

Dynamics of Inequalities in Geometric Function Theory

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A domain in the complex plane which is star-like with respect to a boundary point can be approximated by domains which are star-like with respect to interior points. This approximation process can be viewed dynamically as an evolution of the null points of the underlying holomorphic functions from the interior of the open unit disk towards a boundary point. We trace these dynamics analytically in terms of the Alexander–Nevanlinna and Robertson inequalities by using the framework of complex dynamical systems and hyperbolic monotonicity.

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The following two inequalities are well-known in geometric function theory:

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0, \quad z \in \Delta, \quad (1)$$

$$\operatorname{Re} \left\{ 2 \frac{zf'(z)}{f(z)} + \frac{1+z}{1-z} \right\} > 0, \quad z \in \Delta, \quad (2)$$

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where $\Delta \subset \mathbb{C}$ is the open unit disk in the complex plane and $f: \Delta \rightarrow \mathbb{C}$ is a univalent holomorphic function.

The first inequality is due to Nevanlinna [6] and Alexander [1]. It characterizes those univalent holomorphic functions which are star-like with respect to $f(0) = 0$. The second one was suggested by Robertson [9] as a characterization of those univalent holomorphic $f: \Delta \rightarrow \mathbb{C}$ with $f(0) = 1$ such that $f(\Delta)$ is star-like with respect to the boundary point $f(1) := \lim_{r \rightarrow 1^-} f(r) = 0$ and lies in the right half-plane. This characterization was partially proved by Robertson himself while his full conjecture was established by Lyzzaik [5]. A generalization of these results was later given by Silverman and Silvia [10].

A domain which is star-like with respect to a boundary point can be approximated by domains which are star-like with respect to interior points. This approximation process can be viewed dynamically as an evolution of the null points of the underlying functions from the interior towards a boundary point. So a natural question is how to trace these dynamics analytically in terms of the inequalities (1) and (2). In the present paper we answer this question in the framework of complex dynamical systems and hyperbolic monotonicity.

We begin by recalling two classical definitions.

DEFINITION 1 A set $\Omega \subset \mathbb{C}$ is called star-shaped if given any $\omega \in \Omega$, the point $t\omega$ belongs to Ω for every $t \in (0, 1]$.

DEFINITION 2 A univalent holomorphic function $f: \Delta \rightarrow \mathbb{C}$ is said to be star-like if $f(\Delta)$ is a star-shaped set. In addition, if $0 \in f(\Delta)$, then f is called star-like with respect to the interior point $f(a) = 0$, for some $a \in \Delta$. If $0 \in \partial f(\Delta)$, the boundary of $f(\Delta)$, then f is called star-like with respect to the boundary point 0.

Using Proposition 2.14 and Corollary 2.17 in [7], we see that if $f: \Delta \rightarrow \mathbb{C}$ is star-like with respect to the boundary point $0 \in \partial f(\Delta)$, then there is a unique point $\tau \in \partial \Delta$ such that $\lim_{r \rightarrow 1^-} f(r\tau) = 0$. Therefore in the sequel we will sometimes write that f is star-like with respect to the boundary point $f(\tau) := \lim_{r \rightarrow 1^-} f(r\tau) = 0$. We will also denote the set of all holomorphic functions $f: \Delta \rightarrow \mathbb{C}$ by $\text{Hol}(\Delta, \mathbb{C})$.

THEOREM 3 *Let $f: \Delta \rightarrow \mathbb{C}$ be a univalent holomorphic function on Δ . Then f is star-like if and only if it satisfies the following equation:*

$$f(z) = f'(z)(z - \tau)(1 - z\bar{\tau})p(z), \quad z \in \Delta, \quad (3)$$

where $\tau \in \bar{\Delta}$ and $p: \Delta \rightarrow \mathbb{C}$ is holomorphic with $\text{Re } p(z) \geq 0$ for all $z \in \Delta$. Thus, if $\tau \in \Delta$, then f is star-like with respect to $f(\tau) = 0$, and if $\tau \in \partial\Delta$, then f is star-like with respect to the boundary point $f(\tau) = 0$.

Remark 1 Equation (3) can be rewritten in the form

$$\text{Re} \frac{f'(z)(z - \tau)(1 - z\bar{\tau})}{f(z)} \geq 0, \quad z \in \Delta. \tag{3'}$$

In addition, if $\tau \in \Delta$, then differentiating (3) at $z = \tau$ we get $p(\tau) = (1/(1 - |\tau|^2)) > 0$ which means that inequality (3') is actually strict.

Moreover, setting $\tau = 0$, we obtain Nevanlinna's condition for the star-likeness of f with respect to $f(0) = 0$ ((condition (1)). If $\tau \in \Delta$, $\tau \neq 0$, one can derive the condition obtained by Wald [11]. In Theorem 7 below we will show that for $\tau = 1$ Eq. (3) (hence inequality (3')) is equivalent to a generalized form of Robertson's inequality (2).

As a matter of fact, we will show in the sequel that Theorem 3 is a consequence of the following connections between geometric function theory, complex evolution equations, and the concept of hyperbolic monotonicity for holomorphic functions.

To explain our approach we recall that a mapping $f: D \rightarrow \mathbb{C}$, $D \subset \mathbb{C}$, is said to be monotone if for each pair of points $z, w \in D$ we have

$$\text{Re}[(f(z) - f(w))(\bar{z} - \bar{w})] \geq 0.$$

It is easy to see that this inequality is equivalent to the following condition:

$$|z + rf(z) - (w + rf(w))| \geq |z - w|$$

for all $r \geq 0$ and $z, w \in D$. This condition motivates our next definition.

DEFINITION 4 Let f be a mapping from Δ into the complex plane \mathbb{C} and let ρ be the Poincaré metric on Δ . The mapping f is called ρ -monotone (*i.e.*, monotone with respect to the metric ρ) if for each pair of points $z, w \in \Delta$,

$$\rho(z + rf(z), w + rf(w)) \geq \rho(z, w)$$

for all $r \geq 0$ such that the points $z + rf(z)$ and $w + rf(w)$ belong to Δ .

The notion of ρ -monotonicity (in Hilbert space) was introduced in Section 2 of [8]. The following proposition is a special case of the results and proofs there.

PROPOSITION 5 *Let $g: \Delta \rightarrow \mathbb{C}$ be a holomorphic function on Δ . The following are equivalent:*

- (i) g is ρ -monotone;
- (ii) for each $r > 0$, the function $J_r = (I + rg)^{-1}$ is well-defined on Δ and is a holomorphic, hence ρ -nonexpansive, self-mapping of Δ , i.e.,

$$\rho(J_r(z), J_r(w)) \leq \rho(z, w)$$

for all $z, w \in \Delta$;

- (iii) for each $z \in \Delta$, the Cauchy problem

$$\begin{cases} (\partial u(t, z)/\partial t) + g(u(t, z)) = 0 \\ u(0, z) = z \end{cases} \tag{4}$$

has a unique solution $\{u(t, z)\} \subset \Delta$ for all $t \geq 0$ and the family $\{S(t) := u(t, \cdot)\}$, $t \geq 0$, consists of holomorphic (hence ρ -nonexpansive) self-mappings of Δ ;

- (iv) for each pair of points $z, w \in \Delta$, the following inequality holds:

$$\operatorname{Re} \left[\frac{g(z)\bar{z}}{1 - |z|^2} + \frac{g(w)\bar{w}}{1 - |w|^2} \right] \geq \operatorname{Re} \frac{\bar{z}g(w) + w\overline{g(z)}}{1 - \bar{z}w}.$$

Remark 2 It can be shown by using (ii) and Banach’s fixed point theorem that if $G: \Delta \rightarrow \Delta$ is a ρ -nonexpansive self-mapping, then $I - G$ is ρ -monotone.

Remark 3 Condition (iv) implies that the set $\mathcal{G}(\Delta)$ of all ρ -monotone functions on Δ is a closed real cone with respect to the topology of pointwise convergence on Δ .

Now we are able to formulate a key result.

THEOREM 6 *Let f be a holomorphic univalent function on Δ . Then $f(\Delta)$ is star-shaped if and only if $(f')^{-1}f$ is ρ -monotone.*

Proof Suppose $f(\Delta)$ is star-shaped. Then for each $t \in [0, 1)$, the function $G_t = f^{-1}[(1 - t)f]$ is a well-defined holomorphic (hence

ρ -nonexpansive) self-mapping of Δ . It follows by Remarks 2 and 3 that

$$g_t = \frac{1}{t}(I - G_t)$$

is ρ -monotone for each $t \in (0, 1)$. Consequently,

$$\lim_{t \rightarrow 0^+} g_t = -\frac{d}{dt}f^{-1}[(1-t)f] \Big|_{t=0^+} = (f')^{-1}f$$

is also ρ -monotone.

Conversely, let $g = f/f'$ be ρ -monotone and let $u(t, \cdot)$ be the solution of the Cauchy problem defined in Proposition 5 (iii). Then the family of functions $\{\Psi(t, \cdot)\}$ defined by the formula

$$\Psi(t, w) = f(u(t, f^{-1}(w))), \quad t \geq 0,$$

consists of holomorphic self-mappings of the domain $\Omega = f(\Delta)$. At the same time, direct calculations show that $\Psi(\cdot, \cdot)$ is the solution of the Cauchy problem

$$\begin{cases} (\partial\Psi(t, w)/\partial t) + \Psi(t, w) = 0 \\ \Psi(0, w) = w \in \Omega. \end{cases}$$

Hence $\Psi(t, w) = e^{-t}w \in \Omega$ for all $t \geq 0$ and each $w \in \Omega$.

Proof of Theorem 3 Let f be a univalent star-like function on Δ . Define the holomorphic function g on Δ by

$$g(z) = [f'(z)]^{-1}f(z), \quad z \in \Delta. \tag{5}$$

Since f is univalent it follows that g has at most one null point in Δ . Suppose first that f is star-like with respect to an interior point, *i.e.*, there is $\tau \in \Delta$ such that $f(\tau) = 0$. Then we get by Theorem 6 that g is a ρ -monotone function with $g(\tau) = 0$ and $g'(\tau) = 1$. Setting $\tau = w$ in condition (iv) of Proposition 5 we get

$$\operatorname{Re} \left[\frac{g(z)\bar{z}}{1 - |z|^2} \right] \geq \operatorname{Re} \frac{g(z)\bar{\tau}}{1 - z\bar{\tau}}, \quad z \in \Delta. \tag{6}$$

Calculations show that (6) combined with (5) is equivalent to (3'). Hence f satisfies Eq. (3).

Now if f is a univalent star-like function with no null point in Δ , then g defined by (5) also has no null point in Δ . In this case, fix $\varepsilon > 0$ and consider the mapping $g_\varepsilon \in \text{Hol}(\Delta, \mathbb{C})$ defined by

$$g_\varepsilon(z) = \varepsilon z + g(z), \quad z \in \Delta.$$

It is clear that g_ε belongs to the cone $\mathcal{G}(\Delta)$, *i.e.*, g_ε is a ρ -monotone function. Also, it follows by Proposition 5 (condition (ii)) that for each $r \geq 0$ and $w \in \Delta$ the equation

$$z + rg(z) = w$$

has a unique solution $z = z_r(w) \in \Delta$. Setting $w = 0$ and $r = (1/\varepsilon)$ we get that $\tau_\varepsilon := z_{(1/\varepsilon)}(0)$ is the unique null point of g_ε , *i.e.*, $g_\varepsilon(\tau_\varepsilon) = 0$. Since g has no null point in Δ , we can apply Corollary 1.4 in [8] and conclude that there is a unimodular point $\tau \in \partial\Delta$ such that for each $w \in \Delta$ the net $\{z_r(w)\}$ converges to τ when r tends to infinity. In particular, we have that $\{\tau_\varepsilon\}$ tends to τ when $\varepsilon \rightarrow 0^+$.

Since g_ε satisfies inequality (6) with τ replaced by τ_ε and $\{g_\varepsilon\}$ converges to g as ε tends to 0^+ , we see that g also satisfies this inequality with $\tau = \lim_{\varepsilon \rightarrow 0^+} \tau_\varepsilon$.

Conversely, let f be a univalent holomorphic function on Δ which satisfies Eq. (3) with some $\tau \in \bar{\Delta}$ and $p \in \text{Hol}(\Delta, \mathbb{C})$ such that $\text{Re } p \geq 0$ everywhere. Again we first assume that $\tau \in \Delta$. Consider the function

$$f_1(z) = f(M_\tau(z))$$

where $M_\tau(z) = ((\tau - z)/(1 - \bar{\tau}z))$ is a Möbius transformation of the unit disk. It is sufficient, of course, to show that $f_1(\Delta)$ is star-shaped.

Since

$$f'(M_\tau(z)) = f'_1(z)[M'_\tau(z)]^{-1},$$

substituting $M_\tau(z)$ instead of z in Eq. (3) we get by calculations

$$f_1(z) = f'_1(z)zp_1(z),$$

where $p_1(z) = (1 - |\tau|^2)p(M_\tau(z))$ has a positive real part for all $z \in \Delta$ (see Remark 1). Hence f_1 is star-like by Nevanlinna's theorem (condition (1)).

Let now f satisfy Eq. (3) with some $\tau \in \partial\Delta$. Then it is sufficient to prove that the function g defined by (5),

$$g(z) = \frac{f(z)}{f'(z)} = (z - \tau)(1 - z\bar{\tau})p(z),$$

is ρ -monotone. To this end, we take a sequence $\{\tau_n\} \subset \Delta$ such that $\tau_n \rightarrow \tau$ as $n \rightarrow \infty$. Since the functions g_n defined by

$$g_n(z) := (z - \tau_n)(1 - z\bar{\tau}_n)p(z)$$

are ρ -monotone and the sequence $\{g_n\} \subset \mathcal{G}(\Delta)$ converges to g as $n \rightarrow \infty$, we conclude that g also belongs to $\mathcal{G}(\Delta)$.

The proof of Theorem 3 is complete.

Finally, we will concentrate on star-like functions with respect to a boundary point and show that conditions derived from the above assertions are equivalent to a generalized form of Robertson’s inequality (2). In addition, we will relate these conditions to some geometric considerations in the spirit of Silverman and Silvia [10].

Recall that a holomorphic function $s: \Delta \rightarrow \mathbb{C}$ is said to be star-like of order $\lambda \in [0, 1)$ if $s(0) = 0$, $s'(0) = 1$ and $\text{Re}(zs'(z)/s(z)) > \lambda$ for all $z \in \Delta$.

THEOREM 7 *Let $f: \Delta \rightarrow \mathbb{C}$ be holomorphic and let $\lambda \in [0, 1)$. If f is not a constant and $f(0) = 1$, then the following conditions are equivalent.*

- (i) $\text{Re} [(1/(1 - \lambda))(zf'(z)/f(z) + ((1 + z)/(1 - z)))] > 0$ for all $z \in \Delta$.
- (ii) There exists a star-like function $s: \Delta \rightarrow \mathbb{C}$ of order λ such that

$$zf(z) = (1 - z)^{2-2\lambda}s(z), \quad z \in \Delta.$$

- (iii) The function f is univalent and $f(\Delta)$ is star-like with respect to $f(1) = 0$ and lies in a wedge of angle $2\pi(1 - \lambda)$.
- (iv) The function f is univalent, the mapping $g: \Delta \rightarrow \mathbb{C}$ defined by $g(z) := f(z)/f'(z)$, $z \in \Delta$, is ρ -monotone,

$$\exists \lim_{z \rightarrow 1} \frac{g(z)}{z - 1} =: g'(1)$$

when z approaches 1 in any wedge of the form $W = \{z \in \Delta: (|\text{Im } z|/(1 - \text{Re } z)) < K\}$, and $\text{Re } g'(1) \geq (1/(2 - 2\lambda))$.

Proof We first establish the equivalence of (a) and (b).

(a) \Rightarrow (b). Assuming (a) holds, we define $s: \Delta \rightarrow \mathbb{C}$ by $s(z) := z(1 - z)^{2\lambda - 2}f(z)$.

It is easy to see that $s(0) = 0$ and $s'(0) = 1$. Moreover,

$$\frac{zs'(z)}{s(z)} - \lambda = (1 - \lambda) \left\{ \frac{1}{1 - \lambda} \frac{zf'(z)}{f(z)} + \frac{1 + z}{1 - z} \right\}$$

and therefore

$$\operatorname{Re} \left[\frac{zs'(z)}{s(z)} \right] > \lambda \quad \text{for all } z \in \Delta,$$

as claimed.

(b) \Rightarrow (a). If (b) holds, then $f(z) = (s(z)/z)(1 - z)^{2-2\lambda}$, and

$$\frac{1}{1 - \lambda} \frac{zf'(z)}{f(z)} + \frac{1 + z}{1 - z} = \frac{1}{1 - \lambda} \left(\frac{zs'(z)}{s(z)} - \lambda \right).$$

Before we continue, we note the following facts. The first fact follows from the Riesz–Herglotz representation of functions belonging to the class of Carathéodory.

We denote by G_λ the class of non-vanishing holomorphic functions $f: \Delta \rightarrow \mathbb{C}$ with $f(0) = 1$ which satisfy condition (a).

Fact 1 The set

$$\bigcup_{n=1}^{\infty} \left\{ \prod_{j=1}^n \left(\frac{1 - z}{1 - z\zeta_j} \right)^{\lambda_j} : \sum_{j=1}^n \lambda_j = 2 - 2\lambda, |\zeta_j| = 1 \right\}$$

is dense in G_λ in the topology of uniform convergence on compact subsets of Δ .

Fact 2 If $0 \leq \lambda, \mu < 1$, then $f \in G_\lambda$ if and only if

$$[f(z)]^{((1-\mu)/(1-\lambda))} \in G_\mu.$$

This fact follows from the observation that $s \in S^*(\lambda)$, the family of star-like functions of order λ , if and only if

$$z \left(\frac{s(z)}{z} \right)^{((1-\mu)/(1-\lambda))} \in S^*(\mu).$$

Fact 3 If $f \in G_\lambda \setminus \{1\}$, then $\lim_{z \rightarrow 1} f(z) = 0$ when z approaches 1 in any wedge of the form $W = \{z \in \Delta: (|\operatorname{Im} z|/(1 - \operatorname{Re} z)) < K\}$.

To see this, recall that Cochrane and MacGregor [2] showed that if $s \in S^*(\lambda)$ and $s(z) \neq z/(1-xz)$, where $|x|=1$, then

$$|s(z)| = O(1 - |z|)^{\delta-1}$$

for some $\delta > 0$.

If $f \in G_{1/2}$, and $f(z) = (1-z)s(z)/z$ with such an $s \in S^*(1/2)$, then it follows that $|f(z)| = (|1-z|)O(1 - |z|)^{\delta-1}$ as $z \rightarrow 1$. If z is confined to a wedge, then $((|1-z|)/(1-|z|)) < A$ and therefore $\lim_{z \rightarrow 1} f(z) = 0$, as claimed.

If $s(z) = z/(1-xz)$, then $f(z) = ((1-z)/(1-xz))$, where $|x|=1$, but $x \neq 1$. In this case it is clear that $\lim_{z \rightarrow 1} f(z) = 0$.

Finally, if $f \in G_\lambda$ with $\lambda \neq 1/2$, then we obtain the result by using Fact 2 with $\mu = 1/2$.

Fact 4 If $f \in G_\lambda \setminus \{1\}$, then the function $h(z) := -\log f(z)$, $-\log f(0) = 0$ is univalent and close-to-convex in Δ .

Since $z/(1-z)^2$ is a star-like function, this fact will be proved once we show that

$$\operatorname{Re} \left(- (1-z)^2 \frac{f'(z)}{f(z)} \right) > 0, \quad z \in \Delta.$$

To this end, let $0 < \rho < 1$ and define $f_\rho: \Delta \rightarrow \mathbb{C}$ by

$$f_\rho(z) := f(\rho z) \left(\frac{1-z}{1-\rho z} \right)^{2(1-\lambda)}, \quad z \in \Delta.$$

If we use the corresponding function $s \in S^*(x)$, we can write equivalently that

$$f_\rho(z) = \left(\frac{s(\rho z)}{\rho} \right) \left(\frac{1}{z} \right) (1-z)^{2(1-\lambda)}.$$

This last representation of f_ρ shows that it belongs to G_λ . Its definition shows that $f_\rho \rightarrow f$ as $\rho \rightarrow 1^-$ and that f_ρ is continuous on the closed disk $\bar{\Delta}$. Therefore the claimed inequality will follow if we check it for f_ρ and for $z = e^{i\varphi} \in \partial\Delta$.

Indeed, for such z we have

$$\begin{aligned} \operatorname{Re}\left(- (1-z)^2 \frac{f'_\rho(z)}{f_\rho(z)}\right) &= \operatorname{Re}\left[-\frac{(1-z)^2}{z}\left(\frac{zf'_\rho(z)}{f_\rho(z)} + (1-\lambda)\frac{1+z}{1-z}\right)\right. \\ &\quad \left. + \frac{(1-z)^2}{z}(1-\lambda)\frac{1+z}{1-z}\right] \\ &= \operatorname{Re}\left[(2-z-\bar{z})\left(\frac{zf'_\rho(z)}{f_\rho(z)} + (1-\lambda)\frac{1+z}{1-z}\right)\right. \\ &\quad \left.+ (1-\lambda)(\bar{z}-z)\right] \\ &= 2(1-\cos\varphi)\operatorname{Re}\left[\frac{zf'_\rho(z)}{f_\rho(z)} + (1-\lambda)\frac{1+z}{1-z}\right] \geq 0, \end{aligned}$$

as claimed.

Fact 5 If $f \in G_\lambda \setminus \{1\}$, then $1-f$ is close-to-convex, hence univalent.

This is true because the function

$$h(z) := \frac{zf(z)}{(1-z)^2} = \frac{s(z)}{(1-z)^{2\lambda}}$$

is star-like and

$$\operatorname{Re}\left(\frac{-zf'(z)}{h(z)}\right) > 0 \quad \text{for all } z \in \Delta.$$

We now continue with the proof of Theorem 7.

(a) \Rightarrow (c). By Fact 1 we may assume that f is of the form

$$f(z) = \prod_{j=1}^n \left(\frac{1-z}{1-z\bar{\zeta}_j}\right)^{\lambda_j},$$

where $|\zeta_j| = 1$, $\zeta_j \neq 1$ and $\sum_{j=1}^n \lambda_j = 2(1-\lambda)$. Each function $\omega_j(z) := (1-z)/(1-z\bar{\zeta}_j)$ maps the open unit disk Δ onto a half-plane. In other words, $\operatorname{Re}(e^{i\beta_j}\omega_j(z)) > 0$ for some β_j .

Denoting $\sum_{j=1}^n \lambda_j\beta_j$ by β , we have, for each $z \in \Delta$, $|\arg e^{i\beta}f(z)| = |\arg e^{i\beta} \prod_{j=1}^n \omega_j^{\lambda_j}| = |\sum_{j=1}^n \lambda_j(\arg e^{i\beta_j}\omega_j)| < \sum_{j=1}^n \lambda_j(\pi/2) = \pi(1-\lambda)$. Hence $f(\Delta)$ is contained in a wedge of angle $2\pi(1-\lambda)$. To show that $f(\Delta)$ is star-shaped, we use Fact 4 to write $f(z) = f'(z)(-(1-z)^2p(z))$,

where $\operatorname{Re} p(z) > 0$ for all $z \in \Delta$. Theorem 3 now shows that $f(\Delta)$ is indeed star-shaped, as claimed.

(c) \Rightarrow (a). Let $f_0(z) = f(z)^{1/(1-\lambda)}$. Then $f_0(0) = 1$, $f_0(1) = 0$, f_0 is univalent and $f_0(\Delta)$ is star-shaped with respect to $f_0(1) = 0$. Set

$$D_n = f_0(\Delta) \cup \left\{ z \in \Delta: |z| < \frac{1}{n} \right\},$$

$n = 1, 2, \dots$, and for each n let $f_n: \Delta \rightarrow D_n$ be the conformal mapping of Δ onto D_n such that $f_n(0) = 1$ and $\arg f'_n(0) = \arg f'_0(0)$. By Carathéodory's kernel theorem we know that

$$\lim_{n \rightarrow \infty} f_n = f_0,$$

uniformly on each compact subset of Δ . Since each $f_n(\Delta)$ is star-shaped, there are star-like functions h_n with $h_n(0) = 0$ and numbers τ_n , $|\tau_n| < 1$, such that

$$f_n(z) = \frac{h_n(z)}{z} (z - \tau_n)(1 - \bar{\tau}_n z), \quad z \in \Delta,$$

(cf. [4] and [5]).

Note that $1 = f_n(0) = -\tau_n h'_n(0)$ and that

$$f'_n(0) = \frac{1}{2} h''_n(0)/h'_n(0) + h'_n(0)(1 + |\tau_n|^2)$$

for all n . If the sequence $\{h'_n(0)\}$ were unbounded, then we would reach a contradiction because $f'_n(0) \rightarrow f'_0(0)$ and $|h''_n(0)/h'_n(0)| \leq 4$.

Thus $\{h'_n(0)\}$ is bounded and we can extract a convergent subsequence of $\{h_n\}$. We can and will assume that the corresponding subsequence of $\{\tau_n\}$ converges to a point $\tau \in \bar{\Delta}$. Denoting the limit function of the convergent subsequence of $\{h_n\}$ by h , we see that

$$f_0(z) = \frac{h(z)}{z} (z - \tau)(1 - \bar{\tau}z), \quad z \in \Delta.$$

Letting z approach 1 we conclude that $\tau = 1$. Hence

$$f_0(z) = \left(-\frac{h(z)}{z} \right) (1 - z)^2$$

and

$$f(z) = \left(-\frac{h(z)}{z} \right)^{1-\lambda} (1-z)^{2-2\lambda},$$

where $h: \Delta \rightarrow \mathbb{C}$ is star-like with $h(0) = 0$. Since the function

$$s(z) := z \left(-\frac{h(z)}{z} \right)^{1-\lambda}$$

is star-like of order λ , we conclude by the equivalence (b) \Leftrightarrow (a) that $f \in G_\lambda$, as claimed.

(c) \Rightarrow (d). Let the smallest wedge in which $f(\Delta)$ lies be of angle $2\pi(1-\lambda_1)$. Then $\lambda_1 \geq \lambda$,

$$\operatorname{Re} \left[\frac{1}{1-\lambda_1} \frac{zf'(z)}{f(z)} + \frac{1+z}{1-z} \right] > 0, \quad z \in \Delta, \quad (7)$$

and this inequality no longer holds when λ_1 is replaced with any number $\lambda_1 < \lambda_2 < 1$. By the Riesz–Herglotz representation theorem we can write

$$\frac{1}{1-\lambda_1} \frac{zf'(z)}{f(z)} + \frac{1+z}{1-z} = \int_{|\zeta|=1} \frac{1+z\bar{\zeta}}{1-z\bar{\zeta}} d\mu(\zeta), \quad z \in \Delta, \quad (8)$$

where μ is a probability measure on the unit circle. After some calculations we get

$$f(z) = (1-z)^{2(1-\lambda_1)} \exp \left(-2(1-\lambda_1) \int_{|\zeta|=1} \log(1-z\bar{\zeta}) d\mu(\zeta) \right). \quad (9)$$

Again we note that (9) no longer holds when λ_1 is replaced with any number $\lambda_1 < \lambda_2 < 1$. Let δ denote the Dirac measure at $\zeta = 1 \in \partial\Delta$. Decomposing μ relative to δ , we can write $\mu = (1-a)\nu + a\delta$, where $0 \leq a \leq 1$, and ν and δ are mutually singular probability measures. It follows that

$$f(z) = (1-z)^{2(1-\lambda_2)} \exp \left(-2(1-\lambda_2) \int_{|\zeta|=1} \log(1-z\bar{\zeta}) d\nu(\zeta) \right),$$

where $\lambda_2 = 1 - (1-\lambda_1)(1-a)$.

If $a > 0$, then we reach a contradiction because $\lambda_2 > \lambda_1$. Thus $a = 0$ and $\mu = \nu$. Let $g = f/f'$. Then g is ρ -monotone by Theorem 6. Using (8) or (9) we see that

$$\frac{z - 1}{g(z)} = 2(1 - \lambda_1) \int_{|\zeta|=1} \frac{1 - \bar{\zeta}}{1 - z\bar{\zeta}} d\nu(\zeta), \quad z \in \Delta. \tag{10}$$

Let W be any wedge of the form

$$W = \left\{ z \in \Delta: \frac{|\operatorname{Im} z|}{1 - \operatorname{Re} z} < K \right\}$$

and let $\{z_n\}$ be any sequence in W which tends to 1. Consider the functions $f_n: \partial\Delta \rightarrow \mathbb{C}$, $n = 1, 2, \dots$, defined by

$$f_n(\zeta) := \frac{1 - \bar{\zeta}}{1 - z_n\bar{\zeta}}, \quad \zeta \in \partial\Delta.$$

The function f_n maps the unit circle $\partial\Delta$ onto the circle $|\xi - c_n| = |c_n|$, where

$$c_n = (1 - \bar{z}_n)/(1 - |z_n|^2), \quad n = 1, 2, \dots$$

Hence there is a natural number \mathcal{N} , independent of $\zeta \in \partial\Delta$, such that

$$\begin{aligned} |f_n(\zeta)|^2 &\leq 4|c_n|^2 = 4 \frac{1 + (\operatorname{Im} z_n/(1 - \operatorname{Re} z_n))^2}{[1 + \operatorname{Re} z_n - (\operatorname{Im} z_n) (\operatorname{Im} z_n/(1 - \operatorname{Re} z_n))]^2} \\ &\leq 2(1 + K^2) \end{aligned}$$

for all $n \geq \mathcal{N}$. Using (10) and applying Lebesgue's bounded convergence theorem we now obtain

$$\begin{aligned} \lim_{z \rightarrow 1, z \in W} \frac{z - 1}{g(z)} &= 2(1 - \lambda_1) \lim_{n \rightarrow \infty} \int_{|\zeta|=1} f_n(\zeta) d\nu(\zeta) \\ &= 2(1 - \lambda_1) \leq 2(1 - \lambda). \end{aligned}$$

In other words, condition (d) holds.

(d) \Rightarrow (c). First we note that by Proposition 5 and [3], $g'(1)$ is real and therefore

$$\lim_{z \rightarrow 1, z \in W} \frac{g(z)}{z - 1} \geq \frac{1}{2(1 - \lambda)}. \tag{11}$$

Moreover, $g(z) = f(z)/f'(z) = -(1-z)^2 p(z)$, where $p: \Delta \rightarrow \mathbb{C}$ is holomorphic with $\operatorname{Re} p(z) \geq 0$ for all $z \in \Delta$. Theorem 3 now implies that f is star-like with respect to the boundary point $f(1) = 0$. Let the smallest wedge in which $f(\Delta)$ lies be of angle $2\pi(1-\lambda_1)$, where $\lambda_1 \in [0, 1)$. As we saw in the proof of the implication (c) \Rightarrow (d), it follows that

$$\lim_{z \rightarrow 1, z \in W} \frac{g(z)}{z-1} = \frac{1}{2(1-\lambda_1)}.$$

Comparing the latter equality with (11), we see that $\lambda \leq \lambda_1$. Thus $f(\Delta)$ lies in a wedge of angle $2\pi(1-\lambda)$, as claimed.

This concludes the proof of Theorem 7.

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