0) = f(x), \quad V(x,0) = -f'(x), \quad a \le x \le b. \tag{2.6}$$

The collocation method is applied to find approximate solutions of the system (2.5). Collocation approximant can be expressed for $U(x,t)$ and $V(x,t)$ in terms of element parameters δ_m and σ_m, respectively, and quadratic B-splines $Q_m(x)$, $m = -1,\dots,N$:

$$U_N(x,t) = \sum_{m=-1}^{N} \delta_m(t) Q_m(x), \qquad V_N(x,t) = \sum_{m=-1}^{N} \sigma_m(t) Q_m(x). \tag{2.7}$$

Element parameters δ_m and σ_m are found by requiring that U_N and V_N satisfy the system of (2.5) at knots x_m, $m = 0,\dots,N$. The nodal variables U_m, V_m and their derivatives U'_m, V'_m have the following representation obtained with substitution of the knots in (2.7) and their first derivative in terms of elements parameters:

$$U_m = U(x_m) = \delta_{m-1} + \delta_m,$$
$$hU'_m = hU'(x_m) = 2(\delta_m - \delta_{m-1}),$$
$$V_m = V(x_m) = \sigma_{m-1} + \sigma_m,$$
$$hV'_m = hV'(x_m) = 2(\sigma_m - \sigma_{m-1}). \tag{2.8}$$

Substituting the collocation approximants (2.7)-(2.8) in the system (2.5) and its evaluation at the knots give a nonlinear system of equations:

$$h(\overset{\circ}{\delta}_{m-1} + \overset{\circ}{\delta}_m) - hz_m(\sigma_{m-1} + \sigma_m) + 2\nu(-\sigma_{m-1} + \sigma_m) = 0,$$
$$h(\sigma_{m-1} + \sigma_m) + 2(\delta_m - \delta_{m-1}) = 0, \tag{2.9}$$

where $\circ$ denotes differentiation with respect to time and $z_m = \delta_{m-1} + \delta_m$ is known as nonlinear term.

Time discretization of parameters δ_m and σ_m of the obtained system (2.9) is done with interpolation between two successive time levels n and $n + 1$. So, replacing the following time centering on $t = (n + 1/2)\Delta t$ and its time derivative with a Crank-Nicholson approach into (2.9),

$$
\begin{aligned}
\delta_m &= \frac{\delta_m^{n+1} + \delta_m^n}{2}, & \mathring{\delta}_m &= \frac{\delta_m^{n+1} - \delta_m^n}{\Delta t}, \\
\sigma_m &= \frac{\sigma_m^{n+1} + \sigma_m^n}{2}, & \mathring{\sigma}_m &= \frac{\sigma_m^{n+1} - \sigma_m^n}{\Delta t},
\end{aligned}
\tag{2.10}
$$

leads to nonlinear system of $2N + 2$ algebraic equations in the $2N + 4$ unknowns:

$$
\begin{aligned}
2h\delta_{m-1}^{n+1} - \beta_{m1}\sigma_{m-1}^{n+1} + 2h\delta_m^{n+1} + \beta_{m2}\sigma_m^{n+1} &= 2h\delta_{m-1}^n + \beta_{m1}\sigma_{m-1}^n + 2h\delta_m^n - \beta_{m2}\sigma_m^n, \\
-2\delta_{m-1}^{n+1} + h\sigma_{m-1}^{n+1} + 2\delta_m^{n+1} + h\sigma_m^{n+1} &= 2\delta_{m-1}^n - h\sigma_{m-1}^n - 2\delta_m^n - h\sigma_m^{n+1}, \quad m = 0,\dots,N,
\end{aligned}
\tag{2.11}
$$

where

$$
\beta_{m1} = z_m h\Delta t + 2\nu\Delta t, \quad \beta_{m2} = -z_m h\Delta t + 2\nu\Delta t, \quad z_m = \delta_{m-1} + \delta_m.
\tag{2.12}
$$

The boundary conditions $U_0 = \delta_{-1} + \delta_0$ and $V_N = \sigma_{N-1} + \sigma_N$ are imposed to eliminate parameters δ_{-1} and σ_N from the system (2.11) to have an equal solvable $(2N + 2) \times (2N + 2)$ penta-diagonal matrix system.

To iterate the system (2.11), we need to compute the initial parameters δ_m^0, σ_m^0. To do so, the following requirements of initial and boundary conditions at time $t = 0$ are used:

$$
\begin{aligned}
(U_N)_x(a,0) &= 0, & (V_N)_x(a,0) &= 0, \\
U_N(x,0) &= U(x_m,0), & V_N(x,0) &= V(x_m,0), \quad m = 0,\dots,N,
\end{aligned}
\tag{2.13}
$$

these requirements gives the following determination of the unknown element parameters:

$$
\begin{aligned}
\delta_{-1}^0 &= \frac{U(a,0)}{2}, & \delta_0^0 &= \frac{U(a,0)}{2}, & \delta_m^0 &= U(x_m,0) - \delta_{m-1}^0, \\
\sigma_{-1}^0 &= \frac{V(a,0)}{2}, & \sigma_0^0 &= \frac{V(a,0)}{2}, & \sigma_m^0 &= V(x_m,0) - \sigma_{m-1}^0, \quad m = 0,\dots,N.
\end{aligned}
\tag{2.14}
$$

Once element parameters are determined, time evolutions of the δ_m^n, σ_m^n are found from the system. Any nodal value and its derivatives can be recovered from (2.8) during the running of the program. To cope with the nonlinearity of the system (2.11), solutions can be made better by using the following two or three iterations at each time $n + 1$ for the parameters δ_m^{n+1}, σ_m^{n+1}:

$$
(\delta^*)^{n+1} = \delta^n + \frac{1}{2}(\delta^{n+1} - \delta^n), \quad (\sigma^*)^{n+1} = \sigma^n + \frac{1}{2}(\sigma^{n+1} - \sigma^n),
\tag{2.15}
$$

where

$$
\delta^n = (\delta_{-1}^n, \delta_0^n, \dots, \delta_N^n)^T, \quad \sigma^n = (\sigma_{-1}^n, \sigma_0^n, \dots, \sigma_N^n)^T.
\tag{2.16}
$$

2.2. The cubic B-spline collocation method (CBCM). The Burgers' equation is splitted for the time variable into

$$U_t + 2UU_x = 0,$$
$$U_t - 2\nu U_{xx} = 0. \tag{2.17}$$

The time-splitted Burgers' equation includes the second-order derivatives. So we should select the cubic B-splines for the trial functions in the collocation method. This selection provides the continuity of up to second order of the trial functions.

The forms of the cubic B-splines Q_m, $m = -1,\ldots,N+1$, are defined over the interval $[a,b]$ as follows:

$$Q_m(x) = \frac{1}{h^3} \begin{cases} (x - x_{m-2})^3, & [x_{m-2},x_{m-1}], \\ h^3 + 3h^2(x - x_{m-1}) + 3h(x - x_{m-1})^2 - 3(x - x_{m-1})^3, & [x_{m-1},x_m], \\ h^3 + 3h^2(x_{m+1} - x) + 3h(x_{m+1} - x)^2 - 3(x_{m+1} - x)^3, & [x_m,x_{m+1}], \\ (x_{m+2} - x)^3, & [x_{m+1},x_{m+2}], \\ 0, & \text{otherwise.} \end{cases} \tag{2.18}$$

We seek an approximating solution of the time-splitted Burgers' equation of this form:

$$U_N(x,t) = \sum_{m=-1}^{N+1} \delta_m(t)Q_m(x). \tag{2.19}$$

The coefficients δ_m are found by requiring that U_N satisfies (2.17) at $N+1$ collocation points and boundary conditions.

Nodal value U, the first derivative U', and the second derivative U'' at the knots x_m are obtained using the expression (2.19) and cubic B-splines $Q_m(x)$ (2.18) in terms of the element parameters by

$$U_m = \delta_{m-1} + 4\delta_m + \delta_{m+1},$$
$$hU'_m = 3(\delta_{m+1} - \delta_{m-1}),$$
$$h^2 U''_m = 6(\delta_{m-1} - 2\delta_m + \delta_{m+1}), \tag{2.20}$$

where $'$, $''$ denote the first and the second differentiations with respect to x, respectively.

To apply the collocation method, collocation points are selected to coincide with knots and then substituting nodal values U_m and first two successive derivatives U'_m, U''_m into (2.17). This yields the following coupled matrix system of the first-order ordinary differential equations:

$$\overset{\circ}{\delta}_{m-1} + 4\overset{\circ}{\delta}_m + \overset{\circ}{\delta}_{m+1} + \frac{6}{h}z_m(-\delta_{m-1} + \delta_{m+1}) = 0, \tag{2.21}$$

$$\overset{\circ}{\delta}_{m-1} + 4\overset{\circ}{\delta}_m + \overset{\circ}{\delta}_{m+1} - \frac{12}{h^2}\nu(\delta_{m-1} - 2\delta_m + \delta_{m+1}) = 0, \tag{2.22}$$

where $\circ$ denotes derivative with respect to time and $z_m = \delta_{m-1} + 4\delta_m + \delta_{m+1}$ is the nonlinear term of (2.21).

Assume that the vector of parameters δ_m and their time derivatives are linearly inter-polated between two time levels n and $n + 1/2$ for (2.21) as

$$\delta_m = \frac{\delta_m^n + \delta_m^{n+1/2}}{4}, \qquad \overset{\circ}{\delta}_m = \frac{1}{\Delta t}(\delta_m^{n+1/2} - \delta_m^n), \tag{2.23}$$

and parameters δ_m and their time derivatives are interpolated between two time levels $n + 1/2$ and $n + 1$ for (2.22) as

$$\delta_m = \frac{\delta_m^{n+1} + \delta_m^{n+1/2}}{4}, \qquad \overset{\circ}{\delta}_m = \frac{\delta_m^{n+1} - \delta_m^{n+1/2}}{\Delta t}. \tag{2.24}$$

Plugging expressions (2.23)-(2.24) up into (2.21)-(2.22), respectively, leads to a non-linear system of equations each having $N + 1$ equations in $N + 3$ unknown parameters:

$$\alpha_1 \delta_{m-1}^{n+1/2} + \alpha_2 \delta_m^{n+1/2} + \alpha_3 \delta_{m+1}^{n+1/2} = \alpha_3 \delta_{m-1}^n + \alpha_2 \delta_m^n + \alpha_1 \delta_{m+1}^n, \tag{2.25}$$

$$\alpha_4 \delta_{m-1}^{n+1} + \alpha_5 \delta_m^{n+1} + \alpha_4 \delta_{m+1}^{n+1} = \alpha_6 \delta_{m-1}^{n+1/2} + \alpha_7 \delta_m^{n+1/2} + \alpha_6 \delta_{m+1}^{n+1/2}, \tag{2.26}$$

where

$$\begin{aligned}
\alpha_1 &= 4h - 6d\Delta t, & \alpha_2 &= 16h + 24\mu, & \alpha_3 &= 4h + 6d\Delta t, \\
\alpha_4 &= h^2 - 3\nu\Delta t, & \alpha_5 &= 4h^2 + 6\nu\Delta t, & \alpha_6 &= h^2 + 3\nu\Delta t.
\end{aligned} \tag{2.27}$$

To have solvable system (2.25)-(2.26), application of the boundary conditions $U(a, t) = U_0$, $U(b,t) = U_N$ helps us to eliminate the parameters $\delta_{-1}^{n+1/2} = U_0 - \delta_0^{n+1/2}$, $\delta_N^{n+1/2} = U_N - \delta_{N-1}^{n+1/2}$ from (2.25)-(2.26) so that $(N + 1) \times (N + 1)$ tridiagonal band matrix equation can be solved with the Thomas algorithm. By having found the parameters $\delta_m^{n+1/2}$, $m = -1, \ldots, N+1$, from the system (2.25), solution parameters δ_m^{n+1} are obtained from the system (2.26).

Before carrying on obtaining solution parameters, we have to find initial parameters δ_m^0 by using initial and boundary conditions:

$$(U_x)_N(x_0, 0) = \frac{3}{h}(\delta_1 - \delta_{-1}) = U_x(x_0, 0),$$

$$U_N(x_j, 0) = \delta_{m-1} + 4\delta_m + \delta_{m+1} = U(x_j, 0), \quad j = 0, \ldots, N, \tag{2.28}$$

$$(U_x)_N(x_N, 0) = \frac{3}{h}(\delta_{N+1} - \delta_{N-1}) = U_x(x_N, 0).$$

The above equations yield a tridiagonal band matrix system whose solution can also be found using the Thomas algorithm. Once we find an approximation δ_m^0 using the systems (2.28), the rest of the time parameters δ_m^n is computed from the algebraic systems (2.25)-(2.26). Before proceeding to the each next time step δ^{n+1}, the iteration (2.15) should be repeated two or three times for improving the solution of the nonlinear algebraic equation system.

3. Numerical examples and conclusion

In this section, we will present numerical results of the Burgers' equation for two test problems. Accuracy of the methods will be measured with discrete L_2 and L_∞ error norms

by

$$|U - U_N|_\infty = \max_j |U_j - (U_N^n)_j|, \qquad \|U - U_N\|_2^2 = h \sum_{j=0}^{N} |(U_j - (U_N^n)_j)^2|. \quad (3.1)$$

(a) Shock-like solution of the Burgers' equation has analytic solution of the form

$$U(x,t) = \frac{x/t}{1 + \sqrt{(t/t_0)} \exp(x^2/4\nu t)}, \quad t \geq 1, \ 0 \leq x \leq 1, \quad (3.2)$$

where $t_0 = \exp(1/8\nu)$. Initial condition evaluated from (3.2) at time $t = 1$ is taken and boundary conditions are used $U(0,t) = U(1,t) = 0$. To test effectiveness of the method, the problem is solved with the earlier used data set $h = 0.005$ and $\Delta t = 0.01$, $\nu = 0.005$ over the interval $[0,1]$. Program is run up to time $t = 3.25$. Results of the QBCM, CBCM and the analytical solution at some times are documented in Table 3.1. Numerical solutions reflect analytical solutions within very acceptable limits. It is seen that results of the QBCM are better than those of the CBCM in terms of the norms. Error norms of earlier times of the results are much better than that of later times. Comparison of the presented results with those referenced in the paper [1, 2], when the equation is splitted, shows that there is advantage of using the QBCM due to having less error than results of the cubic B-spline collocation method to the nonsplitted Burgers' equation, but we get the same results with using CBCM. Graphical solutions of both schemes are also given at time $t = 3.25$ in Figures 3.1(a)–3.1(b). Absolute error distributions are depicted at time $t = 3.25$ in Figure 3.2, from which errors are concentrated at right boundary. This means that selection of the artificial right boundary causes an increase in error.

With the smaller viscosity constant $\nu = 0.0005$, corresponding computational experiments are carried out. Tabulated results are documented in Table 3.2. Once again, QBCM provided less error than the CBCM. Figures 3.3(a)–3.3(b) show us behavior of the numerical solutions at some times for the presented schemes. For the smaller viscosity value, the propagation value is steeper. Those graphs agree with some of the previous studies within plotting accuracy. As the smaller viscosity value is used, errors increase. In Figures 3.4(a)–3.4(b), error distributions of both schemes at time $t = 3.25$ are plotted, from which maximum errors are seen at about the center of the shock and measured as about 0.014 for the QBCM and as about 0.021 for CBCM. As for the comparison with the results given in the papers [1, 2], a direct application of both the cubic B-spline collocation method and the B-spline Galerkin finite element method to the Burgers' equation provided less error than both of the proposed schemes.

(b) For our second test example, we consider the particular solution of Burgers' equation

$$U(x,t) = \frac{\alpha + \mu + (\mu - \alpha)\exp\eta}{1 + \exp\eta}, \quad 0 \leq x \leq 1, \ t \geq 0, \quad (3.3)$$

where

$$\eta = \frac{\alpha(x - \mu t - \gamma)}{\nu} \quad (3.4)$$

Table 3.1. Comparison of results at different times for $\nu = 0.005$ with $h = 0.005$ and $\Delta t = 0.01$.

x	QBCM	CBCM	Exact	QBCM	CBCM	Exact	QBCM	CBCM	Exact
	$t = 1.7$	$t = 1.7$	$t = 1.7$	$t = 2.5$	$t = 2.5$	$t = 2.5$	$t = 3.25$	$t = 3.25$	$t = 3.25$
0.1	.05882	0.05882	0.05882	0.04000	0.04000	0.04000	0.03077	0.03077	0.03077
0.2	0.11764	0.11764	0.11765	0.08000	0.08000	0.08000	0.06154	0.06154	0.06154
0.3	0.17646	0.17646	0.17646	0.12000	0.12000	0.12000	0.09230	0.09230	0.09231
0.4	0.23517	0.23517	0.23517	0.15998	0.15998	0.15998	0.12307	0.12307	0.12307
0.5	0.29190	0.29192	0.29190	0.19982	0.19983	0.19983	0.15380	0.15380	0.15380
0.6	0.29572	0.29492	0.29591	0.23811	0.23812	0.23812	0.18430	0.18430	0.18430
0.7	0.04207	0.04299	0.04193	0.25302	0.25275	0.25310	0.21269	0.21269	0.21270
0.8	0.00063	0.00066	0.00065	0.10228	0.10269	0.10210	0.21838	0.21817	0.21844
0.9	0.00000	0.00000	0.00000	0.00553	0.00568	0.00554	0.10170	0.10124	0.10126
$L_2 \times 10^3$	0.07215	2.46642		0.05103	2.11187		1.24901	1.92482	
$L_\infty \times 10^3$	0.31153	27.5770		0.18902	25.1517		8.98390	21.0489	
			$t = 1.7$			$t = 2.4$			$t = 3.1$
$L_2 \times 10^3$ [1]			0.857			0.423			0.235
$L_\infty \times 10^3$ [1]			2.576			1.242			0.688
$L_2 \times 10^3$ [2]			0.857			0.423			0.235
$L_\infty \times 10^3$ [2]			2.576			1.242			0.688

Table 3.2. Comparison of results at different times for $v = 0.005$ with $h = 0.005$ and $\Delta t = 0.01$.

| x	QBCM	CBCM	Exact	QBCM	CBCM	Exact	QBCM	CBCM	Exact
	$t = 1.7$	$t = 1.7$	$t = 1.7$	$t = 2.5$	$t = 2.5$	$t = 2.5$	$t = 3.25$	$t = 3.25$	$t = 3.25$
0.1	0.05882	0.05882	0.05882	0.04000	0.04000	0.04000	0.03077	0.03077	0.03077
0.2	0.11765	0.11765	0.11765	0.08000	0.08000	0.08000	0.06154	0.06154	0.06154
0.3	0.17646	0.17647	0.17647	0.11999	0.12000	0.12000	0.09232	0.09231	0.09231
0.4	0.23525	0.23529	0.23529	0.15998	0.16000	0.16000	0.12309	0.12308	0.12308
0.5	0.29414	0.29412	0.29412	0.20002	0.20000	0.20000	0.15383	0.15385	0.15385
0.6	0.35303	0.35294	0.35294	0.24005	0.24000	0.24000	0.18457	0.18461	0.18462
0.7	0.00001	0.00000	0.00000	0.28002	0.28000	0.28000	0.21536	0.21538	0.21538
0.8	0.00000	0.00000	0.00000	0.00719	0.00845	0.00977	0.24615	0.24615	0.24615
0.9	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.12021	0.11037	0.12434
$L_2 \times 10^3$	1.24624	2.46642		1.43951	2.11186		1.24624	1.92482	
$L_\infty \times 10^3$	13.8155	27.5770		16.7712	25.1517		13.8155	21.0489	
	$t = 1.75$			$t = 2.5$			$t = 3.25$		
$L_2 \times 10^3$ [1]	0.235			0.567			0.308		
$L_\infty \times 10^3$ [1]	0.688			5.880			2.707		
$L_2 \times 10^3$ [2]	0.567			0.308			0.239		
$L_\infty \times 10^3$ [2]	5.880			2.705			2.291		
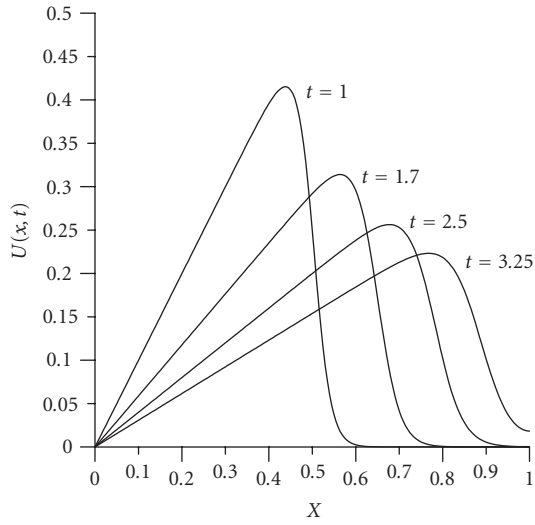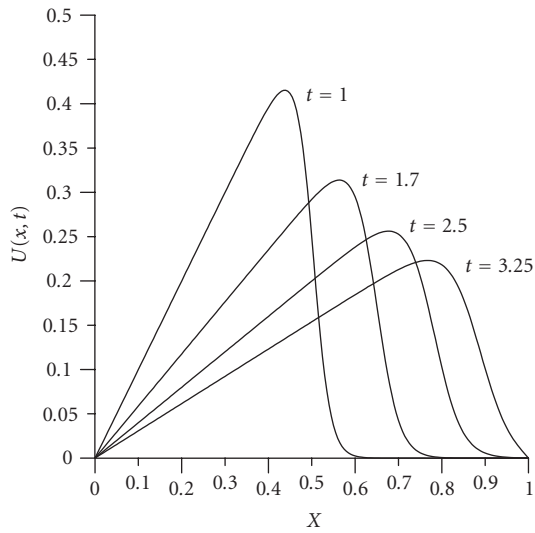
(a) QBCM
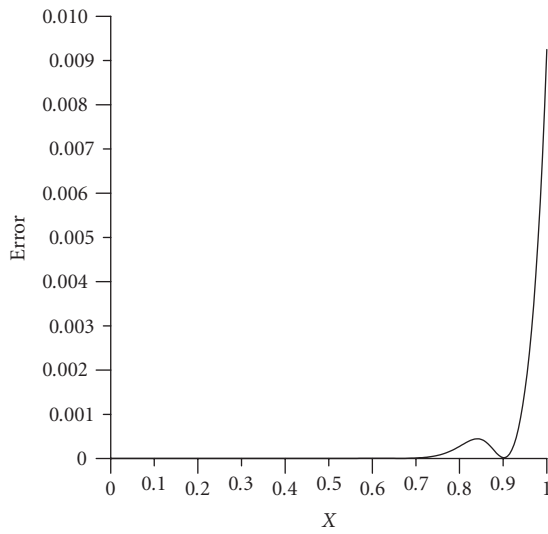

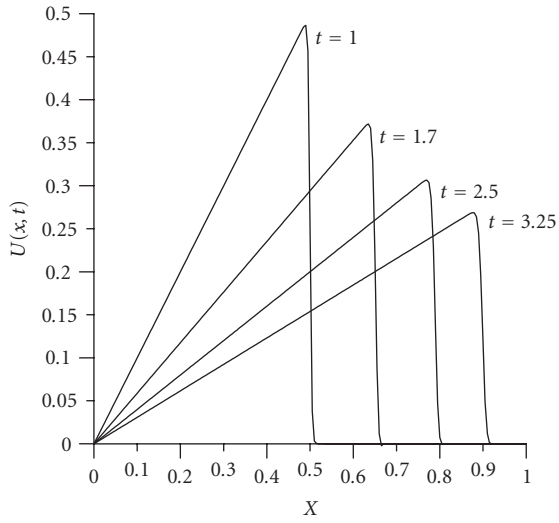(b) CBCM

Figure 3.1. $\nu = 0.005$, $h = 0.005$, $\Delta t = 0.01$.
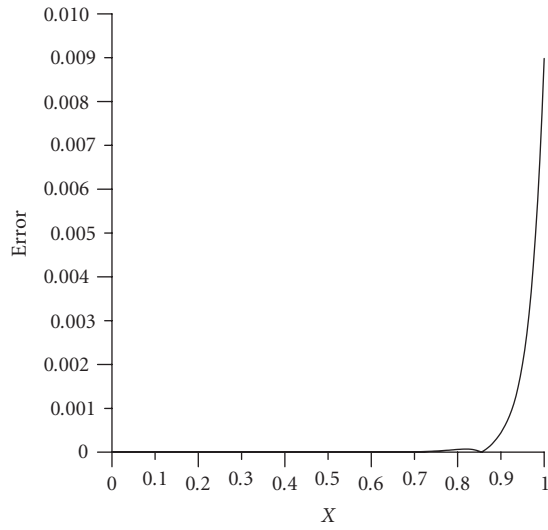
(a) QBCM
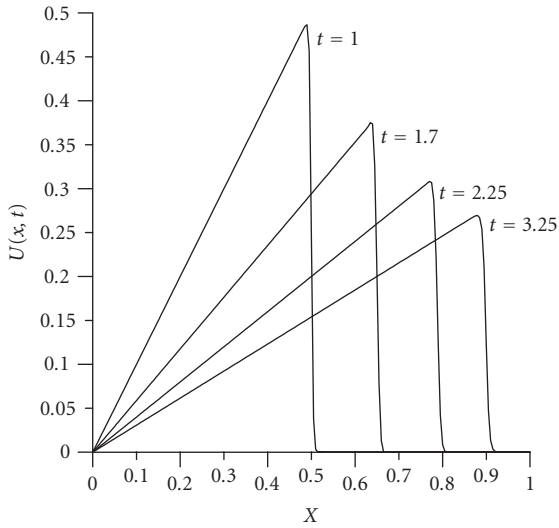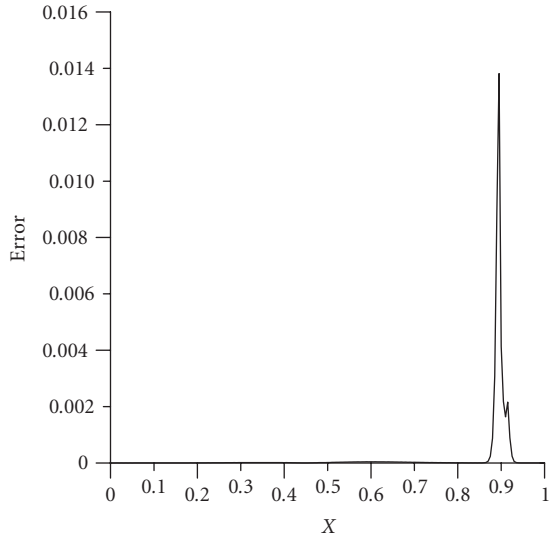

(b) CBCM

Figure 3.2. Errors (|Numerical−Analytical|) at time $t = 3.25$ with $v = 0.005$.

(a) QBCM
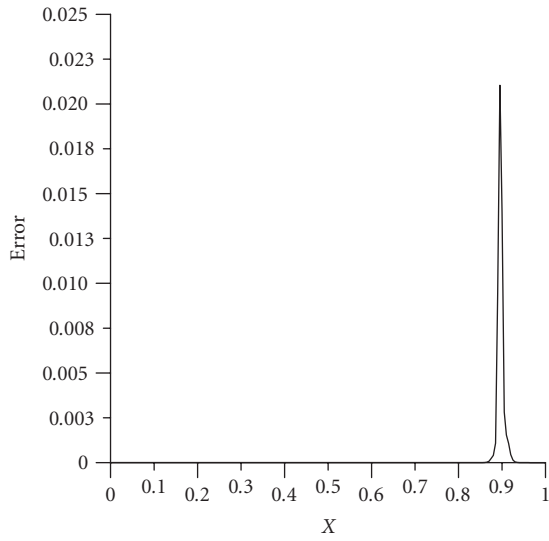

(b) CBCM

Figure 3.3. $\nu = 0.0005$, $h = 0.005$, $\Delta t = 0.01$.

(a) QBCM

(b) CBCM

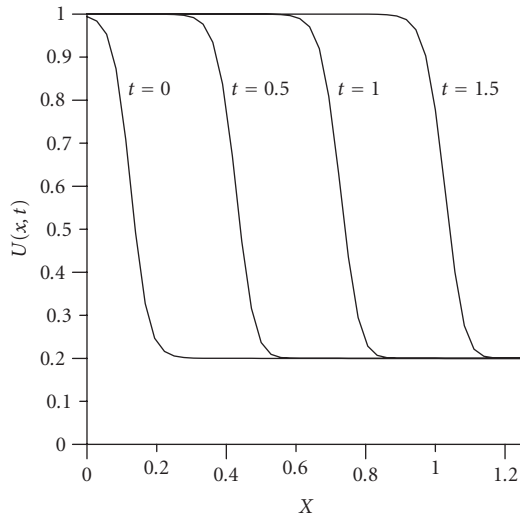Figure 3.4. Errors (|Numerical−Analytical|) at time $t = 3.25$ with $\nu = 0.0005$.

Table 3.3. Comparison of results at time $t = 0.5$ for $\nu = 0.01$ with $h = 1/36$ and $\Delta t = 0.01$.

x	QBCM	CBCM	Exact
0.000	1.000	1.000	1.000
0.056	1.000	1.000	1.000
0.111	1.000	1.000	1.000
0.167	1.000	1.000	1.000
0.222	1.000	1.000	1.000
0.278	0.997	1.000	0.998
0.333	0.977	0.983	0.980
0.389	0.838	0.825	0.847
0.444	0.472	0.465	0.452
0.500	0.237	0.244	0.238
0.556	0.202	0.204	0.204
0.611	0.200	0.200	0.200
0.667	0.200	0.200	0.200
0.722	0.200	0.200	0.200
0.778	0.200	0.200	0.200
0.833	0.200	0.200	0.200
0.889	0.200	0.200	0.200
0.944	0.200	0.200	0.200
1.000	0.200	0.200	0.200
$L_2 \times 10^3$	4.48881	5.86664	
$L_\infty \times 10^3$	19.80734	22.23450	

and α, μ, and γ are constants, which we choose for our experiments $\alpha = 0.4$, $\mu = 0.6$, $\gamma = 0.125$. This solution represents a travelling wave and initial condition obtained from the analytic solution moves to the right with speed η. So the initial condition is determined from (3.3) when $t = 0$. The boundary conditions are

$$U(0,t) = 1, \quad U(1,t) = 0.2, \quad t \geq 0. \tag{3.5}$$

The simulations are run to time $t = 0.5$. Viscosity coefficient $\nu = 0.01$, space time $h = 1/36$, and time step $\Delta t = 0.01$ are chosen for computation. Both numerical solutions and analytical solution are given in Table 3.3. It is seen that the agreement between our two numerical solutions and the analytical solution appears satisfactory. Numerical results at various times are graphed in Figures 3.5(a)–3.5(b) for both schemes. The almost same resemblance of wave fronts which was produced from both scheme is noticeable. The QBCM produced slightly better results than the CBCM. Errors of difference between the analytical and numerical results are visualized in Figures 3.6(a)–3.6(b).
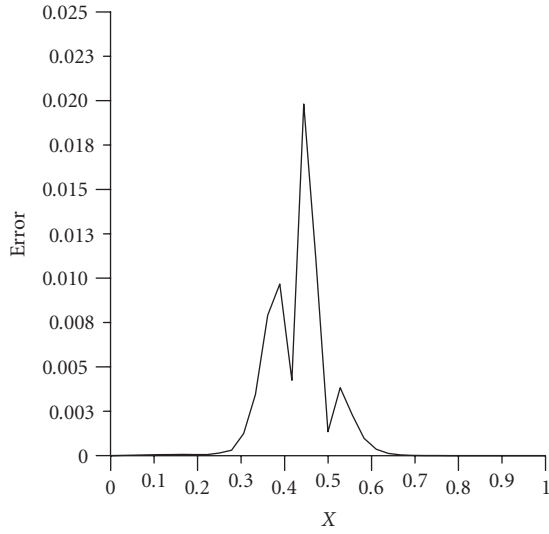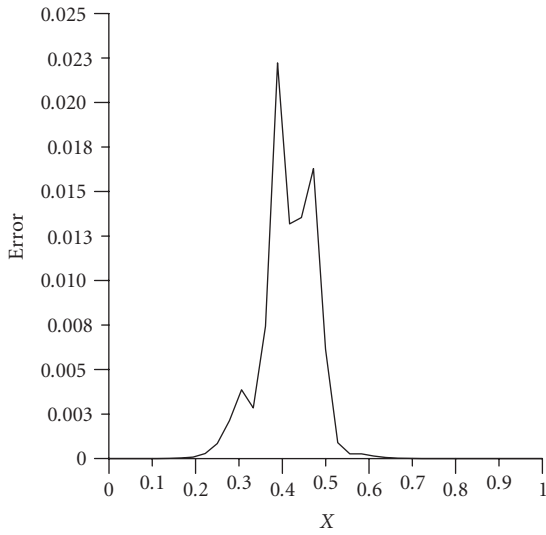
(a) QBCM
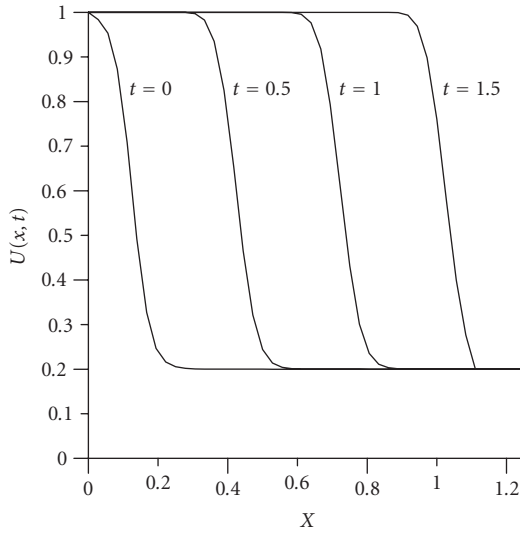
(b) CBCM

Figure 3.5. $\nu = 0.01$, $h = 1/36$, $\Delta t = 0.01$.

(a) QBCM

(b) CBCM

Figure 3.6. Errors ($|$Numerical$-$Analytical$|$) at time $t = 0.5$.

Numerical solutions of the time-space-splitted Burgers' equation are illustrated when both cubic and quadratic B-spline collocation methods are used. Although both schemes give satisfactory results, the comparison of the two schemes shows that the cubic collocation method for time-splitted Burgers' equation gives less error than the quadratic B-spline collocation method for the space-splitted Burgers' equation. We can also make further comparison of the presented schemes with some of recent previous methods referenced in the paper [1, 2]. A direct application of those methods to the Burgers' equation produces worse results than the QBCM and almost same results with CBCM. Unfortunaly, the splitting tecnique for the Burgers' equation together with numerical tecnique gives worse results than those of the proposed methods when the smaller values of the viscosity are used. However, when the differential equation involves higher derivatives, time-space-splitted scheme accompanied with low-order polynomial enables to construct some approximate functions in the numerical tecniques so that time-space-splitted numerical methods can be preferable in getting the numerical solution of those differential equations due to providing easy algorithm.

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