J. of Inequal. & Appl., 2002, Vol. 7(1), pp. 61–77

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Lyapunov Inequalities for Time Scales

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(Received 10 April 2000; In final form 26 June 2000)

The theory of time scales has been introduced in order to unify discrete and continuous analysis. We present a Lyapunov inequality for Sturm-Liouville dynamic equations of second order on such time scales, which can be applied to obtain a disconjugacy criterion for these equations. We also extend the presented material to the case of a general linear Hamiltonian dynamic system on time scales. Some special cases of our results contain the classical Lyapunov inequalities for differential equations as well as only recently developed Lyapunov inequalities for difference equations.

Keywords and Phrases: Time scales; Lyapunov inequality; Dynamic equations; Disconjugacy; Hamiltonian system

1991 Mathematics Subject Classifications: 34A40, 39A10, 39A13, 26D15

1. INTRODUCTION

Lyapunov inequalities have proved to be useful tools in oscillation theory, disconjugacy, eigenvalue problems, and numerous other applications in the theory of differential and difference equations. A nice summary of continuous and discrete Lyapunov inequalities and their applications can be found in the survey paper [8] by Chen. In this paper we present several versions of Lyapunov inequalities that are

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ISSN: 1025-5834. Online ISSN: 1029-242X.
DOI: 10.1080/10755830290013407

valid on so-called time scales. The calculus of time scales has been introduced by Hilger [13] in order to unify discrete and continuous analysis. Hence our results presented cover (among other cases) both the continuous (see [8, Theorem 1.1] and also [20]) and discrete (see [8, Theorem 6.1] and also [1, 11.10.11], [10, Theorem 4.1]) Lyapunov inequalities. For convenience we now recall the following easiest versions of Lyapunov's inequality.

THEOREM 1.1 (Continuous Lyapunov Inequality) *Let* $p:[a,b]\to\mathbb{R}_+$ *be positive-valued and continuous. If the Sturm-Liouville (differential) equation*

$$\ddot{x}+p(t)x=0$$

has a nontrivial solution satisfying $x(a)=x(b)=0$, *then the Lyapunov inequality*

$$\int_a^b p(t)dt\geq\frac{4}{b-a}$$

holds.

THEOREM 1.2 (Discrete Lyapunov Inequality) *Let* $\{p_k\}_{0\leq k\leq N}\subset\mathbb{R}_+$ *be positive-valued. If the Sturm-Liouville difference equation*

$$\Delta^2 x_k+p_k x_{k+1}=0$$

has a nontrivial solution satisfying $x_0=x_N=0$, *then the Lyapunov inequality*

$$\sum_{k=0}^{N-1}p_k\geq\begin{cases}(2/m+1) & \text{if } N=2(m+1)\\ ((2m+1)/m(m+1)) & \text{if } N=2m+1\end{cases}$$

holds.

In this paper we prove a Lyapunov inequality that contains both Theorems 1.1 and 1.2 as special cases. It is valid for an arbitrary time scale, and it reads as follows.

THEOREM 1.3 (Dynamic Lyapunov Inequality) *Suppose* $\mathbb{T}$ *is a time scale and* $a,b\in\mathbb{T}$ *with* $a<b$. *Let* $p:\mathbb{T}\to\mathbb{R}_+$ *be positive-valued and rd-continuous. If the Sturm-Liouville dynamic equation*

$$x^{\Delta^2}+p(t)x^{\sigma}=0 \qquad \text{(SL)}$$

has a nontrivial solution x with $x(a)=x(b)=0$, then the Lyapunov inequality

$$\int_a^b p(t)\Delta t \geq \frac{b-a}{f(d)} \tag{1.1}$$

holds, where $f\colon \mathbb{T}\to\mathbb{R}$ is defined by $f(t)=(t-a)(b-t)$, and where $d\in\mathbb{T}$ is such that

$$\left|\frac{a+b}{2}-d\right| = min\left\{\left|\frac{a+b}{2}-s\right| : s\in[a,b]\cap\mathbb{T}\right\}.$$

To see how Theorems 1.1 and 1.2 follow as special cases from Theorem 1.3, it is at this point only important to know that

- $\mathbb{T}=\mathbb{R}$ corresponds to the continuous case, and $x^\sigma=x$, $x^\Delta=\dot{x}$, $\int_a^b f(t)\Delta t=\int_a^b f(t)dt$, and an rd-continuous function is the same as a continuous function in this case;
- $\mathbb{T}=\mathbb{Z}$ corresponds to the discrete case, and $x^\sigma(t)=x(t+1)$, $x^\Delta=x^\sigma-x$, $\int_a^b f(t)\Delta t=\sum_{t=a}^{b-1} f(t)$, and any function is rd-continuous in this case.

A short introduction to the time scales calculus is given in Section 2. In Section 3 we prove the above Theorem 1.3, and for the proof several lemmas on quadratic functionals connected to the Sturm-Liouville dynamic equation (SL) are needed. In the time scales calculus, the concept of a zero of a function is replaced by a so-called generalized zero, and (as in the classical case), a Lyapunov inequality leads immediately to disconjugacy criteria as presented in Section 3. Two extensions which we have not considered in this paper are the cases when p is not necessarily positive-valued and when the endpoints are not necessarily zeros but generalized zeros. Finally, in Section 4, we extend the theory to linear Hamiltonian dynamic systems of the form

$$x^\Delta = A(t)x^\sigma + B(t)u, \quad u^\Delta = -C(t)x^\sigma - A^*(t)u, \tag{H}$$

where A, B and C are square-matrix-valued functions satisfying the properties as given in Section 4 below. Such Hamiltonian systems contain in particular Sturm-Liouville equations of higher order, and in particular also equations (SL) as presented in Section 3. Several lemmas concerning certain quadratic functionals connected to the system (H) are needed, and a Lyapunov inequality for Hamiltonian systems (H) is presented, as well as a disconjugacy criterion as an immediate application of the inequality. We also consider so-called right-focal boundary conditions and offer a Lyapunov inequality for this case, too.

2. PRELIMINARIES ON TIME SCALES

In this section we briefly introduce the time scales calculus. For proofs and further explanations and results we refer to the papers by Hilger [6, 13, 14], to the book by Kaymakçalan, Lakshmikantham and Sivasundaram [18], and to the more recent papers [3, 4, 7, 12]. A *time scale* $\mathbb{T}$ is a closed subset of $\mathbb{R}$, and the (forward and backward) *jump operators* σ, $\rho: \mathbb{T} \to \mathbb{T}$ are defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}$$

(supplemented by $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$), while the *graininess* $\mu : \mathbb{T} \to \mathbb{R}_+$ is given by

$$\mu(t) = \sigma(t) - t.$$

For a function $f: \mathbb{T} \to \mathbb{R}$ we define the derivative f^{Δ} as follows: Let $t \in \mathbb{T}$. If there exists a number $\alpha \in \mathbb{R}$ such that for all $\varepsilon > 0$ there exists a neighborhood U of t with

$$|f(\sigma(t)) - f(s) - \alpha(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s| \quad \text{for all } s \in U,$$

then f is said to be (delta) *differentiable* at t, and we call α the derivative of f at t and denote it by $f^{\Delta}(t)$. Moreover, we denote $f^{\sigma} = f \circ \sigma$. The following formulas are useful:

- $f^{\sigma} = f + \mu f^{\Delta}$;
- $(fg)^{\Delta} = f^{\Delta} g + f^{\sigma} g^{\Delta}$ ("Product Rule");
- $(f/g)^{\Delta} = (f^{\Delta} g - f g^{\Delta})/(g g^{\sigma})$ ("Quotient Rule").

A function F with $F^{\Delta} = f$ is called an *antiderivative* of f, and then we define

$$\int_a^b f(t) \Delta t = F(b) - F(a),$$

where a, $b \in \mathbb{T}$. If a function is *rd-continuous* (*i.e.*, continuous in points t with $\sigma(t) = t$ and left-hand limit exists in points t with $\rho(t) = t$), then it possesses an antiderivative (see [6, Theorem 6]). We have that (see *e.g.*, [6, Theorem 7])

$$f(t) \geq 0, \quad a \leq t < b \text{ implies } \int_a^b f(t)\Delta t \geq 0.$$

Throughout this paper we assume a, $b \in \mathbb{T}$ with $a < b$. The two most popular cases of time scales are $\mathbb{T} = \mathbb{R}$, where $\int_a^b f(t)\Delta t = \int_a^b f(t)dt$, and $\mathbb{T} = \mathbb{Z}$, where $\int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t)$. Other examples of time scales (to which our inequalities apply as well) are *e.g.*

$$h\mathbb{Z} = \{hk : k \in \mathbb{Z}\} \quad \text{for some } h > 0,$$
$$q^{\mathbb{Z}} = \{q^k : k \in \mathbb{Z}\} \cup \{0\} \quad \text{for some } q > 1$$

(which produces so-called *q-difference equations*),

$$\mathbb{N}^2 = \{k^2 : k \in \mathbb{N}\}, \quad \left\{\sum_{k=1}^{n} \frac{1}{k} : n \in \mathbb{N}\right\}, \quad \bigcup_{k \in \mathbb{Z}} [2k, 2k+1],$$

or the Cantor set.

3. STURM-LIOUVILLE EQUATIONS

We let $\mathbb{T} \subset \mathbb{R}$ be any time scale, $p : \mathbb{T} \to \mathbb{R}$ be rd-continuous with $p(t) > 0$ for all $t \in \mathbb{T}$, and consider the Sturm-Liouville dynamic equation (SL) together with the quadratic functional

$$\mathcal{F}(x) = \int_a^b \{(x^\Delta)^2 - p(x^\sigma)^2\}(t)\Delta t.$$

Our first auxiliary result reads as follows.

Lemma 3.1 *If x solves (SL) and if $\mathcal{F}(y)$ is defined, then*

$$\mathcal{F}(y) - \mathcal{F}(x) = \mathcal{F}(y - x) + 2(y - x)(b)x^\Delta(b) - 2(y - x)(a)x^\Delta(a).$$

Proof Under the above assumptions we find

$$
\begin{aligned}
&\mathcal{F}(y) - \mathcal{F}(x) - \mathcal{F}(y-x)\\
&\quad = \int_a^b \{(y^\Delta)^2 - p(y^\sigma)^2 - (x^\Delta)^2 + p(x^\sigma)^2\\
&\qquad\qquad - (y^\Delta - x^\Delta)^2 + p(y^\sigma - x^\sigma)^2\}(t)\Delta t\\
&\quad = \int_a^b \{(y^\Delta)^2 - p(y^\sigma)^2 - (x^\Delta)^2 + p(x^\sigma)^2 - (y^\Delta)^2 + 2y^\Delta x^\Delta - (x^\Delta)^2\\
&\qquad\qquad + p(y^\sigma)^2 - 2py^\sigma x^\sigma + p(x^\sigma)^2\}(t)\Delta t\\
&\quad = 2\int_a^b \{y^\Delta x^\Delta - py^\sigma x^\sigma + p(x^\sigma)^2 - (x^\Delta)^2\}(t)\Delta t\\
&\quad = 2\int_a^b \{y^\Delta x^\Delta + y^\sigma x^{\Delta^2} - x^\sigma x^{\Delta^2} - (x^\Delta)^2\}(t)\Delta t\\
&\quad = 2\int_a^b \{yx^\Delta - xx^\Delta\}^\Delta \Delta t = 2\int_a^b \{(y-x)x^\Delta\}^\Delta \Delta t\\
&\quad = 2(y(b) - x(b))x^\Delta(b) - 2(y(a) - x(a))x^\Delta(a),
\end{aligned}
$$

where we have used the product rule from Section 2. ∎

LEMMA 3.2 *If $\mathcal{F}(y)$ is defined, then for any $r, s \in \mathbb{T}$ with $a \le r < s \le b$*

$$\int_r^s (y^\Delta(t))^2 \Delta t \ge \frac{(y(s) - y(r))^2}{s - r}.$$

Proof Under the above assumptions we define

$$x(t) = \frac{y(s) - y(r)}{s - r} t + \frac{sy(r) - ry(s)}{s - r}.$$

We then have

$$x(r) = y(r), \quad x(s) = y(s), \quad x^\Delta(t) = \frac{y(s) - y(r)}{s - r}, \quad \text{and} \quad x^{\Delta^2}(t) = 0.$$

Hence x solves the special Sturm-Liouville equation (SL) where $p = 0$ and therefore we may apply Lemma 3.1 to $\mathcal{F}_0$ defined by

$$\mathcal{F}_0(x) = \int_r^s (x^\Delta)^2(t)\Delta t$$

to find

$$\begin{aligned}\mathcal{F}_0(y) &= \mathcal{F}_0(x) + \mathcal{F}_0(y-x) + (y-x)(s)x^{\Delta}(s) - (y-x)(r)x^{\Delta}(r)\\ &= \mathcal{F}_0(x) + \mathcal{F}_0(y-x)\\ &\geq \mathcal{F}_0(x)\\ &= \int_r^s \left\{\frac{y(s)-y(r)}{s-r}\right\}^2 \Delta t\\ &= \frac{(y(s)-y(r))^2}{s-r},\end{aligned}$$

and this proves our claim. ■

Using the above Lemma 3.2, we now can prove one of the main results of this paper, Theorem 1.3 as stated as in Section 1.

Proof of Theorem 1.3 Suppose x is a solution of (SL) with $x(a)=x(b)=0$. But then we have from Lemma 3.1 (with $y=0$) that

$$\mathcal{F}(x) = \int_a^b \{(x^{\Delta})^2 - p(x^{\sigma})^2\}(t)\Delta t = 0.$$

Since x is nontrivial, we have that M defined by

$$M = \max\{x^2(t) : t \in [a,b] \cap \mathbb{T}\} \tag{3.1}$$

is positive. We now let $c \in [a,b]$ be such that $x^2(c) = M$. Applying the above as well as Lemma 3.2 twice (once with $r=a$ and $s=c$ and a second time with $r=c$ and $s=b$) we find

$$\begin{aligned}M\int_a^b p(t)\Delta t &\geq \int_a^b \{p(x^{\sigma})^2\}(t)\Delta t\\ &= \int_a^b (x^{\Delta})^2(t)\Delta t = \int_a^c (x^{\Delta})^2(t)\Delta t + \int_c^b (x^{\Delta})^2(t)\Delta t\\ &\geq \frac{(x(c)-x(a))^2}{c-a} + \frac{(x(b)-x(c))^2}{b-c}\\ &= x^2(c)\left\{\frac{1}{c-a} + \frac{1}{b-c}\right\}\\ &= M\frac{b-a}{f(c)} \geq M\frac{b-a}{f(d)},\end{aligned}$$

where the last inequality holds because of $f(d) = \max\{f(t): t \in [a,b] \cap \mathbb{T}\}$. Hence, dividing by $M > 0$ yields the desired inequality. ■

Example 3.3 Here we shortly wish to discuss the two popular cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$. We use the notation from the proof of Theorem 1.3.

(i) If $\mathbb{T} = \mathbb{R}$, then

$$\min\left\{\left|\frac{a+b}{2} - s\right| : s \in [a,b]\right\} = 0 \quad \text{so that } d = \frac{a+b}{2}.$$

Hence $f(d) = ((b-a)^2/4)$ and the Lyapunov inequality from Theorem 1.3 reads

$$\int_a^b p(t)dt \geq \frac{4}{b-a}.$$

(ii) If $\mathbb{T} = \mathbb{Z}$, then we consider two cases. First, if $a+b$ is even, then

$$\min\left\{\left|\frac{a+b}{2} - s\right| : s \in [a,b] \cap \mathbb{Z}\right\} = 0 \quad \text{so that } d = \frac{a+b}{2}.$$

Hence $f(d) = ((b-a)^2/4)$ and the Lyapunov inequality reads

$$\sum_{t=a}^{b-1} p(t) \geq \frac{4}{b-a}.$$

If $a+b$ is odd, then

$$\min\left\{\left|\frac{a+b}{2} - s\right| : s \in [a,b] \cap \mathbb{Z}\right\} = \frac{1}{2} \quad \text{so that } d = \frac{a+b-1}{2}.$$

This time we have $f(d) = ((b-a)^2 - 1/4)$ and the Lyapunov inequality reads

$$\sum_{t=a}^{b-1} p(t) \geq \frac{4}{b-a}\left\{\frac{1}{1-(1/(b-a)^2)}\right\}.$$

As an application of the above Theorem 1.3 we now prove a sufficient criterion for disconjugacy of (SL).

DEFINITION 3.4 Equation (SL) is called *disconjugate* on $[a,b]$ if the solution $\tilde{x}$ of (SL) with $\tilde{x}(a)=0$ and $\tilde{x}^{\Delta}(a)=1$ satisfies

$$\tilde{x}\tilde{x}^{\sigma}>0 \text{ on } (a,\rho(b)].$$

LEMMA 3.5 *Equation (SL) is disconjugate on* $[a,b]$ *if and only if*

$$\mathcal{F}(x)=\int_a^b\{(x^{\Delta})^2-p(x^{\sigma})^2\}(t)\Delta t>0$$

for all nontrivial x *with* $x(a)=x(b)=0$.

Proof This is a special case of [2, Theorem 5]. ■

THEOREM 3.6 (Sufficient Condition for Disconjugacy of (SL)) *If* p *satisfies*

$$\int_a^b p(t)\Delta t<\frac{b-a}{f(d)},\tag{3.2}$$

then (SL) is disconjugate on $[a,b]$.

Proof Suppose that (3.2) holds. For the sake of contradiction we assume that (SL) is not disconjugate. But then, by Lemma 3.5, there exists a nontrivial x with $x(a)=x(b)=0$ such that $\mathcal{F}(x)\le 0$. Using this x, we now define M by (3.1) to find

$$\begin{aligned}M\int_a^b p(t)\Delta t&\ge\int_a^b\{p(x^{\sigma})^2\}(t)\Delta t\\&\ge\int_a^b(x^{\Delta})^2(t)\Delta t\\&\ge\frac{M(b-a)}{f(d)},\end{aligned}$$

where the last inequality follows precisely as in the proof of Theorem 3.1. Hence, after dividing by $M>0$, we arrive at

$$\int_a^b p(t)\Delta t\ge\frac{b-a}{f(d)}$$

which contradicts (3.2) and hence completes the proof. ■

Remark 3.7 Note that in both conditions (1.1) and (3.2) we could

$$\text{replace } \frac{b-a}{f(d)} \text{ by } \frac{4}{b-a},$$

and Theorems 1.3 and 3.6 would remain true. This is because for $a \le c \le b$ we have

$$\frac{1}{c-a}+\frac{1}{b-c}=\frac{(a+b-2c)^2}{(b-a)(c-a)(b-c)}+\frac{4}{b-a}\ge\frac{4}{b-a}.$$

4. LINEAR HAMILTONIAN SYSTEMS

In this section we consider the linear Hamiltonian dynamic system (H), where A, B and C are rd-continuous $n \times n$-matrix-valued functions on $\mathbb{T}$ such that $I-\mu(t)A(t)$ is invertible and $B(t)$ and $C(t)$ are positive semidefinite for all $t \in \mathbb{T}$. For the continuous case of this theory we refer to [19] (in particular for Lyapunov inequalities [9]) while [5] is a good reference for the discrete case. A corresponding quadratic functional is given by

$$\mathcal{F}(x,u)=\int_a^b \{u^*Bu-(x^\sigma)^*Cx^\sigma\}(t)\Delta t.$$

A pair (x,u) is called *admissible* if it satisfies the equation of motion

$$x^\Delta = A(t)x^\sigma + B(t)u.$$

As in the previous section we start with the following auxiliary lemma.

Lemma 4.1 *If (x,u) solves (H) and if (y,v) is admissible, then*

$$\mathcal{F}(y,v)-\mathcal{F}(x,u)=\mathcal{F}(y-x,v-u) + 2\,Re[(y-x)^*(b)u(b)-(y-x)^*(a)u(a)].$$

Proof Under the above assumptions we calculate

$$
\begin{aligned}
&\mathcal{F}(y,v) - \mathcal{F}(x,u) - \mathcal{F}(y-x, v-u) \\
&= \int_a^b \{v^*Bv - (y^\sigma)^*Cy^\sigma - u^*Bu + (x^\sigma)^*Cx^\sigma \\
&\qquad - [(v-u)^*B(v-u) - (y^\sigma - x^\sigma)^*C(y^\sigma - x^\sigma)]\}(t)\Delta t \\
&= \int_a^b \{-2u^*Bu + v^*Bu + u^*Bv \\
&\qquad + 2(x^\sigma)^*Cx^\sigma - (y^\sigma)^*Cx^\sigma - (x^\sigma)^*Cy^\sigma\}(t)\Delta t \\
&= \int_a^b \{-2u^*Bu + 2\mathrm{Re}[u^*Bv] + 2(x^\sigma)^*Cx^\sigma - 2\mathrm{Re}[(y^\sigma)^*Cx^\sigma]\}(t)\Delta t \\
&= 2\mathrm{Re}\left(\int_a^b \{u^*(Bv - Bu) + [(x^\sigma)^* - (y^\sigma)^*]Cx^\sigma\}(t)\Delta t\right) \\
&= 2\mathrm{Re}\left(\int_a^b \{u^*(y^\Delta - Ay^\sigma - x^\Delta + Ax^\sigma)\right. \\
&\qquad \left. + [(x^\sigma)^* - (y^\sigma)^*][-u^\Delta - A^*u]\}(t)\Delta t\right) \\
&= 2\mathrm{Re}\left(\int_a^b \{u^*(y^\Delta - x^\Delta) + (y^\sigma - x^\sigma)^*u^\Delta\right. \\
&\qquad \left. + 2i\,\mathrm{Im}[u^*Ax^\sigma + (y^\sigma)^*A^*u]\}(t)\Delta t\right) \\
&= 2\mathrm{Re}\left(\int_a^b \{u^*(y^\Delta - x^\Delta) + (y^\sigma - x^\sigma)^*u^\Delta\}(t)\Delta t\right) \\
&= 2\mathrm{Re}\left(\int_a^b \{u^*(y^\Delta - x^\Delta) + (u^\Delta)^*(y^\sigma - x^\sigma)\}(t)\Delta t\right) \\
&= 2\mathrm{Re}\left(\int_a^b \{[u^*(y-x)]^\Delta\}(t)\Delta t\right) \\
&= 2\mathrm{Re}\{u^*(b)[y(b) - x(b)] - u^*(a)[y(a) - x(a)]\} \\
&= 2\mathrm{Re}\{[y-x]^*(b)u(b) - [y-x]^*(a)u(a)\},
\end{aligned}
$$

which is the conclusion we sought. ■

NOTATION 4.2 For the remainder of this section we denote by $W(\cdot, r)$ the unique (see [6, Section 6]) solution of the initial value problem

$$W^\Delta = -A^*(t)W, \quad W(r) = I,$$

where $r \in [a,b]$ is given. We also write

$$F(s,r) = \int_r^s W^*(t,r)B(t)W(t,r)\Delta t.$$

Observe that $W(t,r) \equiv I$ provided $A(t) \equiv 0$.

LEMMA 4.3 *Given are W and F as introduced in Notation* 4.2. *If (y,v) is admissible and if $r,\ s \in \mathbb{T}$ with $a \leq r < s \leq b$ such that $F(s,r)$ is invertible, then*

$$\int_r^s (v^*Bv)(t)\Delta t \geq [W^*(s,r)y(s) - y(r)]^* F^{-1}(s,r)[W^*(s,r)y(s) - y(r)].$$

Proof Under the above assumptions we define

$$x(t) = W^{*^{-1}}(t,r)\{y(r) + F(t,r)F^{-1}(s,r)[W^*(s,r)y(s) - y(r)]\}$$

and

$$u(t) = W(t,r)F^{-1}(s,r)[W^*(s,r)y(s) - y(r)].$$

Then we have

$$x(r) = y(r), \quad x(s) = y(s), \ u^\Delta(t) = -A^*(t)u(t),$$

and

$$\begin{aligned} x^\Delta(t) &= -W^{*^{-1}}(\sigma(t),r)(W^\Delta(t,r))^*x(t) + W^{*^{-1}}(\sigma(t),r)W^*(t,r)B(t)u(t) \\ &= W^{*^{-1}}(\sigma(t),r)W^*(t,r)A(t)x(t) + W^{*^{-1}}(\sigma(t),r)W^*(t,r)B(t)u(t) \\ &= [W(t,r)W^{-1}(\sigma(t),r)]^*[A(t)x(t) + B(t)u(t)]. \end{aligned}$$

But

$$\begin{aligned} W(t,r)W^{-1}(\sigma(t),r) &= [W(\sigma(t),r) - \mu(t)W^\Delta(t,r)]W^{-1}(\sigma(t),r) \\ &= I + \mu(t)A^*(t)W(t,r)W^{-1}(\sigma(t),r) \end{aligned}$$

and therefore $[I - \mu(t)A^*(t)]W(t,r)W^{-1}(\sigma(t),r) = I$ so that

$$[I - \mu(t)A(t)]x^\Delta(t) = A(t)x(t) + B(t)u(t)$$

and hence

$$\begin{aligned} x^{\Delta}(t) &= A(t)x(t) + \mu(t)A(t)x^{\Delta}(t) + B(t)u(t) \\ &= A(t)x^{\sigma}(t) + B(t)u(t). \end{aligned}$$

Thus (x, u) solves the special Hamiltonian system (H) where $C = 0$ and we may apply Lemma 4.1 to $\mathcal{F}_0$ defined by

$$\mathcal{F}_0(x, u) = \int_r^s (u^* Bu)(t)\Delta t$$

to obtain

$$\begin{aligned} \mathcal{F}_0(y, v) &= \mathcal{F}_0(x, u) + \mathcal{F}_0(y - x, v - u) \\ &\quad + 2\mathrm{Re}\{u^*(s)[y(s) - x(s)] - u^*(r)[y(r) - x(r)]\} \\ &= \mathcal{F}_0(x, u) + \mathcal{F}_0(y - x, v - u) \geq \mathcal{F}_0(x, u) \\ &= \int_r^s (u^* Bu)(t)\Delta t \\ &= [W^*(s, r)y(s) - y(r)]^* F^{-1}(r, s)[W^*(s, r)y(s) - y(r)]. \end{aligned}$$

which shows our claim. ■

Remark 4.4 The assumption in Lemma 4.3 that $F(s, r)$ is invertible if $r < s$ can be dropped in case B is positive definite rather than positive semidefinite.

As before in Section 3, we now may use Lemma 4.3 to derive a Lyapunov inequality for Hamiltonian systems.

THEOREM 4.5 [Lyapunov's Inequality] *Assume (H) has a solution (x, u) such that x is nontrivial and satisfies $x(a) = x(b) = 0$. With W and F introduced in Notation 4.2, suppose that $F(b, c)$ and $F(c, a)$ are invertible, where $\|x(c)\| = \max_{t \in [a,b] \cap \mathbb{T}} \|x(t)\|$. Let λ be the biggest eigenvalue of*

$$F = \int_a^b W^*(t, c)B(t)W(t, c)\Delta t,$$

and let $\nu(t)$ be the biggest eigenvalue of $C(t)$. Then the Lyapunov inequality

$$\int_a^b \nu(t)\Delta t \geq \frac{4}{\lambda}$$

holds.

Proof Suppose we are given a solution (x, u) *of* (H) such that $x(a) = x(b) = 0$. Lemma 4.1 then yields (using $y = v = 0$) that

$$\mathcal{F}(x,u) = \int_a^b \{u^* Bu - (x^\sigma)^* Cx^\sigma\}(t)\Delta t = 0.$$

So we apply Lemma 4.3 twice (once with $r = a$ and $s = c$ and a second time with $r = c$ and $s = b$) to obtain

$$\begin{aligned}\int_a^b [(x^\sigma)^* Cx^\sigma](t)\Delta t &= \int_a^b (u^* Bu)(t)\Delta t \\ &= \int_a^c (u^* Bu)(t)\Delta t + \int_c^b (u^* Bu)(t)\Delta t \\ &\geq x^*(c)W(c,a)F^{-1}(c,a)W^*(c,a)x(c) \\ &\quad + x^*(c)F^{-1}(b,c)x(c) \\ &= x^*(c)[F^{-1}(b,c) - F^{-1}(a,c)]x(c) \\ &\geq 4x^*(c)F^{-1}x(c).\end{aligned}$$

Here we have used the relation $W(t,r)W(r,s) = W(t,s)$ (see [6, Theorem 9 (i)]) as well as the inequality $M^{-1} + N^{-1} \geq 4(M+N)^{-1}$ (see [11, Lemma 11, page 63] or [21]). Now, by applying the Rayleigh-Ritz theorem [17, page 176] we conclude

$$\begin{aligned}\int_a^b \nu(t)\Delta t &\geq \int_a^b \nu(t)\frac{\|x^\sigma(t)\|^2}{\|x(c)\|^2}\Delta t \\ &= \frac{1}{\|x(c)\|^2}\int_a^b \nu(t)(x^\sigma(t))^* x^\sigma(t)\Delta t \\ &\geq \frac{1}{\|x(c)\|^2}\int_a^b (x^\sigma(t))^* C(t)x^\sigma(t)\Delta t \\ &\geq \frac{1}{\|x(c)\|^2}4x^*(c)F^{-1}x(c) \\ &\geq 4\min_{x\neq 0}\frac{x^* F^{-1}x}{x^* x} \\ &= \frac{4}{\lambda},\end{aligned}$$

and this finishes the proof. ■

Remark 4.6 If $A \equiv 0$, then $W \equiv I$ and $F = \int_a^b B(t)\Delta t$. If, in addition $B \equiv 1$, then $F = b - a$. Note how the Lyapunov inequality $\int_a^b \nu(t)\Delta t \geq (4/\lambda)$ reduces to $\int_a^b p(t)\Delta t \geq (4/b - a)$ for the scalar case as discussed in Section 3.

It is possible to provide a slightly better bound than the one given in Theorem 4.5, similarly as in Theorem 1.3, but we shall not do so here. Without introducing the notion of disconjugacy for systems (H) we now state the following corollary of Theorem 4.5 whose proof is similar to the one of Theorem 3.6. For the definition of disconjugacy and the result analoguous to Lemma 3.5 we refer to the recent work of Roman Hilscher [15, 16] (see also [2]).

THEOREM 4.7 [Sufficient Condition for Disconjugacy of (H)] *Using the notation from Theorem* 4.5, *if*

$$\int_a^b \nu(t)\Delta t < \frac{4}{\lambda},$$

then (H) is disconjugate on $[a, b]$.

We conclude this paper with a result concerning so-called *right-focal* boundary conditions, *i.e.*, $x(a) = u(b) = 0$.

THEOREM 4.8 *Assume (H) has a solution* (x, u) *with* x *nontrivial and* $x(a) = u(b) = 0$. *With the notation as in Theorem* 4.5, *the Lyapunov inequality*

$$\int_a^b \nu(t)\Delta t \geq \frac{1}{\lambda}$$

holds.

Proof Suppose (x, u) is a solution of (H) such that $x(a) = u(b) = 0$ with $a < b$. Choose the point c in $(a, b]$ where $\|x(t)\|$ is maximal. Apply Lemma 4.1 with $y = v = 0$ to see that $\mathcal{F}(x, u) = 0$. Therefore,

$$\int_a^b [(x^\sigma)^* C x^\sigma](t)\Delta t = \int_a^b (u^* Bu)(t)\Delta t \geq \int_a^c (u^* Bu)(t)\Delta t.$$

Using Lemma 4.3 with $r=a$ and $s=c$, we get

$$\begin{aligned}\int_a^c (u^*Bu)(t)\Delta t &\geq [W^*(c,a)x(c) - x(a)]^* F^{-1}(c,a)[W^*(c,a)x(c) - x(a)] \\ &= x^*(c)W(c,a)F^{-1}(c,a)W^*(c,a)x(c) \\ &= -x^*(c)F^{-1}(a,c)x(c) \\ &= x^*(c)\left(\int_a^c W^*(t,c)B(t)W(t,c)\Delta t\right)^{-1} x(c) \\ &\geq x^*(c)\left(\int_a^b W^*(t,c)B(t)W(t,c)\Delta t\right)^{-1} x(c) \\ &= x^*(c)F^{-1}x(c).\end{aligned}$$

Hence,

$$\int_a^b [(x^\sigma)^* C x^\sigma](t)\Delta t \geq x^*(c)F^{-1}x(c),$$

and the same arguments as in the proof of Theorem 4.5 lead us to our final conclusion. ■

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