J. of Inequal. & Appl., 2002, Vol. 7(5), pp. 727–746

Weighted Integral Inequalities with the Geometric Mean Operator

LARS-ERIK PERSSON[a,*] and VLADIMIR D. STEPANOV[b,†]

[a]*Department of Mathematics, Luleå University, S-97 187 Luleå, Sweden;*
[b]*Computer Centre, Russian Academy of Sciences, Far-Eastern Branch, Khabarovsk 680042, Russia*

(Received 20 March 2001; Revised 29 June 2001)

The geometric mean operator is defined by

$$Gf(x) = \exp\left(\frac{1}{x}\int_0^x \log f(t)\,\mathrm{d}t\right).$$

A precise two-sided estimate of the norm

$$\|G\| = \sup_{f\neq 0}\frac{\|Gf\|_{L^q_u}}{\|f\|_{L^p_v}}$$

for $0 < p,\ q \leq \infty$ is given and some applications and extensions are pointed out.

Keywords: Geometric mean operator; Hardy inequalities

Classification: *AMS Subject Classification* (*1991*): Primary: 26D15, 26D10; Secondary: 47H19

* Corresponding author. E-mail: larserik@sm.luth.se
† E-mail: stepanov@as.fe.ru

ISSN 1025-5834 print; ISSN 1029-242X.
DOI: 10.1080/1025583021000022531

1 INTRODUCTION

Applying the clever Polya's observation to the Hardy inequality, $p > 1$

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\,\mathrm{d}t\right)^p \mathrm{d}x \le \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p, \quad f \ge 0$$

by changing $f \to f^{1/p}$ and tending $p \to \infty$ we obtain the Knopp inequality [8] (c.f. also [2])

$$\int_0^\infty Gf(x)\,\mathrm{d}x \le e \int_0^\infty f$$

with the geometric mean operator

$$Gf(x) := \exp\left(\frac{1}{x}\int_0^x \log f(t)\,\mathrm{d}t\right), \quad f \ge 0.$$

The weighted integral inequality

$$\left(\int_0^\infty (Gf)^q u\right)^{1/q} \le \mathbb{C}\left(\int_0^\infty f^p v\right)^{1/p} \tag{1}$$

was investigated by several authors [3–8, 9, 11, 12] and a most general result was found by P. Gurka, B. Opic and L. Pick [11, 12] with, however, unstable constants pretending to estimate the norm (= the least possible constant $\mathbb{C}$ in (1)) (see (14) and (15) below).

In the present paper we give the precise two-sided estimate of the norm of $G: L_v^p \to L_u^q$ (see Theorems 2 and 4). In the case $0 < p \le q < \infty$ we argue close to the original Polya idea and for $0 < q < p < \infty$ we use the Pick and Opic scheme [12] and a new form of the criterion for the Hardy inequality with weights (Theorem 3) which is of independent interest. Throughout the paper we denote $V(t) = \int_0^t v^{-1/(p-1)}$ and undeterminates $0 \cdot \infty$ are taken to be equal to zero.

2 PICK AND OPIC SCHEME

Put

$$Hf(x) = \frac{1}{x}\int_0^x f(t)\,dt.$$

It is well known that

$$\lim_{\alpha\downarrow 0}\left(\frac{1}{x}\int_0^x f^{\alpha}\right)^{1/\alpha} = Gf(x) \tag{2}$$

and

$$\begin{aligned} &\text{(i) } Gf(x) \le Hf(x)\\ &\text{(ii) } G(f^s) = [G(f)]^s, \quad s \in \mathbb{R}. \end{aligned} \tag{3}$$

Let $0 < p, q < \infty$, $u(x) \ge 0$, $v(x) \ge 0$ and put

$$w := \left[G\left(\frac{1}{v}\right)\right]^{q/p} u.$$

Then it follows from (3)(ii) that (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5), where

$$\left(\int_0^{\infty} (Gf)^q w\right)^{1/q} \le \mathbb{C}\left(\int_0^{\infty} f^p\right)^{1/p}, \tag{4}$$

$$\left(\int_0^{\infty} (Gf)^{qs/p} w\right)^{p/qs} \le \mathbb{C}^{p/s}\left(\int_0^{\infty} f^s\right)^{1/s}, \quad s > 0, \tag{5}$$

$$\|G\|_{L^p_v \to L^q_u} = \|G\|_{L^p \to L^q_w} = \|G\|^{s/p}_{L^s \to L^{sq/p}_w} \tag{6}$$

and

$$\|G\|_{X\to Y} := \sup_{f\neq 0} \frac{\|Gf\|_Y}{\|f\|_X}.$$

It follows from Jensen's inequality (see (3)(i)), that $\|G\|_{X\to Y} \le \|H\|_{X\to Y}$. Therefore the upper bounds for $\|G\|_{L^p_v \to L^q_u}$ can be derived from the following known estimates of $L^p - L^q_w$ norm of H.

(a) $1 < p \le q < \infty$. Then

$$\mathbb{A} \le \|H\|_{L^p \to L^q_w} \le \alpha(p,q)\mathbb{A}, \tag{7}$$

where

$$\mathbb{A} := \mathbb{A}(p,q) = \sup_{t>0} \mathbb{A}(t) = \sup_{t>0} \left(\int_t^\infty \frac{w(x)\,\mathrm{d}x}{x^q} \right)^{1/q} t^{1/p'}, \tag{8}$$

$\alpha(p,p) = p^{1/p}(p')^{1/p'}$ and (Manakov [10])

$$\alpha(p,q) = \left[\frac{\Gamma(q/(\lambda-1))}{\Gamma(\lambda/\lambda-1)\Gamma(q-1/\lambda-1)} \right]^{(\lambda-1)/q}, \quad 1 < p < q < \infty, \ \lambda = q/p. \tag{9}$$

(b) $0 < q < p \le \infty$, $p > 1$, $1/r = 1/q - 1/p$. Then

$$\beta_1(p,q)\mathbb{B} \le \|H\|_{L^p \to L^q_w} \le \beta_2(p,q)\mathbb{B}, \tag{10}$$

where

$$\mathbb{B} := \mathbb{B}(p,q) = \left(\int_0^\infty t^{r/q'} \left(\int_t^\infty \frac{w(x)}{x^q}\,\mathrm{d}x \right)^{r/q} \mathrm{d}t \right)^{1/r},$$

$$\beta_1(p,q) = \begin{cases} q^{1/q}(p')^{1/q'}\dfrac{q}{r}, & 0 < q < p < \infty, \ p > 1, \ q \ne 1, \\ 1, & 1 = q < p < \infty, \ 1 \le q < p = \infty, \end{cases}$$

$$\beta_2(p,q) = \begin{cases} q^{1/q}(p')^{1/q'}, & 1 < q < p < \infty, \\ 1, & 1 = q < p < \infty, \ 1 \le q < p = \infty, \\ r^{1/r}p^{1/p}(p')^{1/q'}, & 0 < q < 1 < p < \infty. \end{cases}$$

This implies the upper bound for $\|G\|$ in the case $p > 1$. For the lower bound the following Lemma can be used.

LEMMA 1 *Let* $0 < p \le q < \infty$, $\|G\| := \|G\|_{L^p \to L^q_w} < \infty$. *Then*

$$\|G\| \ge \sup_{t>0} t^{-1/p} \left(\int_0^t w(x)\, dx \right)^{1/q}, \tag{11}$$

$$\|G\| \ge \sup_{s>1} \left(\frac{(s-1)e^s}{1+(s-1)e^s} \right)^{1/p} \sup_{t>0} t^{(s-1)/p} \left(\int_t^\infty \frac{w(x)\, dx}{x^{sq/p}} \right)^{1/q}. \tag{12}$$

Proof We use a modified test function from the proof of ([5], Theorem 1.4). For $s > 1$, $t > 0$ put

$$f(x) = t^{-1/p} \chi_{[0,t]}(x) + (xe)^{-s/p} t^{(s-1)/p} \chi_{[t,\infty)}(x).$$

Then $(\int_0^\infty f^p)^{1/p} = (1 + ((s-1)e^s)^{-1})^{1/p} =: a_s$ and (5) brings

$$a_s \mathbb{C} \ge \left[t^{-q/p} \int_0^t w(x)\, dx + t^{(s-1)q/p} \int_t^\infty \frac{w(x)\, dx}{x^{sq/p}} \right]^{1/q}.$$

It gives (11) when $s \to \infty$ by omitting the second term on the right hand side and (12) by omitting the first term on the right hand side. ■

The lower bound for the case $0 < q < p < \infty, p > 1$ follows by putting the usual test function

$$f(x) = x^{r/(pq')} \left(\int_x^\infty \frac{w(\tau)\, d\tau}{\tau^q} \right)^{r/(pq)}$$

in (4) ([12], Lemma 3.2). It brings

$$\|G\| \ge e^{-r/(pq')} \left(\frac{q}{p'} \right)^{1/q} \mathbb{B}. \tag{13}$$

Now, on the strength of (6) the upper bound from (7) and (12) imply the result of ([11], (1.3)): if $0 < p \le q < \infty$, then

$$\sup_{s>1} \left(\frac{(s-1)e^s}{1+(s-1)e^s} \right)^{1/p} \mathbb{A}^{s/p}\left(s, \frac{sq}{p}\right) \le \|G\|$$
$$\le \inf_{s>1} \alpha^{s/p}\left(s, \frac{sq}{p}\right) \mathbb{A}^{s/p}\left(s, \frac{sq}{p}\right) \tag{14}$$

with slightly better factors on both sides and the upper bound from (10) and (13) contains the result of ([12], (3.18)): if $0 < q < p \leq \infty$, then

$$\sup_{s>1}\left(\frac{q(s-1)}{p}\right)^{1/q} e^{(1-qs/p)/(p-q)}\mathbb{B}^{s/p}\left(s,\frac{q}{p}s\right) \leq \|G\| \leq \inf_{s>1}\beta_2^{s/p}\left(s,\frac{sq}{p}\right)\mathbb{B}^{s/p}\left(s,\frac{q}{p}s\right) \tag{15}$$

with

$$\mathbb{A}^{s/p}\left(s,\frac{sq}{p}\right) = \sup_{t>0} t^{(s-1)/p}\left(\int_t^\infty \frac{w(x)\,\mathrm{d}x}{x^{sq/p}}\right)^{1/q}$$

and

$$\mathbb{B}^{s/p}\left(s,\frac{sq}{p}\right) = \left(\int_0^\infty t^{(qs-p)/(p-q)}\left(\int_t^\infty \frac{w(x)\,\mathrm{d}x}{x^{sq/p}}\right)^{s/(p-q)}\mathrm{d}t\right)^{1/r}.$$

3 THE CASE $0 < p \leq q < \infty$

We are going to use a limiting consideration originally due to G. Polya. To this end we replace (4) by

$$\left(\int_0^\infty (Hf^\alpha)^{q/\alpha}w\right)^{1/q} \leq \mathbb{C}_\alpha\left(\int_0^\infty f^p\right)^{1/p}, \quad \alpha > 0$$

which is equivalent to the weighted Hardy inequality

$$\left(\int_0^\infty (Hf)^{q/\alpha}w\right)^{\alpha/q} \leq \mathbb{C}_\alpha^\alpha\left(\int_0^\infty f^{p/\alpha}\right)^{\alpha/p}, \quad \alpha > 0$$

and using (3) we reduce the problem to existence of the limits of upper and lower bounds for the norm $\|H\|^{1/\alpha}_{L^{p/\alpha}\to L^{q/\alpha}_w}$ because

$$\|G\|_{L^p_v\to L^q_w} = \lim_{\alpha\downarrow 0}\|H\|^{1/\alpha}_{L^{p/\alpha}\to L^{q/\alpha}_w}. \tag{16}$$

To this purpose we need the following alternate criterion for the weighted Hardy inequality ([14], Section 2.3).

THEOREM 1 *Let* $1 < p \le q < \infty$. *Then*

$$\left(\int_0^\infty \left(\int_0^x f\right)^q u(x)\,\mathrm{d}x\right)^{1/q} \le \mathbb{C}\left(\int_0^\infty f^p v\right)^{1/p} \tag{17}$$

is true for all $f \ge 0$ *iff*

$$\infty > \mathcal{A}_1 = \sup_{t>0} V^{-1/p}(t)\left(\int_0^t u(x)V^q(x)\,\mathrm{d}x\right)^{1/q} \tag{18}$$

and

$$\mathcal{A}_1 \le \mathbb{C} \le p'\mathcal{A}_1. \tag{19}$$

Proof With $p' = p/(p-1)$, $q' = q/(q-1)$ inequality (17) is equivalent to

$$\left(\int_0^\infty \left(\int_x^\infty g\right)^{p'} \mathrm{d}V(x)\right)^{1/p'} \le \mathbb{C}\left(\int_0^\infty g^{q'} u^{-1/(q-1)}\right)^{1/q'}$$

with the same constant $\mathbb{C}$. We have for g with supp $g \subset (0,\infty)$

$$\begin{aligned} J &:= \int_0^\infty \left(\int_x^\infty g\right)^{p'} \mathrm{d}V(x) = p'\int_0^\infty \left(\int_x^\infty g\right)^{1/(p-1)} g(x)V(x)\,\mathrm{d}x \\ &\le p'\left(\int_0^\infty g^{q'} u^{-1/(q-1)}\right)^{1/q'} \left(\int_0^\infty \left(\int_x^\infty g\right)^{q/(p-1)} u(x)V^q(x)\,\mathrm{d}x\right)^{1/q} \\ &:= p'\left(\int_0^\infty g^{q'} u^{-1/(q-1)}\right)^{1/q'} J_1^{1/q}. \end{aligned}$$

Now

$$\begin{aligned} J_1 &= \int_0^\infty \int_x^\infty \mathrm{d}\left(-\left(\int_t^\infty g\right)^{q/(p-1)}\right) u(x)V^q(x)\,\mathrm{d}x \\ &= \int_0^\infty \left[\mathrm{d}\left(-\left(\int_t^\infty g\right)^{q/(p-1)}\right)\right] \int_0^t u(x)V^q(x)\,\mathrm{d}x \end{aligned}$$

(applying (18) and Minkowski's inequality)

$$\leq \mathcal{A}_1^q \int_0^\infty \left[\mathrm{d}\left(-\left(\int_t^\infty g \right)^{q/(p-1)} \right) \right] V^{q/p}(t)$$

$$\leq \mathcal{A}_1^q \left(\int_0^\infty \left(\int_x^\infty \left[\mathrm{d}\left(-\left(\int_t^\infty g \right)^{q/(p-1)} \right) \right] \right)^{p/q} \mathrm{d}V(x) \right)^{q/p}$$

$$= \mathcal{A}_1^q \left(\int_0^\infty \left(\int_x^\infty g \right)^{p'} \mathrm{d}V(x) \right)^{q/p}.$$

Thus, $J^{1/p'} \leq p'\mathcal{A}_1 \left(\int_0^\infty g^{q'} u^{-1/(q-1)} \right)^{1/q'} \Rightarrow \mathbb{C} \leq p'\mathcal{A}_1$. Putting $f_t = \chi_{[0,t]} v^{-1/(p-1)}$ in (17) we obtain $\mathcal{A}_1 \leq \mathbb{C}$. ∎

THEOREM 2 *Let* $0 < p \leq q < \infty$. *Then the inequality* (1) *holds for all* $f \geq 0$ *iff*

$$\mathbb{D} := \sup_{t>0} t^{-1/p} \left(\int_0^t w(x)\,\mathrm{d}x \right)^{1/q} < \infty$$

and

$$\mathbb{D} \leq \|G\|_{L_v^p \to L_u^q} \leq e^{1/p} \mathbb{D}. \tag{20}$$

Proof It follows from Theorem 1, that for $0 < \alpha < p \leq q < \infty$

$$\mathbb{D} \leq \|H\|^{1/\alpha}_{L^{p/\alpha} \to L_w^{q/\alpha}} \leq \left(\frac{p}{p-\alpha} \right)^{1/\alpha} \mathbb{D}$$

and (20) is a consequence of (16). The lower bound in (20) was also proved in Lemma 1 (11). ∎

Remark 1 The factor $e^{1/p}$ is the best possible for $p = q$ and attains in the case $u(x) = v(x) = 1$. For $p = q = 1$ an alternate form of Theorem 2 was proved in ([5], Theorem 1.4). The factor p' in (19) is best possible for only $p = q$. When $1 < p < q < \infty$ it can be improved in general according to the following Lemma.

LEMMA 2 *Suppose* $1 < p < q < \infty$ *and* $V(\infty) = \infty$. *Then there exists a weight* $u^*(x) \geq 0$ *such that* $\mathcal{A}_1 = 1$ *and*

$$\left(\int_0^\infty \left(\int_0^x f\right)^q u^*(x)\,\mathrm{d}x\right)^{1/q} \leq \alpha^*(p,q)\left(\int_0^\infty f^p v\right)^{1/p},$$

where $p' > \alpha^*(p,q)$ and $\alpha^*(p,q)$ *is given by*

$$\alpha^*(p,q) = (p-1)^{-1/q}\left[\frac{\Gamma(qp/(q-p))}{\Gamma(q/(q-p))\Gamma((q-1)p/(q-p))}\right]^{(q-p)/qp}.$$

Proof Let

$$V^{-1/p}(t)\left(\int_0^t u^*(x)V^q(x)\,\mathrm{d}x\right)^{1/q} \equiv 1, \quad t > 0.$$

Then

$$u^* = \frac{q}{p}V^{-q/p'-1}v^{-1/(p-1)}. \tag{21}$$

Using the change of variables

$$V(t) = s, f(t)v^{1/(p-1)}(t) = g(s)$$

we find

$$\int_0^\infty f^p v = \int_0^\infty \left[f(t)v^{1/(p-1)}(t)\right]^p \mathrm{d}V(t) = \int_0^\infty g^p$$

and

$$\begin{aligned}\int_0^\infty \left(\int_0^x f\right)^q u^*(x)\,\mathrm{d}x &= \frac{q}{p}\int_0^\infty \left(\int_0^x f(t)v^{1/(p-1)}(t)\,\mathrm{d}V(t)\right)^q V^{-q/p'-1}(x)\,\mathrm{d}V(x)\\ &= \frac{q}{p}\int_0^\infty \left(\int_0^y g\right)^q y^{-q/p'-1}\,\mathrm{d}y.\end{aligned}$$

Thus, inequality (17) with v and u^* satisfying (21) becomes

$$\left(\int_0^\infty \left(\int_0^y g\right)^q y^{-q/p'-1}\,\mathrm{d}y\right)^{1/q} \leq \left(\frac{p}{q}\right)^{1/q} \mathbb{C}\left(\int_0^\infty g^p\right)^{1/p}$$

and by the result of G. A. Bliss [1] we conclude that

$$\left(\frac{p}{q}\right)^{1/q}\mathbb{C} = C_{p,q},$$

where

$$C_{p,q} = \left(\frac{q}{p} - 1\right)^{-1/q}\left(\frac{q(p-1)}{q-p}\right)^{-1/p} \times \left[\frac{\Gamma(qp/(q-p))}{\Gamma(q/(q-p))\Gamma(q(p-1)/(q-p))}\right]^{(q-p)/pq}$$

and the result follows. ■

Now, it makes sense to look what the limiting procedure gives if it starts from (7–9).

PROPOSITION 1 *Let $0 < p \le q < \infty$. Then the following upper bound holds*

$$\|G\|_{L^p_v \to L^q_w} \le \gamma(p,q) \sup_{t>0} t^{1/q-1/p} w^{1/q}(t), \tag{22}$$

where $\gamma(p,p) = e^{1/p}$ and

$$\gamma(p,q) = \left[\left(\frac{q}{p} - 1\right)\left(\Gamma\left(\frac{q}{q-p}\right)\right)^{(q/p)-1}\right]^{-1/q} < e^{1/q}, \quad p < q.$$

Proof Since $\lim_{q\downarrow p}\gamma(p,q) = e^{1/p}$, we consider the case $p < q$ only. Using (7) and (9) we find for $0 < \alpha < p \le q$

$$\|H\|^{1/\alpha}_{L^{p/\alpha} \to L^{q/\alpha}_w} \le \left[\frac{\Gamma(q/(\lambda-1)\alpha)}{\Gamma(\lambda/(\lambda-1))\Gamma(((q/\alpha)-1)/(\lambda-1))}\right]^{(\lambda-1)/q} \mathbb{A}_\alpha,$$

where

$$\mathbb{A}_\alpha = \sup_{t>0} t^{1/\alpha-1/p}\left(\int_t^\infty \frac{w(x)\,dx}{x^{q/\alpha}}\right)^{1/q}.$$

Denote $q/\alpha(\lambda-1)=x$. It is known that

$$\frac{\Gamma(q/(\lambda-1)\alpha)}{\Gamma(q/(\lambda-1)\alpha-1/(\lambda-1))(q/(\lambda-1)\alpha)^{1/(\lambda-1)}} = \frac{\Gamma(x)}{\Gamma(x-1/(\lambda-1))x^{1/(\lambda-1)}} \to 1, \quad x\to\infty.$$

Hence, (16) yields

$$\|G\|_{L^p_v\to L^q_u} \le \gamma(p,q)\sup_{t>0}\mathbb{A}_0(t),$$

where

$$\mathbb{A}_0(t) = \limsup_{\alpha\downarrow 0}\left(\frac{q}{\alpha}t^{-q/p}\int_t^\infty \frac{w(x)\,dx}{(x/t)^{q/\alpha}}\right)^{1/q}.$$

Denoting $q/\alpha = s\uparrow\infty$ when $\alpha\downarrow 0$ we observe that

$$\mathbb{A}_0(t) = t^{1/q-1/p}\limsup_{s\uparrow\infty}\left(\frac{s-1}{t}\int_t^\infty \frac{w(x)\,dx}{(x/t)^s}\right)^{1/q}.$$

Without loss of generality we suppose that $w(x)$ is a step function, and note that

$$\chi_{[1,\infty)}(x)(s-1)x^{-s} \to \delta_1(x), \quad s\uparrow\infty,$$

where $\delta_1(x)$ is the Dirac delta function with the unit mass at $x=1$. Then

$$\mathbb{A}_0(t) = t^{1/q-1/p}w(t) \text{ a.e. } t>0$$

and (22) follows. Finally we prove that

$$\gamma(p,q) < e^{1/q}, \quad p<q.$$

Indeed, this is equivalent to

$$g(\lambda) := (\lambda-1)\Gamma^{\lambda-1}\left(\frac{\lambda}{\lambda-1}\right) > e^{-1}.$$

We have

$$g(\lambda) = \left(\int_0^\infty (x(\lambda-1))^{1/(\lambda-1)} e^{-x}\,\mathrm{d}x\right)^{\lambda-1}$$
$$= \left(\int_0^\infty t^{1/(\lambda-1)} e^{-t/(\lambda-1)} \frac{\mathrm{d}t}{\lambda-1}\right)^{\lambda-1}$$
$$= \left(1 - \frac{1}{\lambda-1}\int_0^\infty (1 - t^{1/(\lambda-1)}) e^{-t/(\lambda-1)}\,\mathrm{d}t\right)^{\lambda-1}.$$

Plainly

$$\int_0^\infty (1 - t^{1/(\lambda-1)}) e^{-t/(\lambda-1)}\,\mathrm{d}t < \int_0^1 (1 - t^{1/(\lambda-1)}) e^{-t/(\lambda-1)}\,\mathrm{d}t < 1.$$

Thus,

$$g(\lambda) > \left(1 - \frac{1}{\lambda-1}\right)^{\lambda-1} > e^{-1}. \qquad \blacksquare$$

4 THE CASE $0 < q < p \leq \infty$

The aim of this section is to find a criterion for the inequality (1) similar to (20) in the opposite case of relation between parameters p and q. It means that we want to replace the Pick and Opic result (15) by a two-sided estimate with stable factor as in (20). For this purpose we need a new criterion for the weighted Hardy inequality in the case $q < p$.

THEOREM 3 *Let* $0 < q < p < \infty$, $p > 1$, $1/r = 1/q - 1/p$. *Then* (17) *is true for all* $f \geq 0$ *iff*

$$\mathcal{B} := \left(\int_0^\infty \left(\int_0^t uV^q\right)^{r/p} u(t) V^{q-r/p}(t)\,\mathrm{d}t\right)^{1/r} < \infty.$$

Moreover, if $V(\infty) = \infty$, *then*

$$(p')^{1/q'} (q/r)^{1/r'} 2^{-1/q} \mathcal{B} \leq C \leq q^{1/p} p^{1/r} p' \mathcal{B} \qquad (23)$$

and if $0 < V(\infty) < \infty$, *then*

$$\left[\frac{q}{r} + 2^{r/q} r^{r-1} q^{-2r/p} (qp')^{-r/q'}\right]^{-1/r} \mathcal{B} \le C$$
$$\le \left(\frac{r}{q}\right)^{1/r} \left[4^q + q(p')^q \left(\frac{p}{r}\right)^{q/r}\right]^{1/q} \mathcal{B} \tag{24}$$

Proof Suppose that $V(\infty) = \infty$. Then

$$\mathcal{B} = \left(\frac{q}{p}\right)^{1/r} \left(\int_0^\infty \left(\int_0^t uV^q\right)^{r/q} V^{-r/q}(t)\, \mathrm{d}V(t)\right)^{1/r} =: \left(\frac{q}{p}\right)^{1/r} \mathcal{B}_0.$$

Indeed, if $\mathcal{B} < \infty$, then

$$\left(\int_0^t uV^q\right)^{r/q} V^{-r/p}(t) = V^{-r/p}(t) \int_0^t \mathrm{d}\left(\int_0^x uV^q\right)^{r/q}$$
$$\le \frac{r}{q} \int_0^t \left(\int_0^x uV^q\right)^{r/p} u(x) V^{q-r/p}(x)\, \mathrm{d}x \to 0, \quad t \to 0.$$

Integrating by parts we find that $\mathcal{B} \ge (q/p)^{1/r} \mathcal{B}_0$. Hence, $\mathcal{B}_0 < \infty$ and

$$\left(\int_0^t uV^q\right)^{r/q} V^{-r/p}(t) = \left(\int_0^t uV^q\right)^{r/q} \int_t^\infty \mathrm{d}(-V^{-r/p}(x))$$
$$\le \frac{r}{p} \int_t^\infty \left(\int_0^x uV^q\right)^{r/q} V^{-r/q}(x)\, \mathrm{d}V(x) \to 0, \quad t \to \infty.$$

Again, integrating by parts, we see that $\mathcal{B}_0 \ge (p/q)^{1/r} \mathcal{B}$. Consequently, $\mathcal{B} = (q/p)^{1/r} \mathcal{B}_0$. The same arguments work if we start with $\mathcal{B}_0 < \infty$.

Observe, that if $0 < V(\infty) < \infty$, then

$$\mathcal{B}^r = \frac{q}{r} V^{-r/p}(\infty) \left(\int_0^\infty uV^q\right)^{r/q} + \frac{q}{p} \mathcal{B}_0^r. \tag{25}$$

For the lower bound we suppose that inequality (17) holds with $\mathbb{C} < \infty$. Then according to [13]

$$\mathbb{C} \ge q^{1/q} (p')^{1/q'} \frac{q}{r} B, \tag{26}$$

where

$$B = \left(\int_0^\infty \left(\int_x^\infty u \right)^{r/q} V^{r/q'}(x)\, \mathrm{d}V \right)^{1/r}.$$

We show that

$$\mathcal{B}_0 \le (2q)^{1/q} (p/r)^{1/r} \tag{27}$$

and the lower bound (23) will be proved. By writing

$$\mathcal{B}_0^r = \int_0^\infty \left(\int_0^x V^q(t)\, \mathrm{d}\left(-\int_t^x u \right) \right)^{r/q} V^{-r/q}(x)\, \mathrm{d}V(x) \tag{28}$$

we find

$$\begin{aligned} \int_0^x V^q(t)\, \mathrm{d}\left(-\int_t^x u \right) &= q \int_0^x \left(\int_t^x u \right) V^{q-1}(t)\, \mathrm{d}V(t) \\ &= q \int_0^x \left\{ \left(\int_t^x u \right) V^{q-1+q/2p}(t) \right\} V^{-q/2p}(t)\, \mathrm{d}V(t) \end{aligned}$$

applying Hölder's inequality with the exponents r/q and p/q

$$\le q \left(\int_0^x \left(\int_t^x u \right)^{r/q} V^{(q-1+q/2p)r/q}\, \mathrm{d}V \right)^{q/r} \left(\int_0^x V^{-1/2}\, \mathrm{d}V \right)^{q/p}.$$

This and (28) imply

$$\begin{aligned} \mathcal{B}_0^r &\le q^{r/q} 2^{r/p} \int_0^\infty \left(\int_0^x \left(\int_t^\infty u \right)^{r/q} V^{(q-1+q/2p)r/q}\, \mathrm{d}V(t) \right) V^{r/2p-r/q}(x)\, \mathrm{d}V(x) \\ &= q^{r/p} 2^{r/p} \int_0^\infty \left(\int_t^\infty u \right)^{r/q} V^{r/q'+r/2p}(t)\, \mathrm{d}V(t) \int_t^\infty V^{r/2p-r/q}(x)\, \mathrm{d}V(x) \\ &\le \frac{(2q)^{r/q} p}{r} \int_0^\infty \left(\int_t^\infty u \right)^{r/q} V^{r/q'}(t)\, \mathrm{d}V(t) \end{aligned}$$

and (27) follows, which together with (26) gives the lower bound of (23).

For the upper bound we assume first that $\mathcal{B} < \infty$ and $V(\infty) = \infty$. Then

$$\begin{aligned} J &:= \int_0^\infty \left(\int_0^x f \right)^q u(x)\,\mathrm{d}x = \int_0^\infty \left(\int_0^x f \right)^q u(x) V^q(x) V^{-q}(x)\,\mathrm{d}x \\ &= q \int_0^\infty \left(\int_0^x f \right)^q u(x) V^q(x) \int_x^\infty V^{-q-1}(s)\,\mathrm{d}V(s)\,\mathrm{d}x =: J_0 \\ &= q \int_0^\infty V^{-q-1}(s) \left(\int_0^s \left(\int_0^x f \right)^q u(x) V^q(x)\,\mathrm{d}x \right) \mathrm{d}V(s) \\ &\le q \int_0^\infty \left\{ \left(\int_0^s f \right)^q V^{-q}(s) \right\} \left\{ \left(\int_0^s u V^q \right) V^{-1}(s) \right\} \mathrm{d}V(s) \end{aligned}$$

(applying Hölder's inequality with the exponents p/q and r/q)

$$\le q \left(\int_0^\infty \left(\int_0^s f \right)^p \frac{\mathrm{d}V(s)}{V^p(s)} \right)^{q/p} \mathcal{B}_0^q.$$

It is easy to see, that by Theorem 1

$$\left(\int_0^\infty \left(\int_0^s f \right)^p \frac{\mathrm{d}V(s)}{V^p(s)} \right)^{1/p} \le p' \left(\int_0^\infty f^p v \right)^{1/p},$$

and the upper bound in (23) follows.

Now, let $0 < V(\infty) < \infty$ and $\mathcal{B} < \infty$. Arguing as above we find

$$J = J_0 + V^{-q}(\infty) \int_0^\infty \left(\int_0^x f \right)^q u(x) V^q(x)\,\mathrm{d}x =: J_0 + J_1.$$

We need to estimate J_1. To this end let $\{x_k\} \subset (0, \infty)$, $k \le N < \infty$ be such a sequence, that

$$\int_0^{x_k} f = 2^k, \quad k \le N,$$

$$\int_0^\infty f \le 2^{N+1}, \qquad x_{N+1} = \infty.$$

The last is possible because

$$\int_0^\infty f \le \left(\int_0^\infty f^p v\right)^{1/p} V^{1/p'}(\infty) < \infty.$$

Thus,

$$V^q(\infty)J_1 = \sum_{k\le N} \int_{x_k}^{x_{k+1}} \left(\int_0^x f\right)^q u(x)V^q(x)\,\mathrm{d}x \le \sum_{k\le N} 2^{(k+1)q} \int_{x_k}^{x_{k+1}} uV^q$$

$$\le 4^q \sum_{k\le N} \left(\int_{x_{k-1}}^{x_k} f^p v\right)^{q/p} \left(\int_{x_{k-1}}^{x_k} v^{-1/(p-1)}\right)^{q/p'} \int_{x_k}^{x_{k+1}} uV^q$$

$$\le 4^q \left(\int_0^\infty f^p v\right)^{q/p} \left(\sum_{k\le N} V^{r/p'}(x_k)\left(\int_{x_k}^{x_{k+1}} uV^q\right)^{r/q}\right)^{q/r}$$

$$\le \left(\frac{r}{q}\right)^{q/r} 4^q \left(\int_0^\infty f^p v\right)^{q/p} \left(\int_0^\infty \left(\int_0^x uV^q\right)^{r/p} u(x)V^{q+r/p'}(x)\,\mathrm{d}x\right)^{q/r}$$

$$\le \left(\frac{r}{q}\right)^{q/r} 4^q V^q(\infty) \left(\int_0^\infty f^p v\right)^{q/p} \mathcal{B}^q.$$

Therefore,

$$J_1^{1/q} \le \left(\frac{r}{q}\right)^{1/r} \left[q(p')^q \left(\frac{p}{r}\right)^{q/r} + 4^q\right]^{1/q} \mathcal{B} \left(\int_0^\infty f^p v\right)^{1/p}$$

and the upper bound in (24) is proved. For the lower bound we note that (17) with $f = v^{-1/(p-1)}$ brings

$$C \ge V^{-1/p}(\infty)\left(\int_0^\infty uV^q\right)^{1/q}.$$

Now, combining this with (25–27), we obtain the left hand side of (24). ■

The main result of this section is the following.

THEOREM 4 *Let* $0 < q < p < \infty$, $1/r = 1/q - 1/p$. *Then*

$$\|G\|_{L_v^p \to L_u^q} \approx \left(\int_0^\infty \left(\frac{1}{x} \int_0^x w \right)^{r/p} w(x)\, \mathrm{d}x \right)^{1/r}$$

with factors of equivalence depending on p and q only.

Proof It follows from (6), (10) and (15) that

$$\|G\|_{L_v^p \to L_u^q} \approx \|H\|_{L^s \to L_w^{sq/p}}^{s/p}, \quad s > 1$$

and from the case $V(\infty) = \infty$ of Theorem 3 we know that

$$\|H\|_{L^s \to L_w^{sq/p}}^{s/p} \approx \left(\int_0^\infty \left(\frac{1}{x} \int_0^x w \right)^{r/p} w(x)\, \mathrm{d}x \right)^{1/r} := \mathcal{B}_w$$

with factors depending on p, q and $s > 1$ but not w. More precisely,

$$\gamma_1(p, q) B_w \le \|G\|_{L_v^p \to L_u^q} \le \gamma_2(p, q) B_w,$$

where

$$\gamma_1(p, q) = 2^{-1/q} \left(\frac{q}{p} \right)^{1/p} \left(1 - \frac{q}{p} \right)^{-1/r} \sup_{s>1} s^{1/p} \left(\frac{s}{s-1} \right)^{s/p - 1/q} \left(1 - \frac{q}{p} \right)^{s/p},$$

$$\gamma_2(p, q) = \left(\frac{q}{p} \right)^{1/p} \inf_{s>1} s^{1/q} \left(\frac{s}{s-1} \right)^{s/p}. \qquad \blacksquare$$

5 CONCLUDING REMARKS

The results obtained in this paper can be formulated in a more general way. Here we just as an example study the operators

$$G_a f(x) = \exp\left(\frac{a}{x^a} \int_0^x t^{a-1} \ln |f(t)|\, \mathrm{d}t \right), \quad a > 0,$$

and put

$$u_a(x) = \frac{1}{a} x^{1/a-1} u(x^{1/a}), \qquad v_a = \frac{1}{a} x^{1/a-1} v(x^{1/a})$$

and

$$w_a = \left[G_a\left(\frac{1}{v_a}\right)\right]^{q/p} u_a.$$

Then Theorem 2 can (formally) be generalized as follows:

THEOREM 5 *Let* $0 < p \leq q < \infty$. *Then the inequality*

$$\left(\int_0^\infty (G_a f)^q u\right)^{1/q} \leq C_a \left(\int_0^\infty f^p v\right)^{1/p}, \quad f \geq 0 \tag{29}$$

is valid if and only if

$$D_a := \sup_{t>0} t^{-1/p} \left(\int_0^t w_a(x)\,\mathrm{d}x\right)^{1/q} < \infty$$

and

$$D_a \leq \|G_a\|_{L_v^p \to L_u^q} \leq e^{1/p} D_a.$$

Proof Note that

$$G_a f(x) = \frac{1}{x^a} \int_0^x \ln f(t)\,\mathrm{d}t^a,$$

make the variable transformation $y = t^a$ and after that $z = x^a$ in (29) and the result follows from Theorem 2.

In particular, by applying Theorem 5 with $v(t) = t^\beta$ and $u(t) = t^\alpha$ we obtain:

Example Let $\alpha, \beta \in \mathbb{R}$, $a > 0$ and $0 < p \leq q < \infty$. Then the inequality

$$\left(\int_0^\infty x^\alpha \left(\exp a x^{-a} \int_0^x x^{a-1} \ln f(t)\,\mathrm{d}t\right)^q\right)^{1/q} \leq C \left(\int_0^\infty (f(x))^p x^\beta\,\mathrm{d}x\right)^{1/p}$$

for some finite $C > 0$ iff

$$\left(\frac{1+\alpha}{a}\right)\frac{1}{q} - \left(\frac{1+\beta}{a}\right)\frac{1}{p} = 0.$$

Moreover

$$C \leq a^{1/q-1/p} e^{((1+\beta)/ap)} \left(\left(1 - \frac{1+\beta}{a}\right)\frac{q}{p} + \frac{(1+\alpha)}{a} \right)^{1/q}.$$

In particular, for the case $p = q = 1$, $\beta = \alpha$ we obtain the following well-known inequality by Cochran and Lee ([3], Theorem 1):

$$\int_0^\infty x^\alpha \exp\left(a x^{-a} \int_0^x x^{a-1} \ln f(t)\, dt \right) dx \leq e^{(\alpha+1)/a} \int_0^\infty x^\alpha f(x)\, dx.$$

c.f. also [4].

In the same way Theorem 4 can be generalized in the following way:

THEOREM 6 *Let* $0 < q < p < \infty$, $1/r = 1/q - 1/p$. *Then the inequality* (29) *holds for all* $f \geq 0$ *iff*

$$C_a = \left(\int_0^\infty \left(\frac{1}{x} \int_0^x w_a \right)^{r/p} w_a(x)\, dx \right)^{1/r} < \infty$$

and

$$\|G_a\|_{L_v^p \to L_u^q} \approx C_a.$$

Acknowledgements

Vladimir D. Stepanov was partially supported by the RFBR grant 00-01-00239.

References

[1] Bliss, G. A. (1930) "An integral inequality", *J. London Math. Soc.* **5**, 40–46.

[2] Carleman, T. (1923) "Sur les fonctions quasi-analytiques", Conferences faites au cinquieme congres des mathematiciens Scandinaves, Helsingfors, pp. 181–196.

[3] Cochran, J. A. and Lee, C.-S. (1984) "Inequalities related to Hardy's and Heinig's", *Math. Proc. Camb. Phil. Soc.* **96**, 1–7.

[4] Heinig, H. P. (1975) "Some extensions of Hardy's inequality", *SIAM J. Math. Anal.* **6**, 698–713.

[5] Heinig, H. P. (1990) "Weighted inequalities in Fourier analysis", In: Krbec, M., *et al.* (Eds.), *Nonlinear Analysis, Function Spaces and Applications*, Vol. 4. Proceedings of the Spring School at Roudnice nad Labem, 1990, Teubner Texte 119, Leipzig, pp. 42–85.
[6] Jain, P., Persson, L. E. and Wedestig, A. (2000) "From Hardy to Carleman and general mean-type inequalities", *Function Spaces and Applications* (Narosa Publ. House, New Delhi), pp. 117–130.
[7] Jain, P., Persson, L. E. and Wedestig, A. "Carleman–Knopp type inequalities via Hardy inequalities", *Math. Ineq. Appl.* 4 (2001), 343–355.
[8] Knopp, K. (1928) "Uber reihen mit positiven Gliedern", *J. London Math. Soc.* **3**, 205–211.
[9] Love, E. R. (1986) "Inequalities related to those of Hardy and of Cochran and Lee", *Math. Proc. Camb. Phil. Soc.* **99**, 395–408.
[10] Manakov, V. M. (1992) "On the best constant in weighted inequalities for Riemann–Liouville integrals", *Bull. London Math. Soc.* **24**, 442–448.
[11] Opic, B. and Gurka, P. (1994) "Weighted inequalities for geometric means", *Proc. Amer. Math. Soc.* **120**, 771–779.
[12] Pick, L. and Opic, B. (1994) "On geometric mean operator", *J. Math. Anal. Appl.* **183**, 652–662.
[13] Sinnamon, G. and Stepanov, V. D. (1996) "The weighted Hardy inequality: new proofs and the case $p = 1$", *J. London Math. Soc.* **54**(2), 89–101.
[14] Stepanov, V. D. (1994) "Weighted norm inequalities of Hardy type for a class of integral operators", *J. London Math. Soc.* **50**(2), 105–120.