*J. of Inequal. & Appl.*, 2001, Vol. 6, pp. 339–357
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# Maximum Principles and Uniqueness Results for Phi-Laplacian Boundary Value Problems

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*(Received 7 July 1999; Revised 20 December 1999)*

In this paper we derive maximum and comparison principles for second order differential operators, and apply them to prove uniqueness and comparison results for second order boundary value problems of generalized phi-Laplacian equations involving discontinuities.

*Keywords:* Phi-Laplacian; Maximum principle; Uniqueness; Discontinuity

*1991 Mathematics Subject Classification:* Primary 34A40, 34B15; Secondary 34C11, 34L30

## 1 INTRODUCTION

Recently, existence of solutions for boundary value problems of second order ordinary differential equations involving a phi-Laplacian operator have been studied extensively (cf. e.g. [1–4]). On the other hand, maximum or minimum principles and uniqueness have received less attention. In this paper we generalize some results of [4,8] and derive several others in a more general setting.

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First we derive maximum principles for a differential operator combined by a generalized phi-Laplacian operator and a part containing lower order terms, both of which may involve discontinuities. These maximum principles are then applied to prove uniqueness and comparison results for second order boundary value problems with separated, periodic, Neumann or Dirichlet boundary conditions. Some special cases including $p$-Laplacian problems are also considered. Examples and counter-examples are given to illustrate the obtained results.

## 2 MAXIMUM PRINCIPLES

Given intervals $I_0$ and $J=[t_0,t_1]$ and functions $\varphi: J\times I_0\to\mathbb{R}$ and $q: J\times\mathbb{R}^2\to\mathbb{R}$, we derive maximum principles for the differential operator $A$, defined by

$$\begin{aligned} Au(t) &:= -\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t,u'(t)) - q(t,u(t),u'(t)), \quad t\in J,\\ u\in Y &:= \{u\in C^1(J)|u'[J]\subseteq I_0 \text{ and } \varphi(\cdot,u'(\cdot))\in AC(J)\}. \end{aligned} \tag{2.1}$$

We assume that the functions $\varphi$ and $q$ satisfy either the hypotheses

($\varphi$1) to each choice of $s_1, s_2\in I_0$, $s_0<s_1$, there corresponds such an $M>0$ that $\varphi(t,y)-\varphi(t,z)\geq M(y-z)$ whenever $t\in J$ and $s_0\leq z<y\leq s_1$;

($q$1) $q(t,x,z)\leq q(t,y,z)$ for a.a. $t\in J$ and for all $x,y,z\in\mathbb{R}$, $x\geq y$;

($q$2) $|q(t,x,y)-q(t,x,z)|\leq p(t)\phi(|y-z|)$ for a.a. $t\in J$ and for all $x,y,z\in\mathbb{R}$, $0<|y-z|\leq r$, where $r>0$, $p\in L^1_+(J)$, the function $\phi:(0,r]\to(0,\infty)$ is increasing and $\int_{0+}^r(\mathrm{d}z/\phi(z))=\infty$;

or the hypotheses ($q$1) and

($\varphi$0) $\varphi(t,z)<\varphi(t,y)$ whenever $t\in J$, $y,z\in I_0$ and $z<y$.

($q$0) $|q(t,x,y)-q(t,x,z)|\leq p(t)\phi(|\varphi(t,y)-\varphi(t,z)|)$ for a.a. $t\in J$ and for all $x\in\mathbb{R}$, $y,z\in I_0$, $0<|\varphi(t,y)-\varphi(t,z)|\leq r$, where $r>0$, $p\in L^1_+(J)$, $\phi:(0,r]\to(0,\infty)$ is increasing and satisfies $\int_{0+}^r(\mathrm{d}z/\phi(z))=\infty$.

*Remark 2.1* A special feature of the above hypotheses is that the only continuity assumption for the functions $\varphi$ and $q$ is imposed on $q(t,x,\cdot)$ in condition ($q$2).

LEMMA 2.1 *Assume that functions* $q: J \times \mathbb{R}^2 \to \mathbb{R}$ *and* $\varphi: J \times I_0 \to \mathbb{R}$ *satisfy conditions* ($\varphi 1$), ($q1$) *and* ($q2$), *or conditions* ($\varphi 0$) ($q0$) *and* ($q1$). *If* $u, w \in Y$ *satisfy* $Au(t) \leq Aw(t)$ *a.e. in* $J$, *and if* $u - w$ *attains a positive maximum* $c$ *in the open interval* $(t_0, t_1)$, *then* $u(t) - w(t) \equiv c$ *on* $J$.

*Proof* Assume first that conditions ($\varphi 1$), ($q1$) and ($q2$) hold, and that $u - w$ attains a positive maximum $c$ at $t_2 \in (t_0, t_1)$. Let $r > 0$, $p \in L^1_+(J)$ and $\phi: (0, r] \to (0, \infty)$ be as in condition ($q2$). The proof is divided into two steps.

(i) Let $t_3$ be the greatest number on $(t_2, t_1]$ such that $u(t) \geq w(t)$ for each $t \in [t_2, t_3]$. To prove that $w'(t) \leq u'(t)$ for each $t \in [t_2, t_3]$, assume on the contrary: there is a subinterval $[a, b]$ of $[t_2, t_3]$ such that

$$0 < w'(t) - u'(t), \quad t \in (a, b], \ w'(a) - u'(a) = 0.$$

By condition ($\varphi 1$) there exists a $K > 0$ such that

$$w'(t) - u'(t) \leq K(\varphi(t, w'(t)) - \varphi(t, u'(t))) \tag{2.2}$$

for all $t \in [a, b]$. Denote $x(t) = K(\varphi(t, w'(t)) - \varphi(t, u'(t)))$, $t \in J$. Since $u, w \in Y$, then $x \in AC(J)$. Moreover, $x(a) = 0$, whence we may choose $b$ above so that $m = \max\{x(t) \mid t \in [a, b]\} \leq r$. Using the assumption $Au(t) \leq Aw(t)$ a.e. in $J$, we then have by (2.1), (2.2), ($q1$) and ($q2$),

$$\begin{aligned}
x'(t) &= K\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t, w'(t)) - K\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t, u'(t)) \\
&\leq K(q(t, u(t), u'(t)) - q(t, w(t), w'(t))) \\
&\leq K|q(t, w(t), u'(t)) - q(t, w(t), w'(t))| \\
&\leq Kp(t)\phi(|u'(t) - w'(t)|) = Kp(t)\phi(w'(t) - u'(t)) \\
&\leq Kp(t)\phi(K(\varphi(t, w'(t)) - \varphi(t, u'(t)))) = Kp(t)\phi(x(t))
\end{aligned}$$

for a.a. $t \in (a, b]$. Thus we have

$$x'(t) \leq Kp(t)\phi(x(t)) \ \text{a.e. in } (a, b], \quad x(a) = 0,$$

so that, by change of variables (cf. [6, 38.1]),

$$\int_{0+}^{x(b)} \frac{\mathrm{d}x}{\phi(x)} = \lim_{s \to a+} \int_s^b \frac{x'(t)\,\mathrm{d}t}{\phi(x(t))} \leq \int_a^b Kp(t)\,\mathrm{d}t < \infty.$$

This contradicts with the hypotheses given for $\phi$ in condition ($q2$). Consequently, $w'(t) \leq u'(t)$ on $[t_2, t_3]$, whence

$$u(t) - w(t) = u(t_2) - w(t_2) + \int_{t_2}^{t} (u'(s) - w'(s))\, \mathrm{d}s$$
$$\geq u(t_2) - w(t_2), \quad t \in [t_2, t_3].$$

Because $t_2$ was the maximum point of $u(t) - w(t)$, then $u(t) - w(t) \equiv c$ on $[t_2, t_3]$. This and the choice of $t_3$ imply that $t_3 = t_1$. Thus $u(t) - w(t) \equiv c$ in $[t_2, t_1]$.

(ii) Choose next $t_4$ to be the least number on $[t_0, t_2)$ such that $u(t) \geq w(t)$ for each $t \in [t_4, t_2]$. To prove that $u'(t) \leq w'(t)$ for each $t \in [t_4, t_2]$, assume on the contrary: there is a subinterval $[a, b]$ of $[t_4, t_2]$ such that

$$0 < u'(t) - w'(t), \ t \in [a, b), \quad u'(b) = w'(b).$$

Choose $K > 0$ and $b$ above so that

$$u'(t) - w'(t) \leq K(\varphi(t, u'(t)) - \varphi(t, w'(t))) \tag{2.3}$$

holds for all $t \in [a, b]$, and that the function $x(t) = K(\varphi(t, u'(t)) - \varphi(t, w'(t)))$, $t \in J$, satisfies $m = \max\{x(t) \mid t \in [a, b]\} \leq r$. Noticing that $Au(t) \leq Aw(t)$ a.e. in $J$ we obtain, by applying (2.1), (2.3), ($q1$) and ($q2$),

$$\begin{aligned}
-x'(t) &= K\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t, w'(t)) - K\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t, u'(t)) \\
&\leq K(q(t, u(t), u'(t)) - q(t, w(t), w'(t))) \\
&\leq K|q(t, w(t), u'(t)) - q(t, w(t), w'(t))| \\
&\leq Kp(t)\phi(|u'(t) - w'(t)|) = Kp(t)\phi(u'(t) - w'(t)) \\
&\leq Kp(t)\phi(K(\varphi(t, u'(t)) - \varphi(t, w'(t)))) = Kp(t)\phi(x(t))
\end{aligned}$$

for a.a. $t \in [a, b)$. Because $x(b) = 0$, we obtain

$$-x'(t) \leq Kp(t)\phi(x(t)) \ \text{a.e. in } [a, b), \quad x(b) = 0,$$

which implies a contradiction

$$\begin{aligned}
\infty &= \int_{0+}^{x(a)} \frac{\mathrm{d}x}{\phi(x)} = \int_{b-}^{a} \frac{x'(t)\,\mathrm{d}t}{\phi(x(t))} = \int_{a}^{b-} \frac{-x'(t)\,\mathrm{d}t}{\phi(x(t))} \\
&\leq \int_a^b Kp(t)\,\mathrm{d}t < \infty.
\end{aligned}$$

Thus $u'(t) \le w'(t)$ on $[t_4, t_2]$, whence

$$u(t_2) - w(t_2) = u(t) - w(t) + \int_t^{t_2} (u'(s) - w'(s))\,\mathrm{d}s$$
$$\le u(t) - w(t), \quad t \in [t_4, t_2].$$

Because $t_2$ was the maximum point of $u(t) - w(t)$, then $u(t) - w(t) \equiv c$ on $[t_4, t_2]$. This and the choice of $t_4$ imply that $t_4 = t_0$. Thus $u(t) - w(t) \equiv c$ on $[t_0, t_2]$.

The results of (i) and (ii) imply that $u(t) - w(t) \equiv c$ on $J$.

In the case when conditions ($\varphi 0$), ($q0$) and ($q1$) hold we do not need inequalities (2.2) and (2.3), and the proof is almost the same as above with $K = 1$.

Given $a_j, b_j \in \mathbb{R}_+, j = 0, 1$, and $u \in C^1(J)$, denote

$$B_0 u(t_0) = a_0\, u(t_0) - b_0\, u'(t_0), \quad B_1 u(t_1) = a_1\, u(t_1) + b_1\, u'(t_1). \tag{2.4}$$

As an application of Lemma 2.1 we obtain the following result.

LEMMA 2.2 *Let the hypotheses of Lemma* 2.1 *hold, and assume that functions* $u, w \in Y$ *satisfy inequalities*

$$Au(t) \le Aw(t) \text{ a.e. } in\ J, \quad B_j u(t_j) \le B_j w(t_j), j = 0, 1, \tag{2.5}$$

*where* $A, B_0$ *and* $B_1$ *are defined by* (2.1) *and* (2.4) *with* $a_j, b_j \in \mathbb{R}_+$ *and* $a_j + b_j > 0$, $j = 0, 1$. *If* $c = \max\{u(t) - w(t) \mid t \in J\}$ *is positive, then* $u(t) - w(t) \equiv c$ *on* $J$ *and* $a_0 = a_1 = 0$.

*Proof* Let $u(t) - w(t)$ attain its positive maximum $c$ at a point $t_2 \in J$. Assume first that $t_0 < t_2 < t_1$. It follows from Lemma 2.1 that $u(t) - w(t) \equiv c$ on $J$, so that

$$u(t_0) = w(t_0) + c, \text{ and } u'(t_0) = w'(t_0),$$
$$u(t_1) = w(t_1) + c \text{ and } u'(t_1) = w'(t_1).$$

Thus $B_j u(t_j) = B_j w(t_j) + a_j c$, $j = 0,1$, which imply by (2.5) that $a_j c \le 0$, $j = 0,1$, i.e. $a_0 = a_1 = 0$.

Assume next that the positive maximum $c$ of $u(t) - w(t)$ is attained at $t_0$. Then $u'(t_0) \le w'(t_0)$, so that

$$B_0 u(t_0) = a_0 u(t_0) - b_0 u'(t_0) \ge a_0(w(t_0) + c) - b_0 w'(t_0)$$
$$= B_0 w(t_0) + a_0 c.$$

In view of this result and (2.5) we have $a_0=0$ and $b_0u'(t_0)=b_0w'(t_0)$. Because $a_0+b_0>0$, then $b_0\neq 0$, whence $u'(t_0)=w'(t_0)$. Thus we can choose $t_2=t_0$ in part (i) of the proof of Lemma 2.1, which yields $u(t)-w(t)\equiv c$ on $J$, and hence $B_1u(t_1)=B_1w(t_1)+a_1c$. This and (2.5) imply that $a_1=0$.

In the case when $u(t)-w(t)$ is assumed to obtain its positive maximum at $t_1$ we have $u'(t_1)\geq w'(t_1)$. This and (2.5) imply that $a_1=0$ and $w'(t_1)=u'(t_1)$. Thus the choice $t_2=t_1$ in part (ii) of the proof of Lemma 2.1 yields $u(t)-w(t)\equiv c$ on $J$. Hence $B_0u(t_0)=B_0w(t_0)+a_0c$, so that $a_0=0$ by (2.5).

The proof of Lemma 2.2 contains also the proof of the following result.

LEMMA 2.3 *Let the hypotheses of Lemma* 2.1 *hold, and let* $u,w\in Y$ *satisfy inequalities*

$$Au(t)\leq Aw(t) \text{ a.e. } in\ J,\quad u'(t_0)\geq w'(t_0),\ u'(t_1)\leq w'(t_1), \tag{2.6}$$

*where $A$ is defined by* (2.1). *If* $c=\max\{u(t)-w(t)\mid t\in J\}$ *is positive, then* $u(t)-w(t)\equiv c$ *on $J$.*

## 3 COMPARISON AND UNIQUENESS RESULTS

The maximum principles derived in Section 2 will now be applied to prove comparison and uniqueness results for second order boundary value problems. The following consequence of Lemma 2.2 is used in the proofs.

PROPOSITION 3.1 *Assume that functions* $q:J\times\mathbb{R}^2\to\mathbb{R}$ *and* $\varphi:J\times I_0\to\mathbb{R}$ *satisfy conditions* ($\varphi 1$), ($q1$) *and* ($q2$) *or conditions* ($\varphi 0$), ($q0$) *and* ($q1$), *and that functions* $u,w\in Y$ *satisfy inequalities* (2.5). *Then* $u(t)\leq w(t)$ *for each* $t\in J$ *in the following cases*:

(a) $a_j,b_j\in\mathbb{R}_+,j=0,1$ *and* $a_0a_1+a_0b_1+a_1b_0>0$.

(b) $a_j,b_j\in\mathbb{R}_+,j=0,1$, $b_0b_1>0$, *and there is a nonnegligible subset* $\hat{J}$ *of* $J$ *such that* $q(t,x,z)<q(t,y,z)$ *for all* $t\in\hat{J}$ *and* $x,y,z\in\mathbb{R}$, $x>y$.

(c) $a_j,b_j\in\mathbb{R}_+$, $j=0,1$, $b_0b_1>0$, *and there is a linear functional* $Q:C(J)\to\mathbb{R}$, *satisfying* $Qv>0$ if $v(t)\equiv c>0$, *such that* $Qu\leq Qw$.

*Proof* Assume on the contrary that $c = \max\{u(t) - w(t) \mid t \in J\}$ is positive. In all the cases (a)–(c) the hypotheses of Lemma 2.2 are satisfied, whence $u(t) - w(t) \equiv c > 0$ and $a_0 = a_1 = 0$. But then $a_0 a_1 + a_0 b_1 + a_1 b_0 = 0$ which contradicts with the hypotheses of (a). Because $u(t) = w(t) + c$, $t \in J$, it follows from (2.1) that

$$Au(t) - Aw(t) = q(t, w(t), w'(t)) \\ - q(t, w(t) + c, w'(t)), \text{ a.e. in } J.$$

Since $c > 0$, then $Au(t) > Aw(t)$ a.e. in $\hat{J}$ by the hypotheses given in (b), which contradicts with (2.5). If $Q$ is as in (c), then $Q(u - w) = Qc > 0$, contradicting with $Qu \leq Qw$. Thus the given hypotheses do not allow that $u(t) = w(t) + c$, $c > 0$ for all $t \in J$, which concludes the proof.

*Remark 3.1* Linear functionals $Q : C(J) \to \mathbb{R}$, defined by

$$Qv = \int_J v(s)\,\mathrm{d}s \text{ and } Qv = v(\bar{t}) \quad (\bar{t} \in J \text{ is fixed})$$

satisfy condition $Qv > 0$ if $v(t) \equiv c > 0$, assumed in Proposition 3.1(c).

The results of Proposition 3.1 will now be applied to the differential equation

$$-\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t, u'(t)) = q(t, u(t), u'(t)) \text{ a.e. in } J, \tag{3.1}$$

associated with separated boundary conditions

$$a_0 u(t_0) - b_0 u'(t_0) = c_0, \quad a_1 u(t_1) + b_1 u'(t_1) = c_1. \tag{3.2}$$

We say that a function $u \in Y$ is a *lower solution* of (3.1), (3.2) if

$$\begin{aligned} &-\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t, u'(t)) \leq q(t, u(t), u'(t)) \text{ a.e. in } J, \\ &a_0 u(t_0) - b_0 u'(t_0) \leq c_0, \quad a_1 u(t_1) + b_1 u'(t_1) \leq c_1, \end{aligned} \tag{3.3}$$

and an *upper solution* if the reversed inequalities hold. If equalities hold in (3.3), we say that $u$ is a *solution* of (3.1), (3.2). As a consequence of Proposition 3.1(a) we get the following comparison and uniqueness results.

THEOREM 3.1 *Assume that functions* $q: J \times \mathbb{R}^2 \to \mathbb{R}$ and $\varphi: J \times I_0 \to \mathbb{R}$ *have properties* ($\varphi 1$), ($q1$) *and* ($q2$), *or properties* ($\varphi 0$), ($q0$) *and* ($q1$) *and that constants* $c_j \in \mathbb{R}$, $a_j, b_j \geq 0$, $j = 0,1$, *satisfy condition* $a_0 a_1 + a_0 b_1 + a_1 b_0 > 0$. *If* $u \in Y$ *is a lower solution and* $w \in Y$ *an upper solution of* (3.1), (3.2). *Then* $u(t) \leq w(t)$ *on* $J$. *In particular, the separated problem* (3.1), (3.2) *can have at most one solution.*

*Proof* If $u$ and $w$ are lower and upper solutions of (3.1), (3.2), then the inequalities (2.5) hold for the operators $A$, $B_0$ and $B_1$, defined by (2.1) and (2.4). Thus $u(t) \leq w(t)$ on $J$ by Proposition 3.1(a). The last conclusion follows from the first one.

A special case of problem (3.1), (3.2) where $a_0 = a_1 = 0$ and $b_0 b_1 > 0$ is not covered by Theorem 3.1. In this case boundary conditions (3.2) are reduced to the Neumann conditions

$$u'(t_0) = c_0, \quad u'(t_1) = c_1. \tag{3.4}$$

Condition ($q1$) will now be replaced by the following stronger condition:

($q1'$) There is a nonnegligible subset $\hat{J}$ of $J$ such that $q(t,x,z) < q(t,y,z)$ for all $t \in \hat{J}$ and $x, y, z \in \mathbb{R}$, $x > y$.

The proof of the following result is the same as that of Theorem 3.1 when we apply part (b) of Proposition 3.1, instead of part (a).

THEOREM 3.2 *Assume that functions* $q: J \times \mathbb{R}^2 \to \mathbb{R}$ *with and* $\varphi: J \times I_0 \to \mathbb{R}$ *have properties* ($\varphi 1$), ($q1'$) *and* ($q2$), *or properties* ($\varphi 0$), ($q0$) *and* ($q1$)$'$. *If* $u \in Y$ *is a lower solution and* $w \in Y$ *an upper solution of* (3.1), (3.4), *then* $u(t) \leq w(t)$ *on* $J$. *In particular, the Neumann problem* (3.1), (3.4) *can have at most one solution.*

Consider next Eq. (3.1), equipped with the periodic boundary conditions

$$u(t_0) = u(t_1), \quad u'(t_0) = u'(t_1). \tag{3.5}$$

We say that a function $u \in Y$ is a *lower solution* of (3.1), (3.5) if

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t, u'(t)) &\leq q(t, u(t), u'(t)) \ \text{ a.e. in } J, \\ u(t_0) = u(t_1), &\quad u'(t_0) \geq u'(t_1), \end{aligned} \tag{3.6}$$

an *upper solution* if the reversed inequalities hold in (3.6), and a *solution* of (3.1), (3.5) if equalities hold in (3.6). The following comparison and uniqueness results are consequences of Lemmas 2.1 and 2.3.

THEOREM 3.3 *Let* $q: J \times \mathbb{R}^2 \to \mathbb{R}$ *and* $\varphi: J \times I_0 \to \mathbb{R}$ *have properties* ($\varphi$1), ($q$1′) *and* ($q$2), *or properties* ($\varphi$0), ($q$0) *and* ($q$1′). *If* $u \in Y$ *is a lower solution and* $w \in Y$ *an upper solution of* (3.1), (3.5), *then* $u(t) \le w(t)$ *for each* $t \in J$. *In particular, the periodic problem* (3.1), (3.5) *can have at most one solution.*

*Proof* Assume on the contrary that $c = \max\{u(t) - w(t) \mid t \in J\}$ is positive. If $u(t_2) - w(t_2) = c$ for some $t_2 \in (t_0, t_1)$, then $u(t) - w(t) \equiv c$ by Lemma 2.1. Assume next that $u(t_0) - w(t_0) = c$. This and the definition of upper and lower solution of (3.1), (3.5) imply that $u(t_1) - w(t_1) = c$. Thus $u - w$ attains its positive maximum at $t_0$ and $t_1$, whence $u'(t_0) \le w'(t_0)$ and $u'(t_1) \ge w'(t_1)$. These inequalities and the definition of upper and lower solutions of (3.1), (3.5) imply that

$$u'(t_1) \le u'(t_0) \le w'(t_0) \le w'(t_1), \quad \text{and}$$
$$u'(t_0) \ge u'(t_1) \ge w'(t_1) \ge w'(t_0).$$

But then $u(t) - w(t) \equiv c$ by Lemma 2.3.

The above proof shows that $u(t) - w(t) \equiv c > 0$, which leads to contradiction with ($q$1′) (cf. the proof of Proposition 3.1(b). This concludes the proof.

In the case of Dirichlet boundary conditions

$$u(t_0) = c_0, \quad u(t_1) = c_1, \tag{3.7}$$

instead of the two-sided Osgood condition ($q$0) we will require only the following one-sided condition:

(qa) $q(t,x,z) - q(t,x,y) \le h(t, \varphi(t,y) - \varphi(t,z))$ for a.a. $t \in J$ and for all $x \in \mathbb{R}$, $y, z \in I_0$, $y > z$, $0 < \varphi(t,y) - \varphi(t,z) \le r$, where $r > 0$, $h: J \times [0, r] \to \mathbb{R}_+$, and $x(t) \equiv 0$ is the only function in $AC(J)$ which satisfies

$$x'(t) \le h(t, x(t)) \text{ a.e. in } J, \quad x(t_0) = 0.$$

THEOREM 3.4 *Given functions* $q : J \times \mathbb{R}^2 \to \mathbb{R}$ *and* $\varphi : J \times I_0 \to \mathbb{R}$ *having properties* ($\varphi$0), ($q$1) *and* ($q$a), *assume that* $u, w \in Y$ *satisfy*

$$Au(t) \leq Aw(t) \text{ a.e. } in\ J, \quad u(t_0) \leq w(t_0), \quad u(t_1) \leq w(t_1). \tag{3.8}$$

*Then* $u(t) \leq w(t)$ *for each* $t \in J$. *In particular, with the conditions given for* $q$ *and* $\varphi$, *the Dirichlet problem* (3.1), (3.7) *can have at most one solution.*

*Proof* If the first claim is wrong, then there exists a subinterval $[a, b]$ of $J$ such that $w'(a) - u'(a) = 0$, $w(t) - u(t) < 0$, $w'(t) - u'(t) > 0$ for all $t \in (a, b]$ (cf. [4]), and that the function

$$x(t) = \begin{cases} 0, & t_0 \leq t \leq a, \\ \varphi(t, w'(t)) - \varphi(t, u'(t)), & a \leq t \leq b, \\ x(b), & b \leq t \leq t_1, \end{cases}$$

satisfies $0 \leq x(t) \leq r$. Since $Au(t) \leq Aw(t)$ a.e. in $J$, we obtain (2.1), ($q$a) and ($q$1),

$$\begin{aligned} x'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\varphi(t, w'(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\varphi(t, u'(t)) \\ &\leq q(t, u(t), u'(t)) - q(t, w(t), w'(t)) \\ &\leq q(t, w(t), u'(t)) - q(t, w(t), w'(t)) \\ &\leq h(t, \varphi(t, w'(t)) - \varphi(t, u'(t))) = h(t, x(t)) \end{aligned}$$

for a.a. $t \in [a, b]$. Thus we have proved that

$$x'(t) \leq h(t, x(t)) \text{ a.e. in } [a, b].$$

This inequality holds also when $t \in J \setminus [a, b]$. Because $x(t_0) = 0$, then $x(t) \equiv 0$ on $J$, which contradicts with the fact that $x(t) = \varphi(t, w'(t)) - \varphi(t, u'(t)) > 0$ on $(a, t_1]$. This concludes the proof of the first assertion, and hence also the second one.

By assuming mixed monotonicity of $q$ in its last two arguments we can relax the strict monotonicity of $\varphi(t, \cdot)$, assumed in ($\varphi$0).

THEOREM 3.5 *The results of Theorem* 3.4 *are valid if* $\varphi : J \times \mathbb{R} \to I_0$ *and* $q : J \times \mathbb{R}^2 \to \mathbb{R}$ *satisfy the following hypotheses*:

($\varphi$) $\varphi(t, y) \leq \varphi(t, z)$ *whenever* $t \in J$, $y, z \in I_0$ *and* $y \leq z$;

($q$) $q(t, x, z) < q(t, y, s)$ *for* a.a. $t \in J$ *and for all* $x, y, z \in \mathbb{R}$, $x > y$ *and* $z < s$;

*Proof* Let $u, w \in Y$ satisfy (3.8), and assume on the contrary that $u(t) > w(t)$ for some $t \in J$. As in the proof of Theorem 3.4 we can choose a subinterval $[a, b]$ of $J$ such that $w'(a) - u'(a) = 0$, $w(t) - u(t) < 0$, $w'(t) - u'(t) > 0$ for all $t \in (a, b]$, and that $x(t) = \varphi(t, w'(t)) - \varphi(t, u'(t)) \leq r$, $a \leq t \leq b$. It then follows from (3.8), and $(q)$ that

$$\begin{aligned} x'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\varphi(t, w'(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\varphi(t, u'(t)) \\ &\leq q(t, u(t), u'(t)) - q(t, w(t), w'(t)) < 0 \end{aligned}$$

for a.a. $t \in [a, b]$. Because $x(a) = 0$, then $x(t) < 0$ on $(a, b]$, which contradicts with the fact that $x(t) = \varphi(t, w'(t)) - \varphi(t, u'(t)) \geq 0$ on $(a, b]$ by $(\varphi)$ This concludes the proof.

*Remarks 3.2* Theorem 3.4 generalizes Proposition 9 of [4] as follows:

- The function $\varphi$ depends also on $t$, and need not be measurable;
- One-sided Lipschitz condition is replaced by a more general one-sided condition;
- $q$ need not be a Caratheodory function since conditions $(q\mathrm{a})$ and $(q1)$ allow $q$ to be discontinuous in all variables;
- Monotonicity of $q$ in its second variable is not necessarily strict.

The following one-sided Osgood condition

$(q0')$ $q(t, x, z) - q(t, x, y) \leq p(t)\phi(\varphi(t, y) - \varphi(t, z))$ for a.a. $t \in J$ and for all $x \in \mathbb{R}$, $y, z \in I_0$, $y > z$, $0 < \varphi(t, y) - \varphi(t, z) \leq r$, where $r > 0$, $p \in L^1_+(J)$, $\phi : (0, r] \to (0, \infty)$ is increasing and satisfies $\int_{0+}^{r} (\mathrm{d}z/\phi(z)) = \infty$

is a special case of condition $(q\mathrm{a})$, as shown in [5]. The considerations of [5] imply also that $(q\mathrm{a})$ can be replaced by the following condition.

$(q\mathrm{b})$ $q(t, x, z) - q(t, x, y) \leq p(t)/P(t)(\varphi(t, y) - \varphi(t, z))$ for a.a. $t \in J$ and for all $x \in \mathbb{R}$, $y, z \in I_0$, $y > z$, $0 < \varphi(t, y)\varphi(t, z) \leq r$, where $r > 0$, $p \in L^1(J, \mathbb{R}_+)$ with $P(t) = \int_{t_0}^{t} p(s)\mathrm{d}s > 0$ for $t \in (t_0, t_1]$, and $\sup\{[q(t,\ x, y) - q(t, x, z)]_+ \mid 0 \leq y - z \leq r,\ x \in \mathbb{R}\} = o(p(t))$ as $t \to t_0+$, where $[a]_+ = \max\{a, 0\}$.

When $q$ is constant with respect to its last argument, the comparison results of Theorems 3.4 and 3.5 can be restated in the form which is similar to a comparison principle derived in [8, Section 3].

PROPOSITION 3.2 *Given $\varphi : J \times \mathbb{R} \to \mathbb{R}$ and $q : J \times \mathbb{R} \to \mathbb{R}$, assume that $\varphi(t, \cdot)$ is increasing and $q(t, \cdot)$ is decreasing, one of them being strict, for all $t \in J$. If $u, w \in Y$ satisfy*

$$-\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t, u(t)) - q(t, u(t)) \leq -\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t, w(t)) - q(t, w(t))$$
$$\text{a.e. } in\ J, \quad u(t_0) \leq w(t_0),\ u(t_1) \leq w(t_1),$$

*then $u(t) \leq w(t)$ on $J$.*

*Example 3.1* Denote by $[x]$ the greatest integer $\leq x$, and by $\chi_U$ the characteristic function of a subset $U$ of $J = [0,1]$. The periodic boundary value problem

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(u'(t) + [\chi_U(t)u'(t)]) &= [1 + 2t](u'(t) + [\chi_U(t)u'(t)]) \\ &\quad - u(t) + 1 \text{ a.e. in } J, \\ u(0) = u(1),\ u'(0) &= u'(1), \end{aligned} \tag{3.9}$$

is a special case of problem (3.1), (3.5) with

$$\begin{aligned} \varphi(t, x) &= x + [\chi_U(t)x],\ q(t, x, y) \\ &= [1 + 2t](y + [\chi_U(t)y]) - x + 1, \end{aligned}$$

$t \in J$, $x, y \in \mathbb{R}$. It is easy to see that the functions $\varphi$ and $q$ satisfy the hypotheses of Theorem 3.3. Thus problem (3.9) can have only one solution. Obviously, $u(t) \equiv 1$ is the solution of (3.9). Notice that the functions $\varphi$ and $q$ are discontinuous in all their variables, and even nonmeasurable in $t$ if $U$ is a nonmeasurable subset of $J$.

*Example 3.2* The BVP

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(u'(t) + [\chi_U(t)u'(t)]) &= u'(t) + [\chi_U(t)u'(t)] - [\chi_V(t)u(t)] \\ &\quad - \frac{1}{2} \text{ a.e. in } J = [0, 1], \\ u(0) = 0,\ u(1) &= \frac{1}{2}, \end{aligned} \tag{3.10}$$

where $U, V \subset J$, is a special case of problem (3.1), (3.5) with

$$\varphi(t,x) = x + [\chi_U(t)x],$$
$$q(t,x,y) = y + [\chi_U(t)y] - [\chi_V(t)x] - \tfrac{1}{2},$$

$t \in J$, $x, y \in \mathbb{R}$. It is easy to see that the hypotheses of Theorem 3.4 hold. Thus $u(t) = t/2$, $t \in J$, is the only solution of (3.10). Also in this example the functions $\varphi$ and $q$ are discontinuous in all their variables and may be nonmeasurable in $t$.

## 4 SPECIAL CASES

For each $n = 1, 2, \ldots$ the function

$$\phi_n(z) = z \ln \frac{1}{z} \cdots \ln_n \frac{1}{z}, \quad 0 < z \leq r_n = (\exp_n 1)^{-1},$$

satisfies the hypotheses given for $\phi$ in ($q$0), ($q$2) and ($q$0)′. This is also true when $\phi(z) = z$, $z \in \mathbb{R}_+$, in which case we get the following result.

COROLLARY 4.1 *The results of Lemmas* 2.1 *and* 2.2, *Proposition* 3.1 *and Theorems* 3.1–3.3 *also hold when condition* ($q$2) *is replaced by condition*

($q$3) *there is* $p \in L^1_+(J)$ *such that* $|q(t,x,y) - q(t,x,z)| \leq p(t)|y - z|$ *for a.a.* $t \in J$ *and for all* $x, y, z \in \mathbb{R}$,

*or when condition* ($q$0) *is replaced by*

($q$4) *there is* $p \in L^1_+(J)$ *such that for a.a.* $t \in J$ *and for all* $x \in \mathbb{R}$ *and* $y, z \in I_0$, $|q(t,x,y) - q(t,x,z)| \leq p(t)|\varphi(t,y) - \varphi(t,z)|$.

*The result of Theorem* 3.4 *holds if condition* ($qa$) *is replaced by*

($q$5) *there is* $p \in L^1_+(J)$ *such that* $q(t,x,z) - \mathrm{q}(t,x,y) \leq p(t)(\varphi(t,y) - \varphi(t,z))$ *for a.a.* $t \in J$ *and for all* $x \in \mathbb{R}$ *and* $y, z \in I_0$, $y < z$.

Consider next the special case when $\varphi$ does not depend on $t$, i.e. the *phi-Laplacian* case the differential equation (3.1) is reduced to

$$-\frac{\mathrm{d}}{\mathrm{d}t}\varphi(u'(t)) = q(t,u(t),u'(t)), \tag{4.1}$$

and the conditions ($\varphi 0$) and ($\varphi 1$) are reduced to conditions

($\varphi 0'$) $\varphi : I_0 \to \mathbb{R}$ is strictly increasing.
($\varphi 1'$) If $s_1, s_2 \in I_0$ and $s_1 < s_2$, there exists $M > 0$ such that $\varphi(y) - \varphi(z) \geq M(y - z)$ whenever $s_1 \leq z < y \leq s_2$.

Existence results for (4.1) under various boundary conditions including periodic, Dirichlet and Neuman conditions, are derived e.g, in [1,3,4], whereas in [2] existence results are derived for a more general looking differential equation

$$-\frac{\mathrm{d}}{\mathrm{d}t}\varphi(u'(t)) = \psi(u'(t))q(t, u(t), u'(t)), \tag{4.2}$$

under quite general boundary conditions. We are now looking for uniqueness and comparison results for Eq. (4.2), associated with one of the boundary conditions (3.2), (3.4), (3.5) and (3.7). By assuming that $\psi : \mathbb{R} \to (0,\infty)$ satisfies the following condition:

($\psi$) $\psi, \frac{1}{\psi} \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ and $\psi \circ \varphi^{-1} : I_0 \to \mathbb{R}$ is measurable;

we prove the following result.

PROPOSITION 4.1 *The comparison and uniqueness results of Theorems* 3.1–3.4 *hold, respectively, for the*

(a) *separated problem* (4.2), (3.2) *if conditions* ($\varphi 1'$), ($\psi$), ($q1$) *and* ($q2$) *or conditions* ($\varphi 0'$), ($\psi$), ($q1$) *and* ($q0$) *hold;*
(b) *Neumann problem* (4.2), (3.4) *and periodic problem* (4.2), (3.5) *if condition* ($q1$) *is replaced in* (a) *by condition* ($q1'$);
(c) *Dirichlet problem* (4.2), (3.7) *if conditions* ($\varphi 0'$), ($\psi$), ($q0'$) *and* ($q1$) *are valid.*

*Proof* Define $\bar{\varphi} : I_0 \to \mathbb{R}$ by (cf. [3])

$$\bar{\varphi}(x) = \int_0^{\varphi(x)} \frac{\mathrm{d}z}{\psi(\varphi^{-1}(z))}, \quad x \in I_0.$$

If $u \in Y$, then $u'[J] \subseteq I_0$ and $\varphi \circ u' \in AC(J)$. Condition ($\psi$) ensures that $1/(\psi \circ \varphi^{-1})$ is measurable and locally essentially bounded. Thus an application of [6, 38.3] yields

$$\bar{\varphi}(u'(t)) = \int_0^{\varphi(u'(t))} \frac{\mathrm{d}z}{\psi(\varphi^{-1}(z))} = \int_0^t \frac{\mathrm{d}/\mathrm{d}s(\varphi(u'(s)))\mathrm{d}s}{\psi(u'(s))}, \quad t \in J.$$

This implies that $\bar{\varphi} \circ u' \in AC(J)$, and that

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{\varphi}(u'(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{\mathrm{d}/\mathrm{d}s\varphi(u'(s))\mathrm{d}s}{\psi(u'(s))} = \frac{\mathrm{d}/\mathrm{d}t\varphi(u'(t))}{\psi(u'(t))} \text{ a.e. in } J.$$

Hence, if $u \in Y$ is a lower solution, an upper solution or solution of (4.2) with boundary conditions (3.2), (3.4), (3.5) or (3.7), then $u$ is a lower solution, an upper solution or solution of corresponding problems where the differential equation (4.2) is replaced by

$$-\frac{\mathrm{d}}{\mathrm{d}t}(\bar{\varphi}(u'(t))) = q(t, u(t), u'(t)) \text{ a.e. in } J. \tag{4.3}$$

It is easy to show that all the hypotheses given in Proposition 4.1 hold also when $\varphi$ is replaced by $\bar{\varphi}$. Thus the results of Theorems 3.1–3.4 hold for Eq. (4.3) with corresponding boundary conditions, which implies the assertions.

*Remark 4.1* The conditions ($\psi$) imposed on the function $\psi$ allow the right-hand side of Eq. (4.2) to have discontinuous dependence also on $u'$.

When $\varphi(t,x) = \mu(t)|x|^{p-2}x$, the differential equation (3.1) is reduced to equation containing a *p-Laplacian* operator, i.e.

$$-\frac{\mathrm{d}}{\mathrm{d}t}(\mu(t)|u'(t)|^{p-2}u'(t)) = q(t, u(t), u'(t)) \text{ a.e. in } J. \tag{4.4}$$

If $\mu : J \to [a,b]$, $0 < a < b < \infty$, condition ($\varphi 0$) holds for all $p > 1$, and condition ($\varphi 1$) holds when $1 < p \leq 2$. Thus we get the following corollary.

COROLLARY 4.2 *If* $\mu : J \to [a,b]$, $0 < a < b < \infty$, *then the comparison and uniqueness results of Theorems* 3.1–3.4 *hold, respectively, for the*

(a) *separated problem* (4.4), (3.2) *if* $p \in (1,2]$ *and conditions* ($q1$) *and* ($q2$) *hold, or if* $p > 1$ *and conditions* ($q0$) *and* ($q1$) *hold;*
(b) *Neumann problem* (4.4), (3.4) *and periodic problem* (4.4), (3.5) *if conditions* ($q0$) *and* ($q1'$) *are satisfied and* $p > 1$;
(c) *Dirichlet problem* (4.4), (3.7) *if conditions* ($qa$) *and* ($q1$) *hold and* $p > 1$.

When $\varphi(t,x) = (\mu(t)x)/\sqrt{1+x^2}$, the differential equation (3.1) can be restated as

$$-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mu(t)u'(t)}{\sqrt{1+u'(t)^2}} = q(t,u(t),u'(t)) \text{ a.e. in } J. \tag{4.5}$$

Condition ($\varphi$1) holds, whence we get the following result.

COROLLARY 4.3 *If $\mu : J \to [a,b], 0 < a < b < \infty$, then the comparison and uniqueness results of Theorems* 3.1–3.4 *hold, respectively, for the*

(a) *separated problem* (4.5), (3.2) *if conditions* ($q$1), *and* ($q$2) *or* ($q$0) *hold;*
(b) *Neumann problem* (4.5), (3.4) *and periodic problem* (4.5), (3.5) *if conditions* ($q$0) *and* ($q1'$) *hold;*
(c) *Dirichlet problem* (4.5), (3.7) *if conditions* ($qa$) *and* ($q$1) *are valid.*

In the case when $\varphi(t,x) = \mu(t)x$ we get the following consequence of Proposition 3.1 and Corollary 4.1.

COROLLARY 4.4 *Given $p, q, h \in L^1(J)$ and $\mu \in AC(J,(0,\infty))$, assume that $q(t) \leq 0$ a.e. in $J$, and denote*

$$Lu(t) = -\frac{\mathrm{d}}{\mathrm{d}t}(\mu(t)u'(t)) - p(t)u'(t) - q(t)u(t) - h(t),$$
$$t \in J, \quad u \in AC(J);$$

(a) *If $Lu(t) \leq 0$ a.e. in $J$, and if $u$ attains a positive maximum $c$ at an interior point of $J$, then $u(t) \equiv c$.*
(b) *If $Lu(t) \leq 0$ a.e. in $J$, if $B_ju(t_j) \leq 0, j = 0, 1$, if $c = \max\{u(t) | t \in J\} > 0$, and if $a_j, b_j \in \mathbb{R}_+$ and $a_j + b_j > 0, j = 0, 1$, then $u(t) \equiv c$ and $a_0 = a_1 = 0$.*
(c) *If $Lu(t) \leq 0$ a.e. in $J$, and if $B_ju(t_j) \leq 0, j = 0, 1$, then $u(t) \leq 0$ for each $t \in J$ in the following cases:*

(1) *$a_j, b_j \in \mathbb{R}_+, j = 0, 1$, and $a_0a_1 + a_0b_1 + a_1b_0 > 0$;*
(2) *$a_j, b_j \in \mathbb{R}_+, j = 0, 1$, $b_0b_1 > 0$, and $q$ is not equivalent to zero-function;*
(3) *$a_j, b_j \in \mathbb{R}_+$, $j = 0, 1$, $b_0b_1 > 0$, and there is a linear functional $Q : C(J) \to \mathbb{R}$, satisfying $Qv > 0$ if $v(t) \equiv c > 0$, such that $Qu \leq 0$.*

## 5 REMARKS, EXAMPLES AND COUNTER-EXAMPLES

In Lemma 2.1 and in Corollary 4.4(a) it suffices to assume that $p$ is locally Lebesgue integrable in $(t_0, t_1)$. For instance, if a solution $u$ of

$$-u''(t) = \frac{3}{t}u'(t), \quad t \in (0,1)$$

has a positive maximum $c$, then $u(t) \equiv c$. Moreover, all the results of Corollary 4.4 hold if

$$p(t) = \begin{cases} -\frac{3}{t^\alpha}, & t \neq 0, \\ 0, & t = 0, \end{cases} \quad \text{and } \alpha > 1.$$

Thus $p$ needs not to be bounded, as assumed, e.g., in [7]. On the other hand, the Dirichlet problem

$$\begin{aligned} &-u''(t) = p(t)u'(t), \quad t \in J = [-1,1], \ u(-1) = u(1) = 0, \\ &p(t) = \begin{cases} -\frac{3}{t}, & t \neq 0, \\ 0, & t = 0, \end{cases} \end{aligned}$$

has solutions $u(t) \equiv 0$ and $u(t) = 1 - t^4$. Thus the results of Lemmas 2.1 and 2.2, Proposition 3.1, Theorems 3.1 and 3.4 and Corollary 4.4(a) do not hold in general if $p \notin L^1(J)$. This is true also in the case when $\int_{0+}^{r}(\mathrm{d}x/\phi(x)) < \infty$ in conditions $(q2)$ and $(q0)'$, because the BVP

$$-u''(t) = 3(2u'(t))^{2/3}, \quad t \in J = [-1,1], \ u(-1) = u(1) = 0 \tag{5.1}$$

has also solutions $u(t) \equiv 0$ and $u(t) = 1 - t^4$. However, if we are interested in such solutions of (5.1) whose derivatives are nonzero, we can rewrite (5.1) in the following forms:

$$\begin{aligned} &-\frac{\mathrm{d}}{\mathrm{d}t}u'(t)^{1/3} = 2^{2/3}, \quad t \in J = [-1,1], \ u(-1) = u(1) = 0, \\ &-\frac{\mathrm{d}}{\mathrm{d}t}(|u'(t)|^{4/3} - 2u'(t)) = 2^{2/3}, \quad t \in J = [-1,1], \\ &u(-1) = u(1) = 0 \end{aligned}$$

The first one is in phi-Laplacian form with $\varphi(x)=x^{1/3}$, whereas the second one is in $p$-Laplacian form with $p=4/3$. In the former case the hypotheses of Proposition 4.1 hold, and in the latter case the hypotheses of Corollary 4.2 hold, whence these problems can have only one solution, which is $u(t)=1-t^4$.

The periodic boundary value problem

$$\frac{d}{dt}(u'(t)+[tu'(t)]) = [1+2t](u'(t)+[tu'(t)]) - [\sin(t)u(t)]$$
$$\text{a.e. in } J=[0,1], \quad u(0)=u(1),\ u'(0)=u'(1)$$

has a continuum of solutions of the form $x(t)\equiv c$, $c\in[0,1/\sin(1)]$. In this example hypotheses of Theorem 3.3 are not valid. On the other hand, the hypotheses of Theorem 3.4 hold for the BVP

$$\frac{d}{dt}(u'(t)+[tu'(t)]) = [1+2t](u'(t)+[tu'(t)]) - [\sin(t)u(t)]$$
$$\text{a.e. in } J=[0,1], \quad u(0)=c_0,\ u(1)=c_1,$$

whence this problem can have only one solution. In fact $u(t)\equiv c$ is the solution when $c_0=c_1=c\in[0,1/\sin(1)]$.

The Dirichlet problem

$$-u''(t)=\mu u(t), \quad t\in J=\left[0,\frac{\pi}{\sqrt{\mu}}\right],\ u(0)=u\left(\frac{\pi}{\sqrt{\mu}}\right)=0$$

has for each $\mu>0$ solutions $u(t)\equiv 0$ and $u(t)=\sin(\sqrt{\mu}t)$. Thus the results of Proposition 3.1 and Theorems 3.1 and 3.4 do not hold in general if condition $(q1)$ is not satisfied.

If $q,h\in L^1(J)$, and if $q(t)=0$ a.e. in $J$, then the Neumann problem

$$-u''(t)-q(t)u(t)=h(t) \text{ a.e. in } J,$$
$$u'(t_0)=1,\ u'(t_1)=-1$$

has solutions if and only if $\int_{t_0}^{t_1} h(t)\,dt=0$. In this case the solutions are of the form $u(t)=-\int_{t_0}^{t}\left(\int_{t_0}^{s}h(\tau)d\tau\right)ds+t+C$. This example shows that the result of Theorem 3.2 does not hold in general if condition $(q1')$ is replaced by $(q1)$, and that the results of Proposition 3.1(b) and (c) and Corollary 4.4. (c)-(2) and -(3) do not hold in general if we only assume that $a_j, b_j \in \mathbb{R}_+, j=0,1$, and $b_0 b_1 > 0$.

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