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On a Minimax Problem of Ricceri

GIUSEPPE CORDARO*

*Dipartimento di Matematica, Università di Messina,
98166 Sant'Agata – Messina, Italy*

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Let E be a real separable and reflexive Banach space, $X \subseteq E$ weakly closed and unbounded, Φ and Ψ two non-constant weakly sequentially lower semicontinuous functionals defined on X, such that $\Phi + \lambda\Psi$ is coercive for each $\lambda \geq 0$. In this setting, if

$$\sup_{\lambda \geq 0} \inf_{x \in X} (\Phi(x) + \lambda(\Psi(x) + \rho)) = \inf_{x \in X} \sup_{\lambda \geq 0} (\Phi(x) + \lambda(\Psi(x) + \rho))$$

for every $\rho \in \mathbf{R}$, then, one has

$$\sup_{\lambda \geq 0} \inf_{x \in X} (\Phi(x) + \lambda\Psi(x) + h(\lambda)) = \inf_{x \in X} \sup_{\lambda \geq 0} (\Phi(x) + \lambda\Psi(x) + h(\lambda)),$$

for every concave function $h : [0, +\infty[\to \mathbf{R}$.

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1. INTRODUCTION

Here and throughout the sequel, E is a real separable and reflexive Banach space, X is a weakly closed unbounded subset of E, and Φ, Ψ are two (non-constant) sequentially weakly lower semicontinuous

* E-mail: cordaro@dipmat.unime.it.

functionals on X such that

$$\lim_{x\in X,\|x\|\to+\infty}(\Phi(x)+\lambda\Psi(x))=+\infty$$

for all $\lambda\geq 0$.

In this setting, the importance of finding a continuous concave function $h:[0,+\infty[\to\mathbf{R}$ such that

$$\sup_{\lambda\geq 0}\inf_{x\in X}(\Phi(x)+\lambda\Psi(x)+h(\lambda))<\inf_{x\in X}\sup_{\lambda\geq 0}(\Phi(x)+\lambda\Psi(x)+h(\lambda)), \qquad (1)$$

has been clearly shown by Ricceri in a series of recent papers ([2–4]). Actually, if that happens, there is an open interval $I\subseteq\,]0,+\infty[$ such that, for each $\lambda\in I$, the functional $\Phi+\lambda\Psi$ has a local non-absolute minimum in the relative weak topology of X. In turn, under further appropriate assumptions, this fact leads to a three critical points theorem (Theorem 1 of [4] improving Theorem 3.1 of [3]) which is a new, useful tool to get multiplicity results for non-linear boundary value problems ([1,2,4]).

In [3], just in view of an application to the Dirichlet problem, Ricceri pointed out a natural way to get (1), with a linear h ([3], Proposition 3.1). At the same time, he asked ([3], Remark 5.2) whether it may happen that for a suitable continuous concave function h, (1) holds, while, for every $\rho\in\mathbf{R}$, one has

$$\sup_{\lambda\geq 0}\inf_{x\in X}(\Phi(x)+\lambda(\Psi(x)+\rho))=\inf_{x\in X}\sup_{\lambda\geq 0}(\Phi(x)+\lambda(\Psi(x)+\rho)).$$

The aim of this paper is to answer, in negative, Ricceri's question.

Our result is as follows.

THEOREM 1 *Under the assumptions above, the following assertions are equivalent:*

(i) *For every* $\rho\in\mathbf{R}$, *one has*

$$\sup_{\lambda\geq 0}\inf_{x\in X}(\Phi(x)+\lambda(\Psi(x)-\rho))=\inf_{x\in X}\sup_{\lambda\geq 0}(\Phi(x)+\lambda(\Psi(x)-\rho)).$$

(ii) *For every* $\rho \in]\inf_X \Psi, \sup_X \Psi[$, *one has*

$$\sup_{x\in\Psi^{-1}(]\rho,+\infty[)} \frac{\Phi(x) - \inf_{\Psi^{-1}(]-\infty,\rho])} \Phi}{\rho - \Psi(x)} \leq \inf_{x\in\Psi^{-1}(]-\infty,\rho[)} \frac{\Phi(x) - \inf_{\Psi^{-1}(]-\infty,\rho])} \Phi}{\rho - \Psi(x)}.$$

(iii) *For every concave function* $h : [0, +\infty[\to \mathbf{R}$, *one has*

$$\sup_{\lambda\geq 0} \inf_{x\in X} (\Phi(x) + \lambda\Psi(x) + h(\lambda)) = \inf_{x\in X} \sup_{\lambda\geq 0} (\Phi(x) + \lambda\Psi(x) + h(\lambda)).$$

2. PRELIMINARY LEMMAS

The proof of Theorem 1 needs some rather delicate lemmas. We prove them in this section. From now on, we denote l.s.c. as lower semicontinuous, u.s.c. as upper semicontinuous, $\chi_{[a,b]}$ the characteristic function of a real interval $[a, b]$.

Except for Lemmas 1 and 4, we also assume that the two functions Φ, Ψ satisfy (ii) of Theorem 1.

LEMMA 1 *Assume that* $\alpha, \beta \in \mathbf{R}_+$ *with* $\alpha < \beta$, *then the following assertions are equivalent:*

(*i1*) *For every* $\rho \in \mathbf{R}$, *one has*

$$\sup_{\lambda\in[\alpha,\beta]} \inf_{x\in X} (\Phi(x) + \lambda(\Psi(x) - \rho)) = \inf_{x\in X} \sup_{\lambda\in[\alpha,\beta]} (\Phi(x) + \lambda(\Psi(x) - \rho)).$$

(*ii1*) *For every* $\rho \in]\inf_X \Psi, \sup_X \Psi[$, *one of the following two pairs of inequalities holds:*

$$\sup_{x\in\Psi^{-1}(]\rho,+\infty[)} \frac{\Phi(x) - a(\rho)}{\rho - \Psi(x)} > \beta, \tag{2}$$

$$\sup_{x\in\Psi^{-1}(]\rho,+\infty[)} \frac{\Phi(x) - b(\rho)}{\rho - \Psi(x)} \leq \inf_{x\in\Psi^{-1}(]-\infty,\rho[)} \frac{\Phi(x) - b(\rho)}{\rho - \Psi(x)} \tag{3}$$

or

$$\sup_{x\in\Psi^{-1}(]\rho,+\infty[)} \frac{\Phi(x)-a(\rho)}{\rho-\Psi(x)} \leq \beta, \tag{4}$$

$$\sup_{x\in\Psi^{-1}(]\rho,+\infty[)} \frac{\Phi(x)-a(\rho)}{\rho-\Psi(x)} \leq \inf_{x\in\Psi^{-1}(]-\infty,\rho[)} \frac{\Phi(x)-a(\rho)}{\rho-\Psi(x)} \tag{5}$$

whereas

$$a(\rho) = \inf_{x\in\Psi^{-1}(]-\infty,\rho])} (\Phi(x) + \alpha(\Psi(x) - \rho))$$

and

$$b(\rho) = \inf_{x\in\Psi^{-1}(]\rho,+\infty[)} (\Phi(x) + \beta(\Psi(x) - \rho)).$$

Proof

(i1) $\Rightarrow$ *(ii1)*

Fix $\rho \in]\inf_X \Psi, \sup_X \Psi[$, we have

$$\inf_{x\in X} \sup_{\lambda\in[\alpha,\beta]} (\Phi(x) + \lambda(\Psi(x) - \rho)) = \min\{a(\rho), b(\rho)\}.$$

We prove that

$$b(\rho) < a(\rho) \ \Rightarrow \ (2) \text{ and } (3)$$
$$a(\rho) \leq b(\rho) \ \Rightarrow \ (4) \text{ and } (5).$$

Suppose $b(\rho) < a(\rho)$, then $\inf_{x\in X}\sup_{\lambda\in[\alpha,\beta]}(\Phi(x) + \lambda(\Psi(x) - \rho)) = b(\rho)$, moreover there exists $\bar{x} \in \Psi^{-1}(]\rho, +\infty[)$ such that $\Phi(\bar{x}) + \beta(\Psi(\bar{x}) - \rho) < a(\rho)$, then

$$\frac{\Phi(\bar{x}) - a(\rho)}{\rho - \Psi(\bar{x})} > \beta$$

that implies (2).

Since $\sup_{\lambda\in[\alpha,\beta]}\inf_{x\in X}(\Phi(x) + \lambda(\Psi(x) - \rho)) = b(\rho)$, there exists $\lambda_\rho \in [\alpha, \beta]$ such that

$$\inf_{x\in X}(\Phi(x) + \lambda_\rho(\Psi(x) - \rho)) = b(\rho),$$

thus we have

$$\sup_{x\in\Psi^{-1}(]\rho,+\infty[)} \frac{\Phi(x)-b(\rho)}{\rho-\Psi(x)} \leq \lambda_\rho \leq \inf_{x\in\Psi^{-1}(]-\infty,\rho[)} \frac{\Phi(x)-b(\rho)}{\rho-\Psi(x)},$$

that implies (3).

Suppose $a(\rho)\leq b(\rho)$, then $\inf_{x\in X}\sup_{\lambda\in[\alpha,\beta]}(\Phi(x)+\lambda(\Psi(x)-\rho))=a(\rho)$. The inequality (4) holds, in fact if it does not hold, we have

$$\sup_{x\in\Psi^{-1}(]\rho,+\infty[)} \frac{\Phi(x)-a(\rho)}{\rho-\Psi(x)} > \beta,$$

that implies $b(\rho)<a(\rho)$ against the hypothesis.

Since $\sup_{\lambda\in[\alpha,\beta]}\inf_{x\in X}(\Phi(x)+\lambda(\Psi(x)-\rho))=a(\rho)$, there exists $\lambda_\rho\in[\alpha,\beta]$ such that

$$\inf_{x\in X}(\Phi(x)+\lambda_\rho(\Psi(x)-\rho))=a(\rho),$$

thus we have

$$\sup_{x\in\Psi^{-1}(]\rho,+\infty[)} \frac{\Phi(x)-a(\rho)}{\rho-\Psi(x)} \leq \lambda_\rho \leq \inf_{x\in\Psi^{-1}(]-\infty,\rho[)} \frac{\Phi(x)-a(\rho)}{\rho-\Psi(x)},$$

that implies (5).

$(ii1)\Rightarrow(i1)$

Let $\rho\in]\inf_X\Psi,\sup_X\Psi[$ be such that (2) and (3) hold. In the previous proof "$(i1)\Rightarrow(ii1)$" we saw that (2) $\Longleftrightarrow$ $b(\rho)<a(\rho)$, then at first we have

$$\inf_{x\in X}\sup_{\lambda\in[\alpha,\beta]}(\Phi(x)+\lambda(\Psi(x)-\rho))=b(\rho).$$

Put

$$A_\rho=\left[\sup_{x\in\Psi^{-1}(]\rho,+\infty[)} \frac{\Phi(x)-b(\rho)}{\rho-\Psi(x)}, \inf_{x\in\Psi^{-1}(]-\infty,\rho[)} \frac{\Phi(x)-b(\rho)}{\rho-\Psi(x)}\right]\cap[\alpha,\beta],$$

it is $A_\rho \neq \emptyset$. In fact, A_ρ is empty iff

$$\inf_{x\in\Psi^{-1}(]-\infty,\rho[)} \frac{\Phi(x)-b(\rho)}{\rho-\Psi(x)} < \alpha \ \text{ or } \ \sup_{x\in\Psi^{-1}(]\rho,+\infty[)} \frac{\Phi(x)-b(\rho)}{\rho-\Psi(x)} > \beta.$$

If it is

$$\inf_{x\in\Psi^{-1}(]-\infty,\rho[)} \frac{\Phi(x)-b(\rho)}{\rho-\Psi(x)} < \alpha,$$

then $a(\rho) < b(\rho)$ that contradicts (2).

If it is

$$\sup_{x\in\Psi^{-1}(]\rho,+\infty[)} \frac{\Phi(x)-b(\rho)}{\rho-\Psi(x)} > \beta,$$

the absurd

$$\Phi(\bar{x}) + \beta(\Psi(\bar{x})-\rho) < \inf_{x\in\Psi^{-1}(]\rho,+\infty[)} (\Phi(x)+\beta(\Psi(x)-\rho)),$$

for some $\bar{x} \in \Psi^{-1}(]\rho,+\infty[)$, is obtained.

Thus we can choose $\lambda_\rho \in A_\rho$ for which $\inf_{x\in X}\Phi(x)+\lambda_\rho(\Psi(x)-\rho) \geq b(\rho)$, that implies the equality

$$\sup_{\lambda\in[\alpha,\beta]} \inf_{x\in X}(\Phi(x)+\lambda(\Psi(x)-\rho)) = b(\rho) = \inf_{x\in X} \sup_{\lambda\in[\alpha,\beta]} (\Phi(x)+\lambda(\Psi(x)-\rho)).$$

Let $\rho \in]\inf_X \Psi, \sup_X \Psi[$ be such that (4) and (5) hold. It is

$$\inf_{x\in X} \sup_{\lambda\in[\alpha,\beta]} (\Phi(x)+\lambda(\Psi(x)-\rho)) = a(\rho).$$

Put

$$B_\rho = \left[\sup_{x\in\Psi^{-1}(]\rho,+\infty[)} \frac{\Phi(x)-a(\rho)}{\rho-\Psi(x)}, \inf_{x\in\Psi^{-1}(]-\infty,\rho[)} \frac{\Phi(x)-a(\rho)}{\rho-\Psi(x)}\right] \cap [\alpha,\beta],$$

it is $B_\rho \neq \emptyset$. In fact, B_ρ is empty iff

$$\inf_{x\in\Psi^{-1}(]-\infty,\rho[)} \frac{\Phi(x)-a(\rho)}{\rho-\Psi(x)} < \alpha \quad \text{or} \quad \sup_{x\in\Psi^{-1}(]\rho,+\infty[)} \frac{\Phi(x)-a(\rho)}{\rho-\Psi(x)} > \beta.$$

The second inequality contradicts (4).

The first inequality implies that, for some $\bar{x} \in \Psi^{-1}(]-\infty,\rho])$,

$$\Phi(\bar{x}) + \alpha(\Psi(\bar{x}) - \rho) < \inf_{x\in\Psi^{-1}(]-\infty,\rho])} (\Phi(x) + \alpha(\Psi(x) - \rho)),$$

and this is absurd.

Thus we can choose $\lambda_\rho \in B_\rho$, then $\inf_{x\in X}(\Phi(x) + \lambda_\rho(\Psi(x) - \rho)) \geq a(\rho)$, that implies the equality

$$\sup_{\lambda\in[\alpha,\beta]} \inf_{x\in X} (\Phi(x) + \lambda(\Psi(x) - \rho)) = a(\rho) = \inf_{x\in X} \sup_{\lambda\in[\alpha,\beta]} (\Phi(x) + \lambda(\Psi(x) - \rho)).$$

It is easily seen that, if $\rho \in \mathbf{R}\setminus]\inf_X \Psi, \sup_X \Psi[\neq \emptyset$, the equality

$$\sup_{\lambda\in[\alpha,\beta]} \inf_{x\in X} (\Phi(x) + \lambda(\Psi(x) - \rho)) = \inf_{x\in X} \sup_{\lambda\in[\alpha,\beta]} (\Phi(x) + \lambda(\Psi(x) - \rho))$$

holds.

COROLLARY 1 *Fix arbitrarily* $\alpha, \beta \in \mathbf{R}_+$ *with* $\alpha < \beta$, *then for any* $\rho \in \mathbf{R}$, *one has*

$$\sup_{\lambda\in[\alpha,\beta]} \inf_{x\in X} (\Phi(x) + \lambda(\Psi(x) - \rho)) = \inf_{x\in X} \sup_{\lambda\in[\alpha,\beta]} (\Phi(x) + \lambda(\Psi(x) - \rho)).$$

Proof Let us consider arbitrary $\alpha, \beta \in \mathbf{R}_+$ with $\alpha < \beta$, by Lemma 1 it is enough to prove that (*ii1*) is true.

Let $\rho \in]\inf_X \Psi, \sup_X \Psi[$, since $a(\rho) \leq \inf_{\Psi^{-1}(]-\infty,\rho])} \Phi$ the following inequalities hold:

$$\sup_{x\in\Psi^{-1}(]\rho,+\infty[)} \frac{\Phi(x)-a(\rho)}{\rho-\Psi(x)} \leq \sup_{x\in\Psi^{-1}(]\rho,+\infty[)} \frac{\Phi(x) - \inf_{\Psi^{-1}(]-\infty,\rho])} \Phi}{\rho-\Psi(x)}$$

and

$$\inf_{x\in\Psi^{-1}(]-\infty,\rho[)} \frac{\Phi(x) - \inf_{\Psi^{-1}(]-\infty,\rho])}\Phi}{\rho - \Psi(x)} \leq \inf_{x\in\Psi^{-1}(]-\infty,\rho[)} \frac{\Phi(x) - a(\rho)}{\rho - \Psi(x)},$$

owing to (*ii*), then we have

$$\sup_{x\in\Psi^{-1}(]\rho,+\infty[)} \frac{\Phi(x) - a(\rho)}{\rho - \Psi(x)} \leq \inf_{x\in\Psi^{-1}(]-\infty,\rho[)} \frac{\Phi(x) - a(\rho)}{\rho - \Psi(x)}.$$

So, if $a(\rho) \leq b(\rho)$ then (4) and (5) hold.

If $b(\rho) < a(\rho)$, then (2) holds, moreover since $b(\rho) < \inf_{\Psi^{-1}(]-\infty,\rho])}\Phi$ and by (*ii*), we also have (3).

LEMMA 2 *Let $\alpha, \beta \in \mathbf{R}_+$ with $\alpha < \beta$ and $\rho \in]\inf_X \Psi, \sup_X \Psi[$ such that $b(\rho) < a(\rho)$, then $\Psi^{-1}(\rho) \neq \emptyset$ and for any $\gamma \in [\alpha, \beta]$ one has*

$$\inf_{x\in\Psi^{-1}(]-\infty,\rho])} (\Phi(x) + \gamma(\Psi(x) - \rho)) = \inf_{x\in\Psi^{-1}(\rho)} \Phi(x).$$

Proof Fix $\alpha, \beta \in \mathbf{R}_+$ with $\alpha < \beta$ and $\rho \in]\inf_X \Psi, \sup_X \Psi[$ such that $b(\rho) < a(\rho)$, by (*ii*), (2) of (*ii1*) Lemma 1 and $a(\rho) \leq \inf_{\Psi^{-1}(]-\infty,\rho])}\Phi$, we have

$$\inf_{x\in\Psi^{-1}(]-\infty,\rho[)} \frac{\Phi(x) - \inf_{\Psi^{-1}(]-\infty,\rho])}\Phi}{\rho - \Psi(x)} > \beta.$$

Then, for every $\gamma \in [0, \beta]$, since $\inf_{x\in\Psi^{-1}(]-\infty,\rho])} (\Phi(x) + \gamma(\Psi(x) - \rho)) \leq \inf_{\Psi^{-1}(]-\infty,\rho])}\Phi$, it turns out that

$$\inf_{x\in\Psi^{-1}(]-\infty,\rho[)} \frac{\Phi(x) - c(\rho)}{\rho - \Psi(x)} > \gamma,$$

where $c(\rho) = \inf_{x\in\Psi^{-1}(]-\infty,\rho])}(\Phi(x) + \gamma(\Psi(x) - \rho))$. Thus for every $x \in \Psi^{-1}(]-\infty, \rho[)$,

$$\Phi(x) + \gamma(\Psi(x) - \rho) > \inf_{x\in\Psi^{-1}(]-\infty,\rho])} (\Phi(x) + \gamma(\Psi(x) - \rho)),$$

moreover there exists $x^* \in \Psi^{-1}(]-\infty,\rho])$ such that $\Phi(x^*)+\gamma(\Psi(x^*)-\rho)=c(\rho)$, so it is necessarily $x^* \in \Psi^{-1}(\rho)$ and

$$\inf_{x\in\Psi^{-1}(]-\infty,\rho])}(\Phi(x)+\gamma(\Psi(x)-\rho)) = \inf_{x\in\Psi^{-1}(\rho)}\Phi(x).$$

LEMMA 3 *Consider $\alpha,\beta\in\mathbf{R}_+$ with $\alpha<\beta$ and a subdivision $\alpha=\alpha_1\leq\alpha_2\leq\cdots\leq\alpha_{n-2}\leq\alpha_{n-1}=\beta$ of the interval with $n\geq 3$. Define the function $h:[\alpha,\beta]\to\mathbf{R}$*

$$h(\lambda)=\sum_{i=1}^{n-1}\chi_{[\alpha_i,\alpha_{i+1}]}(\lambda)(\rho_i\lambda+a_i)\quad \textit{for each } \lambda\in[\alpha,\beta],$$

where $\{\rho_k\}_{1\leq k\leq n-1}$ is a non-increasing finite sequence of real numbers and $a_{i+1}=a_i+(\rho_i-\rho_{i+1})\alpha_{i+1}$ for $1\leq i\leq n-2$, with $a_1\in\mathbf{R}$ arbitrarily chosen.
Then one has

$$\sup_{\lambda\in[\alpha,\beta]}\inf_{x\in X}(\Phi(x)+\lambda\Psi(x)+h(\lambda)) = \inf_{x\in X}\sup_{\lambda\in[\alpha,\beta]}(\Phi(x)+\lambda\Psi(x)+h(\lambda)). \tag{6}$$

Proof By Corollary 1, we have

$$\begin{aligned}&\sup_{\lambda\in[\alpha,\beta]}\inf_{x\in X}(\Phi(x)+\lambda\Psi(x)+h(\lambda))\\ &\quad=\max_{1\leq i\leq n-1}\inf_{x\in X}\sup_{\lambda\in[\alpha_i,\alpha_{i+1}]}(\Phi(x)+\lambda(\Psi(x)+\rho_i)+a_i),\end{aligned}$$

thus there exists $j\in\{1,2,\ldots,n-1\}$ such that

$$\sup_{\lambda\in[\alpha,\beta]}\inf_{x\in X}(\Phi(x)+\lambda(\Psi(x)+h(\lambda))) = \inf_{x\in X}\sup_{\lambda\in[\alpha_j,\alpha_{j+1}]}(\Phi(x)+\lambda(\Psi(x)+\rho_j)+a_j). \tag{7}$$

For the sake of simplicity, we denote

$$f_i(x,\lambda)=\Phi(x)+\lambda(\Psi(x)+\rho_i)+a_i$$

for $1\leq i\leq n-1$ being x and λ on the respective domains.

1st Step

At first, we prove the thesis when $\inf_X \Psi < -\rho_i < \sup_X \Psi$ for every $1 \leq i \leq n-1$. Fix $1 \leq i \leq n-2$ for every $x \in \Psi^{-1}(]-\infty, -\rho_{i+1}])$ and $i \leq k \leq n-1$, we have

$$\sup_{\lambda \in [\alpha_k, \alpha_{k+1}]} (\Phi(x) + \lambda(\Psi(x) + \rho_k) + a_k)$$
$$\geq \sup_{\lambda \in [\alpha_{k+1}, \alpha_{k+2}]} (\Phi(x) + \lambda(\Psi(x) + \rho_{k+1}) + a_{k+1}),$$

hence

$$\max_{i \leq k \leq n-1} \sup_{\lambda \in [\alpha_k, \alpha_{k+1}]} (\Phi(x) + \lambda(\Psi(x) + \rho_k) + a_k)$$
$$= \sup_{\lambda \in [\alpha_i, \alpha_{i+1}]} (\Phi(x) + \lambda(\Psi(x) + \rho_i) + a_i). \tag{8}$$

Fix $2 \leq i \leq n-2$, if $\Psi^{-1}(]-\rho_i, -\rho_{i+1}]) \neq \emptyset$, then for every $x \in \Psi^{-1}(]-\rho_i, -\rho_{i+1}])$ and $2 \leq k \leq i$, we have

$$\sup_{\lambda \in [\alpha_k, \alpha_{k+1}]} (\Phi(x) + \lambda(\Psi(x) + \rho_k) + a_k)$$
$$\geq \sup_{\lambda \in [\alpha_{k-1}, \alpha_k]} (\Phi(x) + \lambda(\Psi(x) + \rho_{k-1}) + a_{k-1}),$$

whence, by (8), it follows that

$$\max_{1 \leq k \leq n-1} \sup_{\lambda \in [\alpha_k, \alpha_{k+1}]} (\Phi(x) + \lambda(\Psi(x) + \rho_k) + a_k)$$
$$= \Phi(x) + \alpha_{i+1}(\Psi(x) + \rho_i) + a_i.$$

For every $x \in \Psi^{-1}(]-\rho_{n-1}, +\infty[)$ and $1 \leq k \leq n-2$, we have

$$\sup_{\lambda \in [\alpha_{k+1}, \alpha_{k+2}]} (\Phi(x) + \lambda(\Psi(x) + \rho_{k+1}) + a_{k+1})$$
$$\geq \sup_{\lambda \in [\alpha_k, \alpha_{k+1}]} (\Phi(x) + \lambda(\Psi(x) + \rho_k) + a_k),$$

hence

$$\max_{1\le k\le n-1}\ \sup_{\lambda\in[\alpha_k,\alpha_{k+1}]}(\Phi(x)+\lambda(\Psi(x)+\rho_k)+a_k)$$
$$=\Phi(x)+\beta(\Psi(x)+\rho_{n-1})+a_{n-1}.$$

We set $N=\{1\le i\le n-2/\Psi^{-1}(]-\rho_i,-\rho_{i+1}])\ne\emptyset\}$ and

$$\delta=\begin{cases}+\infty & \text{if } N=\emptyset\\ \min\limits_{i\in N}\ \inf\limits_{x\in\Psi^{-1}(]-\rho_i,-\rho_{i+1}])} f_i(x,\alpha_{i+1}) & \text{if } N\ne\emptyset,\end{cases}$$

then it follows that

$$\inf_{x\in X}\ \sup_{\lambda\in[\alpha,\beta]}(\Phi(x)+\lambda\Psi(x)+h(\lambda))$$
$$=\min\left\{\delta,\ \inf_{x\in\Psi^{-1}(]-\infty,-\rho_1])} f_1(x,\alpha),\ \inf_{x\in\Psi^{-1}(]-\rho_{n-1},+\infty[)} f_{n-1}(x,\beta)\right\}. \tag{9}$$

Now we state and prove the following assertions:

(a) If

$$\inf_{x\in\Psi^{-1}(]-\rho_j,+\infty[)} f_j(x,\alpha_{j+1})<\min\left\{\delta,\ \inf_{x\in\Psi^{-1}(]-\infty,-\rho_j])} f_j(x,\alpha_j)\right\},$$

then for every $j\le k\le n-1$, we have

$$\inf_{x\in\Psi^{-1}(]-\rho_k,+\infty[)} f_k(x,\alpha_{k+1})\le\inf_{x\in\Psi^{-1}(]-\rho_j,+\infty[)} f_j(x,\alpha_{j+1})$$
$$<\inf_{x\in\Psi^{-1}(]-\infty,-\rho_k])} f_k(x,\alpha_k).$$

(b) If

$$\inf_{x\in\Psi^{-1}(]-\infty,-\rho_j])} f_j(x,\alpha_j)\le\inf_{x\in\Psi^{-1}(]-\rho_j,+\infty[)} f_j(x,\alpha_{j+1})$$

and

$$\inf_{x\in\Psi^{-1}(]-\infty,-\rho_j])} f_j(x,\alpha_j)<\delta,$$

then for every $1 \le k \le j$, we have

$$\inf_{x\in\Psi^{-1}(]-\infty,-\rho_k])} f_k(x,\alpha_k) = \inf_{x\in\Psi^{-1}(]-\infty,-\rho_j])} f_j(x,\alpha_j).$$

Let us prove (a).

If $j=n-1$ the thesis is obvious.

Let $j<n-1$, inequalities obviously hold for $k=j$. Put $T=\{j\le k\le n-1/$ such that the inequalities hold$\}$, then $T\neq\emptyset$ since $j\in T$. Let $m\in T$ with $m\le n-2$, we denote $A_{m,m+1}=\Psi^{-1}(]-\rho_m,-\rho_{m+1}])$, then

$$\begin{aligned}
&\inf_{x\in\Psi^{-1}(]-\infty,-\rho_{m+1}])} f_{m+1}(x,\alpha_{m+1})\\
&\quad= \inf_{x\in\Psi^{-1}(]-\infty,-\rho_{m+1}])} f_m(x,\alpha_{m+1})\\
&\quad= \begin{cases} \displaystyle\inf_{x\in\Psi^{-1}(]-\infty,-\rho_m])} f_m(x,\alpha_{m+1}) \\ \quad \text{if } A_{m,m+1}=\emptyset, \\ \displaystyle\min\left\{\inf_{x\in\Psi^{-1}(]-\infty,-\rho_m])} f_m(x,\alpha_{m+1}),\ \inf_{x\in A_{m,m+1}} f_m(x,\alpha_{m+1})\right\} \\ \quad \text{if } A_{m,m+1}\neq\emptyset. \end{cases}
\end{aligned} \tag{10}$$

Since $m\in T$,

$$\begin{aligned}
&\inf_{x\in\Psi^{-1}(]-\rho_m,+\infty[)} (\Phi(x)+\alpha_{m+1}(\Psi(x)+\rho_m))\\
&\qquad< \inf_{x\in\Psi^{-1}(]-\infty,-\rho_m])} (\Phi(x)+\alpha_m(\Psi(x)+\rho_m)),
\end{aligned}$$

then, by Lemma 2, we have $\Psi^{-1}(-\rho_m)\neq\emptyset$ and

$$\begin{aligned}
&\inf_{x\in\Psi^{-1}(]-\infty,-\rho_m])} (\Phi(x)+\alpha_{m+1}(\Psi(x)+\rho_m))\\
&\qquad= \inf_{x\in\Psi^{-1}(]-\infty,-\rho_m])} (\Phi(x)+\alpha_m(\Psi(x)+\rho_m)),
\end{aligned}$$

so one has

$$\inf_{x\in\Psi^{-1}(]-\rho_j,+\infty[)} f_j(x,\alpha_{j+1}) < \inf_{x\in\Psi^{-1}(]-\infty,-\rho_m])} f_m(x,\alpha_{m+1}).$$

Moreover, if $A_{m,m+1}\neq\emptyset$, then the hypotheses imply that

$$\inf_{x\in\Psi^{-1}(]-\rho_j,+\infty[)} f_j(x,\alpha_{j+1}) < \inf_{x\in\Psi^{-1}(]-\rho_m,-\rho_{m+1}])} f_m(x,\alpha_{m+1}).$$

Consequently, by (10), it follows

$$\inf_{x\in\Psi^{-1}(]-\rho_j,+\infty[)} f_j(x,\alpha_{j+1}) < \inf_{x\in\Psi^{-1}(]-\infty,-\rho_{m+1}])} f_{m+1}(x,\alpha_{m+1}). \tag{11}$$

By (7), we have

$$\inf_{x\in\Psi^{-1}(]-\rho_j,+\infty[)} f_j(x,\alpha_{j+1}) = \max_{1\leq i\leq n-1}\inf_{x\in X}\sup_{\lambda\in[\alpha_i,\alpha_{i+1}]} f_i(x,\lambda),$$

hence

$$\inf_{x\in\Psi^{-1}(]-\rho_j,+\infty[)} f_j(x,\alpha_{j+1}) \geq \inf_{x\in X}\sup_{\lambda\in[\alpha_{m+1},\alpha_{m+2}]} f_{m+1}(x,\lambda)$$

$$=\min\left\{\inf_{x\in\Psi^{-1}(]-\infty,-\rho_{m+1}])} f_{m+1}(x,\alpha_{m+1}),\ \inf_{x\in\Psi^{-1}(]-\rho_{m+1},+\infty[)} f_{m+1}(x,\alpha_{m+2})\right\},$$

therefore, by (11), it turns out that

$$\inf_{x\in\Psi^{-1}(]-\rho_j,+\infty[)} f_j(x,\alpha_{j+1}) \geq \inf_{x\in\Psi^{-1}(]-\rho_{m+1},+\infty[)} f_{m+1}(x,\alpha_{m+2}).$$

Thus we have proved that $m\in T$ with $m\leq n-2 \Rightarrow m+1\in T$, then the thesis of assertion (a) is proved.

Let us prove (b).

If $j=1$ the thesis is obvious.

Let $j>1$, the equality obviously holds when $k=j$. Put $T=\{1\leq k\leq j/$ the equality holds$\}$, then $T\neq\emptyset$ because $j\in T$. Suppose $m\in T$ with $m>1$, we have that

$$
\begin{aligned}
&\inf_{x\in\Psi^{-1}(]-\infty,-\rho_m])} f_m(x,\alpha_m)\\
&\quad= \inf_{x\in\Psi^{-1}(]-\infty,-\rho_m])} f_{m-1}(x,\alpha_m)\\
&\quad= \begin{cases} \inf\limits_{x\in\Psi^{-1}(]-\infty,-\rho_{m-1}])} f_{m-1}(x,\alpha_m) \\ \quad \text{if } A_{m-1,m}=\emptyset, \\ \min\left\{\inf\limits_{x\in\Psi^{-1}(]-\infty,-\rho_{m-1}])} f_{m-1}(x,\alpha_m),\ \inf\limits_{x\in A_{m-1,m}} f_{m-1}(x,\alpha_m)\right\} \\ \quad \text{if } A_{m-1,m}\neq\emptyset, \end{cases}
\end{aligned}
\tag{12}
$$

then it is seen, by similar arguments to those in the proof of assertion (a), that

$$
\inf_{x\in\Psi^{-1}(]-\infty,-\rho_m])} f_m(x,\alpha_m) = \inf_{x\in\Psi^{-1}(]-\infty,-\rho_{m-1}])} f_{m-1}(x,\alpha_m).
$$

Because of (7) and $m\in T$, we also have

$$
\begin{aligned}
&\inf_{x\in\Psi^{-1}(]-\infty,-\rho_{m-1}])} f_{m-1}(x,\alpha_m) = \inf_{x\in\Psi^{-1}(]-\infty,-\rho_j])} f_j(x,\alpha_j)\\
&\quad\geq \min\left\{\inf_{x\in\Psi^{-1}(]-\infty,-\rho_{m-1}])} f_{m-1}(x,\alpha_{m-1}),\ \inf_{x\in\Psi^{-1}(]-\rho_{m-1},+\infty[)} f_{m-1}(x,\alpha_m)\right\},
\end{aligned}
$$

which implies, by Lemma 2,

$$
\inf_{x\in\Psi^{-1}(]-\infty,-\rho_{m-1}])} f_{m-1}(x,\alpha_m) \geq \inf_{x\in\Psi^{-1}(]-\infty,-\rho_{m-1}])} f_{m-1}(x,\alpha_{m-1}).
$$

Since $\alpha_{m-1}\leq\alpha_m$, the opposite inequality holds too, therefore the thesis is proved.

Now we can prove the equality (6) with the further hypothesis we stated at the beginning of this step.

If

$$
\inf_{x\in\Psi^{-1}(]-\rho_j,+\infty[)} f_j(x,\alpha_{j+1}) < \inf_{x\in\Psi^{-1}(]-\infty,-\rho_j])} f_j(x,\alpha_j),
$$

then by (7),

$$\sup_{\lambda\in[\alpha,\beta]} \inf_{x\in X}(\Phi(x)+\lambda\Psi(x)+h(\lambda)) = \inf_{x\in\Psi^{-1}(]-\rho_j,+\infty[)} f_j(x,\alpha_{j+1}),$$

owing to (a), it follows that

$$\inf_{x\in\Psi^{-1}(]-\rho_j,+\infty[)} f_j(x,\alpha_{j+1}) \geq \delta$$

or

$$\inf_{x\in\Psi^{-1}(]-\rho_j,+\infty[)} f_j(x,\alpha_{j+1}) \geq \inf_{x\in\Psi^{-1}(]-\rho_{n-1},+\infty[)} f_{n-1}(x,\beta),$$

hence

$$\sup_{\lambda\in[\alpha,\beta]} \inf_{x\in X}(\Phi(x)+\lambda\Psi(x)+h(\lambda)) \geq \inf_{x\in X} \sup_{\lambda\in[\alpha,\beta]} (\Phi(x)+\lambda\Psi(x)+h(\lambda)),$$

that implies (6).

If

$$\inf_{x\in\Psi^{-1}(]-\infty,-\rho_j])} f_j(x,\alpha_j) \leq \inf_{x\in\Psi^{-1}(]-\rho_j,+\infty[)} f_j(x,\alpha_{j+1}),$$

then the equality (6) is implied by assertion (b).

2nd Step

We prove that the equality (6) holds, when $\inf_{x\in X}\Psi(x) \neq -\infty$ and there exists $1 \leq \bar{k} \leq n-2$ such that

$$\text{for } 1 \leq i \leq \bar{k} \qquad -\rho_i \leq \inf_{x\in X}\Psi(x),$$

$$\text{for } \bar{k}+1 \leq i \leq n-1 \quad \inf_{x\in X}\Psi(x) < -\rho_i < \sup_{x\in X}\Psi(x).$$

We have

$$\sup_{\lambda\in[\alpha,\beta]} \inf_{x\in X}(\Phi(x)+\lambda\Psi(x)+h(\lambda))$$

$$= \max_{\bar{k}+1\leq i\leq n-1} \inf_{x\in X} \sup_{\lambda\in[\alpha_i,\alpha_{i+1}]} (\Phi(x)+\lambda(\Psi(x)+\rho_i)+a_i), \tag{13}$$

and

$$\inf_{x\in X}\sup_{\lambda\in[\alpha,\beta]}(\Phi(x)+\lambda\Psi(x)+h(\lambda))$$
$$=\inf_{x\in X}\max_{\bar{k}+1\le i\le n-1}\sup_{\lambda\in[\alpha_i,\alpha_{i+1}]}(\Phi(x)+\lambda(\Psi(x)+\rho_i)+a_i). \qquad (14)$$

Put $g=h_{|[\alpha_{\bar{k}+1},\beta]}$, then the equality (6) follows from the 1st step, where g takes the place of h, and from (13), (14).

3rd Step

Let for $1\le i\le n-1$, $-\rho_i\le\inf_X\Psi\ne-\infty$, then we have

$$\sup_{\lambda\in[\alpha,\beta]}\inf_{x\in X}(\Phi(x)+\lambda\Psi(x)+h(\lambda))=\inf_{x\in X}f_{n-1}(x,\beta)$$
$$=\inf_{x\in X}\sup_{\lambda\in[\alpha,\beta]}(\Phi(x)+\lambda\Psi(x)+h(\lambda)).$$

Let for $1\le i\le n-1$, $-\rho_i\ge\sup_X\Psi(x)\ne+\infty$, then we have

$$\sup_{\lambda\in[\alpha,\beta]}\inf_{x\in X}(\Phi(x)+\lambda\Psi(x)+h(\lambda))=\inf_{x\in X}f_1(x,\alpha)$$
$$=\inf_{x\in X}\sup_{\lambda\in[\alpha,\beta]}(\Phi(x)+\lambda\Psi(x)+h(\lambda)).$$

4th Step

Suppose that there exists $2\le\bar{k}\le n-1$ such that

$$\text{for } 1\le i\le\bar{k}-1,\quad -\rho_i\le\sup_{x\in X}\Psi(x),$$
$$\text{for } \bar{k}\le i\le n-1,\quad -\rho_i\ge\sup_{x\in X}\Psi(x).$$

We have

$$\sup_{\lambda\in[\alpha,\beta]}\inf_{x\in X}(\Phi(x)+\lambda\Psi(x)+h(\lambda))=\sup_{\lambda\in[\alpha,\alpha_{\bar{k}}]}\inf_{x\in X}(\Phi(x)+\lambda\Psi(x)+g(\lambda))$$

and

$$\inf_{x\in X}\sup_{\lambda\in[\alpha,\beta]}(\Phi(x)+\lambda\Psi(x)+h(\lambda))=\inf_{x\in X}\sup_{\lambda\in[\alpha,\alpha_{\bar{k}}]}(\Phi(x)+\lambda\Psi(x)+g(\lambda)),$$

where $g=h_{|[\alpha,\alpha_{\bar{k}}]}$. Therefore the equality (6) follows from the previous steps, where g takes the place of h.

LEMMA 4 *Let $\alpha, \beta \in \mathbf{R}_+$ with $\alpha < \beta$ and $g:[\alpha,\beta]\to\mathbf{R}$ be a concave function such that* $\max\{|g'_d(\alpha)|, |g'_s(\beta)|\} \neq +\infty$. *There exists a non-increasing sequence of functions $\{g_n\}_{n\in\mathbf{N}}$ pointwise convergent to g on $[\alpha,\beta]$ such that for every $n\in\mathbf{N}$, g_n is formally defined as the function h in Lemma* 3.

Proof Fix $n\in\mathbf{N}$, we set

$$\delta_k^{(n)} = \alpha + k\frac{\beta-\alpha}{2^{n-1}} \quad \text{for } 0 \le k \le 2^{n-1};$$

$\rho_0^{(n)} = g'_d(\alpha)$, $\rho_{2^n}^{(n)} = g'_s(\beta)$ and for $1 \le j \le 2^n - 1$

$$\rho_j^{(n)} = \begin{cases} g'_s(\delta_k^{(n)}) & \text{with } k = \dfrac{j+1}{2}, \quad \text{if } j \text{ is odd,} \\[2ex] g'_d(\delta_k^{(n)}) & \text{with } k = \dfrac{j}{2}, \quad \text{if } j \text{ is even,} \end{cases}$$

$$a_j^{(n)} = g(\delta_k^{(n)}) - \rho_j^{(n)}\delta_k^{(n)} \quad \text{for } 0 \le j \le 2^n,$$

$\alpha_0^{(n)} = \alpha$, $\alpha_{2^n+1}^{(n)} = \beta$ and for $1 \le j \le 2^n$

$$\alpha_j^{(n)} = \begin{cases} \delta_k^{(n)} & \text{if } \rho_j^{(n)} = \rho_{j-1}^{(n)}, \\[2ex] \dfrac{a_j^{(n)} - a_{j-1}^{(n)}}{\rho_{j-1}^{(n)} - \rho_j^{(n)}} & \text{if } \rho_j^{(n)} \neq \rho_{j-1}^{(n)}, \end{cases}$$

with $k=j/2$ if j is even, $k=(j+1)/2$ if j is odd.

Because g is concave, then $\{\rho_k^{(n)}\}_{0\le k\le 2^n}$ is non-increasing, moreover it is easily seen that $\{\alpha_k^{(n)}\}_{0\le k\le 2^n+1}$ is a subdivision of the interval $[\alpha,\beta]$.

Now we define the function:

$$g_n(\lambda) = \sum_{j=0}^{2^n} \chi_{[\alpha_j^{(n)},\alpha_{j+1}^{(n)}]}(\lambda)\left(\rho_j^{(n)}\lambda + a_j^{(n)}\right) \text{ for each } \lambda \in [\alpha,\beta],$$

from the definition of $\alpha_{j+1}^{(n)}$, one has $a_{j+1}^{(n)} = a_j^{(n)} + \left(\rho_j^{(n)} - \rho_{j+1}^{(n)}\right)\alpha_{j+1}^{(n)}$ for every $0 \le j \le 2^n-1$.

We prove that $\{g_n\}$ is pointwise convergent to g on the interval $[\alpha, \beta]$: Fix $\lambda \in [\alpha, \beta]$ and $n \in \mathbf{N}$, we put $k_n = \max\{0 \leq k \leq 2^{n-1}\colon \delta_k^{(n)} \leq \lambda\}$; since $\alpha_{2k_n}^{(n)} = \delta_{k_n}^{(n)}$ and $\alpha_{2k_n+2}^{(n)} = \delta_{k_n+1}^{(n)}$, one has $\lambda \in [\alpha_{2k_n}^{(n)}, \alpha_{2k_n+2}^{(n)}]$, then

$$|g_n(\lambda) - g(\lambda)| \leq \max\{|g_d'(\alpha)|, |g_s'(\beta)|\} \frac{\beta - \alpha}{2^{n-1}} + \max\left\{\left|g\left(\delta_{k_n}^{(n)}\right) - g(\lambda)\right|, \left|g\left(\delta_{k_n+1}^{(n)}\right) - g(\lambda)\right|\right\}.$$

The function g is continuous, then, since

$$\lim_{n\to\infty} \delta_{k_n}^{(n)} = \lim_{n\to\infty} \delta_{k_n+1}^{(n)} = \lambda,$$

we have

$$\lim_{n\to\infty} |g_n(\lambda) - g(\lambda)| = 0.$$

Let $n \in \mathbf{N}$ and $\lambda \in [\alpha, \beta]$, it results that $g_n(\lambda) \geq g_{n+1}(\lambda)$.

There exists $0 \leq j \leq 2^{n+1}$ such that $\lambda \in \left[\alpha_j^{(n+1)}, \alpha_{j+1}^{(n+1)}\right]$, then we have to examine the different cases which can occur.

If j is even and $k = j/2$ is also even, for some $0 \leq m \leq 2^{n-1}$, one has $j = 2k = 4m$. In this case, at first, we have

$$\left[\alpha_j^{(n+1)}, \alpha_{j+1}^{(n+1)}\right] \subseteq \left[\alpha_k^{(n)}, \alpha_{k+1}^{(n)}\right],$$

hence

$$g_n(\lambda) - g_{n+1}(\lambda) = \left(\rho_k^{(n)}\lambda + a_k^{(n)}\right) - \left(\rho_j^{(n+1)}\lambda + a_j^{(n+1)}\right) = 0.$$

If j is even and $k = j/2$ is odd, then $j = 2k = 4m + 2$ for some $0 \leq m \leq 2^{n-1} - 1$. In this case, it results that

$$\left[\alpha_j^{(n+1)}, \alpha_{j+1}^{(n+1)}\right] \subseteq \left[\alpha_{k-1}^{(n)}, \alpha_{k+1}^{(n)}\right].$$

Since for every $\lambda \in \left]\alpha_j^{(n+1)}, \alpha_{j+1}^{(n+1)}\right[\cap \left]\alpha_{k-1}^{(n)}, \alpha_k^{(n)}\right[$, we have

$$g_n'(\lambda) - g_{n+1}'(\lambda) = \rho_{2m}^{(n)} - \rho_{4m+2}^{(n)} \geq 0$$

and

$$g_n\left(\alpha_j^{(n+1)}\right) - g_{n+1}\left(\alpha_j^{(n+1)}\right) = \left(\rho_{2m}^{(n)} - \rho_{4m+1}^{(n+1)}\right)\left(\alpha_{4m+2}^{(n+1)} - \alpha_{4m+1}^{(n+1)}\right) \geq 0,$$

it follows that

$$g_n(\lambda) \geq g_{n+1}(\lambda), \quad \text{if } \lambda \in \left[\alpha_j^{(n+1)}, \alpha_{j+1}^{(n+1)}\right] \cap \left[\alpha_{k-1}^{(n)}, \alpha_k^{(n)}\right].$$

Since for every $\lambda \in \left]\alpha_j^{(n+1)}, \alpha_{j+1}^{(n+1)}\right[\cap \left]\alpha_k^{(n)}, \alpha_{k+1}^{(n)}\right[$, one has

$$g_n'(\lambda) - g_{n+1}'(\lambda) \leq 0,$$

and

$$g_n\left(\alpha_{j+1}^{(n+1)}\right) - g_{n+1}\left(\alpha_{j+1}^{(n+1)}\right) = 0,$$

it follows that

$$g_n(\lambda) \geq g_{n+1}(\lambda) \quad \text{if } \lambda \in \left[\alpha_j^{(n+1)}, \alpha_{j+1}^{(n+1)}\right] \cap \left[\alpha_k^{(n)}, \alpha_{k+1}^{(n)}\right].$$

In the remaining cases, the inequality $g_n(\lambda) \geq g_{n+1}(\lambda)$ also holds, the proof is analogous to the previous ones.

3. PROOF OF THEOREM 1

In the first instance, we remark that if $\rho \in]\inf_X \Psi, \sup_X \Psi[$, then

$$\inf_{x\in X} \sup_{\lambda\geq 0} (\Phi(x) + \lambda(\Psi(x) - \rho)) = \inf_{\Psi^{-1}(]-\infty,\rho])} \Phi.$$

$(i) \Rightarrow (ii)$

Let $\rho \in]\inf_X \Psi, \sup_X \Psi[$, then $\lim_{\lambda\to+\infty} \inf_{x\in X}(\Phi(x) + \lambda(\Psi(x) - \rho)) = -\infty$, moreover the function $\lambda \in [0, +\infty[\to \inf_{x\in X}(\Phi(x) + \lambda(\Psi(x) - \rho))$ is u.s.c. then attains its supremum. Consequently there exists $\lambda_\rho \in [0, +\infty[$

such that

$$\inf_{x\in X}(\Phi(x)+\lambda_\rho(\Psi(x)-\rho)) = \sup_{\lambda\geq 0}\inf_{x\in X}(\Phi(x)+\lambda(\Psi(x)-\rho))$$
$$= \inf_{\Psi^{-1}(]-\infty,\rho])}\Phi,$$

so we have

$$\sup_{x\in\Psi^{-1}(]\rho,+\infty[)} \frac{\Phi(x)-\inf_{\Psi^{-1}(]-\infty,\rho])}\Phi}{\rho-\Psi(x)}$$
$$\leq \lambda_\rho \leq \inf_{x\in\Psi^{-1}(]-\infty,\rho[)} \frac{\Phi(x)-\inf_{\Psi^{-1}(]-\infty,\rho])}\Phi}{\rho-\Psi(x)},$$

then owing to arbitrariness of ρ the thesis is proved.

$(ii)\Rightarrow(i)$
Let $\rho\in\,]\inf_X\Psi,\sup_X\Psi[$, since

$$0\leq \inf_{x\in\Psi^{-1}(]-\infty,\rho[)} \frac{\Phi(x)-\inf_{\Psi^{-1}(]-\infty,\rho])}\Phi}{\rho-\Psi(x)} < +\infty,$$

we can set

$$\lambda_\rho = \inf_{x\in\Psi^{-1}(]-\infty,\rho[)} \frac{\Phi(x)-\inf_{\Psi^{-1}(]-\infty,\rho])}\Phi}{\rho-\Psi(x)},$$

so we have

$$\inf_{x\in X}(\Phi(x)+\lambda_\rho(\Psi(x)-\rho)) \geq \inf_{\Psi^{-1}(]-\infty,\rho])}\Phi$$
$$= \inf_{x\in X}\sup_{\lambda\geq 0}(\Phi(x)+\lambda(\Psi(x)-\rho)),$$

therefore the equality

$$\sup_{\lambda\geq 0}\inf_{x\in X}(\Phi(x)+\lambda(\Psi(x)-\rho)) = \inf_{x\in X}\sup_{\lambda\geq 0}(\Phi(x)+\lambda(\Psi(x)-\rho))$$

holds.

In order to complete the proof we have to prove that, if $\mathbf{R}\setminus]\inf_X\Psi,\sup_X\Psi[\neq\emptyset$, for every $\rho\in\mathbf{R}\setminus]\inf_X\Psi,\sup_X\Psi[$ the equality in (i) holds.

If $\sup_X\Psi\neq+\infty$ and $\rho\geq\sup_X\Psi$ then

$$\inf_{x\in X}\sup_{\lambda\geq 0}(\Phi(x)+\lambda(\Psi(x)-\rho))=\inf_{x\in X}\Phi(x),$$

thus the equality follows because

$$\sup_{\lambda\geq 0}\inf_{x\in X}(\Phi(x)+\lambda(\Psi(x)-\rho))\geq\inf_{x\in X}\Phi(x).$$

Now we suppose that $\inf_X\Psi\neq-\infty$ and $\rho\leq\inf_X\Psi$. It is necessary to distinguish the following two cases:

(1) Ψ does not have absolute minimum.
(2) Ψ has absolute minimum.

Let (1) be true.

Since for every $x\in X$, $\Psi(x)-\rho>0$, it follows that $\inf_{x\in X}\sup_{\lambda\geq 0}(\Phi(x)+\lambda(\Psi(x)-\rho))=+\infty$. We assume that

$$\sup_{\lambda\geq 0}\inf_{x\in X}(\Phi(x)+\lambda(\Psi(x)-\rho))<\inf_{x\in X}\sup_{\lambda\geq 0}(\Phi(x)+\lambda(\Psi(x)-\rho)),$$

then there exists $\alpha\in\mathbf{R}$ such that $\sup_{\lambda\geq 0}\inf_{x\in X}(\Phi(x)+\lambda(\Psi(x)-\rho))=\alpha$. Consequently, for every $n\in\mathbf{N}$, there exists $x_n\in X$ such that $\Phi(x_n)+n(\Psi(x_n)-\rho)<\alpha+1$, since for every $n\in\mathbf{N}$, $\Psi(x_n)-\rho>0$, it follows $\Phi(x_n)<\alpha+1$, then the weak coerciveness of Φ implies that $\{x_n\}_{n\in\mathbf{N}}$ is bounded. Because of the hypotheses about E and X, there exist $x^*\in X$ and a subsequence $\{x_{n_k}\}_{k\in\mathbf{N}}$ such that $x_{n_k}\to x^*$ weakly for $k\to\infty$. The function Φ is weakly sequentially l.s.c., then

$$\begin{aligned}&\Phi(x^*)+\liminf_{k\to\infty}n_k(\Psi(x_{n_k})-\rho)\\&\leq\liminf_{k\to\infty}\Phi(x_{n_k})+n_k(\Psi(x_{n_k})-\rho)\leq\alpha+1,\end{aligned}$$

consequently, it follows $\liminf_{k\to\infty}\Psi(x_{n_k})=\rho$.

Therefore we have the absurd $\rho < \Psi(x^*) \leq \liminf_{k\to\infty} \Psi(x_{n_k}) = \rho$, being Ψ weakly sequentially l.s.c. The absurd follows from the hypothesis that the equality in (*i*) does not hold, so the thesis is proved.

Let (2) be true.

If we choose $\rho < \inf_X \Psi$, since for every $x \in X$, $\Psi(x) > \rho$, we can proceed as in (1).

Let $\rho = \inf_X \Psi$, then

$$\inf_{x\in X}\sup_{\lambda\geq 0}(\Phi(x) + \lambda(\Psi(x) - \rho)) = \inf_{x\in\Psi^{-1}(\rho)} \Phi(x),$$

in fact, if we assume that $\sup_{\lambda\geq 0}\inf_{x\in X}(\Phi(x) + \lambda(\Psi(x) - \rho)) < \inf_{x\in\Psi^{-1}}(\rho)\Phi(x)$, we can choose $\gamma \in \mathbf{R}$ such that

$$\sup_{\lambda\geq 0}\inf_{x\in X}(\Phi(x) + \lambda(\Psi(x) - \rho)) < \gamma < \inf_{x\in\Psi^{-1}(\rho)} \Phi(x).$$

Therefore for every $\lambda \in [0, +\infty[$ it results that

$$\inf_{x\in X,\ \Psi(x)\neq\rho}(\Phi(x) + \lambda(\Psi(x) - \rho)) < \gamma,$$

hence, for every $n \in \mathbf{N}$, there exists $x_n \in X$ with $\Psi(x_n) > \rho$ and $\Phi(x_n) + n(\Psi(x_n) - \rho) < \gamma$. Since for every $n \in \mathbf{N}$, $\Phi(x_n) < \gamma$, there exist $x^* \in X$ and a subsequence $\{x_{n_k}\}_{k\in\mathbf{N}}$ such that $x_{n_k} \to x^*$ weakly when $k \to \infty$, so $\rho \leq \Psi(x^*) \leq \liminf_{k\to\infty} \Psi(x_{n_k}) = \rho$, that implies $x^* \in \Psi^{-1}(\rho)$. We also have

$$\Phi(x^*) \leq \liminf_{k\to\infty} \Phi(x_{n_k}) \leq \gamma < \inf_{x\in\Psi^{-1}(\rho)} \Phi(x),$$

that is absurd since $x^* \in \Psi^{-1}(\rho)$.

(*iii*) $\Rightarrow$ (*i*) is obvious.

(*ii*) $\Rightarrow$ (*iii*)

Consider a concave function $h: [0, +\infty[\to \mathbf{R}$, let $0 < \alpha < \beta$ be arbitrary real numbers and set $g = h_{|[\alpha,\beta]}$, the function g meets the hypotheses of Lemma 4, hence we can consider a non-increasing sequence of functions $\{g_n\}_{n\in\mathbf{N}}$ pointwise convergent to g such that, for every $n \in \mathbf{N}$, by Lemma 3,

$$\sup_{\lambda\in[\alpha,\beta]}\inf_{x\in X}(\Phi(x)+\lambda\Psi(x)+g_n(\lambda))=\inf_{x\in X}\sup_{\lambda\in[\alpha,\beta]}(\Phi(x)+\lambda\Psi(x)+g_n(\lambda)). \tag{15}$$

Since $[\alpha,\beta]$ is compact and $\{g_n\}_{n\in\mathbf{N}}$ is a monotone sequence of functions pointwise convergent to g, it follows that $g_n\to g$ uniformly on $[\alpha,\beta]$ when $n\to+\infty$, by the Dini's theorem. Hence

$$\begin{aligned}
&\sup_{\lambda\in[\alpha,\beta]}\inf_{x\in X}(\Phi(x)+\lambda\Psi(x)+g(\lambda))\\
&\quad=\lim_{n\to\infty}\sup_{\lambda\in[\alpha,\beta]}\inf_{x\in X}(\Phi(x)+\lambda\Psi(x)+g_n(\lambda))\\
&\quad=\lim_{n\to\infty}\inf_{x\in X}\sup_{\lambda\in[\alpha,\beta]}(\Phi(x)+\lambda\Psi(x)+g_n(\lambda))\\
&\quad\geq\inf_{x\in X}\sup_{\lambda\in[\alpha,\beta]}(\Phi(x)+\lambda\Psi(x)+g(\lambda)),
\end{aligned}$$

where the last inequality is due to $g(\lambda)=\inf_{n\in\mathbf{N}}g_n(\lambda)$ for any $\lambda\in[\alpha,\beta]$. Thus the equality follows.

Since $\beta>\alpha$ is arbitrary, it follows that

$$\begin{aligned}
&\sup_{\lambda\in[\alpha,+\infty[}\inf_{x\in X}(\Phi(x)+\lambda\Psi(x)+h(\lambda))\\
&\quad=\sup_{\beta>\alpha}\inf_{x\in X}\sup_{\lambda\in[\alpha,\beta]}(\Phi(x)+\lambda\Psi(x)+h(\lambda)).
\end{aligned}$$

Let us suppose that

$$\begin{aligned}
&\sup_{\beta>\alpha}\inf_{x\in X}\sup_{\lambda\in[\alpha,\beta]}(\Phi(x)+\lambda\Psi(x)+h(\lambda))\\
&\quad<\inf_{x\in X}\sup_{\lambda\in[\alpha,+\infty[}(\Phi(x)+\lambda\Psi(x)+h(\lambda)),
\end{aligned}$$

then we can choose $\gamma\in\mathbf{R}$ such that

$$\begin{aligned}
&\sup_{\beta>\alpha}\inf_{x\in X}\sup_{\lambda\in[\alpha,\beta]}(\Phi(x)+\lambda\Psi(x)+h(\lambda))\\
&\quad<\gamma<\inf_{x\in X}\sup_{\lambda\in[\alpha,+\infty[}(\Phi(x)+\lambda\Psi(x)+h(\lambda)),
\end{aligned}$$

consequently for every $\beta > \alpha$ there exists $x_\beta \in X$ such that

$$\sup_{\lambda\in[\alpha,\beta]} (\Phi(x_\beta) + \lambda\Psi(x_\beta) + h(\lambda)) < \gamma$$

in particular $\Phi(x_\beta) + \alpha\Psi(x_\beta) < \gamma - h(\alpha)$, that implies $\{x_\beta\}_{\beta>\alpha}$ is bounded owing to the weak coerciveness of the functional $\Phi(\cdot) + \alpha\Psi(\cdot)$. Therefore there exist $x^* \in X$ and a subsequence $\{x_{n_k}\}_{k\in\mathrm{N}}$ with $x_{n_k} \to x^*$ when $k \to +\infty$. Fix $\delta > \alpha$, there exists $\bar{k} \in N$ such that $n_k \geq \delta$ for each $k \geq \bar{k}$, moreover the function $x \in X \to \sup_{\lambda\in[\alpha,\delta]} (\Phi(x) + \lambda\Psi(x) + h(\lambda))$ is weakly sequentially l.s.c., then it is

$$\sup_{\lambda\in[\alpha,\delta]} (\Phi(x^*) + \lambda\Psi(x^*) + h(\lambda)) \leq \gamma.$$

Because of the arbitrariness of $\delta > \alpha$, it follows that

$$\sup_{\lambda\in[\alpha,+\infty[} (\Phi(x^*) + \lambda\Psi(x^*) + h(\lambda)) \leq \gamma,$$

from which, we obtain the absurd

$$\inf_{x\in X} \sup_{\lambda\in[\alpha,+\infty[} (\Phi(x) + \lambda\Psi(x) + h(\lambda)) \leq \gamma < \inf_{x\in X} \sup_{\lambda\in[\alpha,+\infty[} (\Phi(x) + \lambda\Psi(x) + h(\lambda)).$$

Thus, at this point, we have for each $\alpha > 0$

$$\sup_{\lambda\in[\alpha,+\infty[} \inf_{x\in X} (\Phi(x) + \lambda\Psi(x) + h(\lambda)) = \inf_{x\in X} \sup_{\lambda\in[\alpha,+\infty[} (\Phi(x) + \lambda\Psi(x) + h(\lambda)),$$

from which, it follows that

$$\sup_{\lambda\in]0,+\infty[} \inf_{x\in X} (\Phi(x) + \lambda\Psi(x) + h(\lambda)) = \sup_{\alpha>0} \inf_{x\in X} \sup_{\lambda\in[\alpha,+\infty[} (\Phi(x) + \lambda\Psi(x) + h(\lambda)).$$

It is easily seen, by similar arguments as above, that

$$\sup_{\alpha>0} \inf_{x\in X} \sup_{\lambda\in[\alpha,+\infty[} (\Phi(x) + \lambda\Psi(x) + h(\lambda))$$
$$= \inf_{x\in X} \sup_{\lambda\in]0,+\infty[} (\Phi(x) + \lambda\Psi(x) + h(\lambda)).$$

Since h is concave, for each $x \in X$ the function $\lambda \in [0, +\infty[\to \Phi(x) + \lambda\Psi(x) + h(\lambda)$ is l.s.c., then

$$\sup_{\lambda\in[0,+\infty[} (\Phi(x) + \lambda\Psi(x) + h(\lambda)) = \sup_{\lambda\in]0,+\infty[} (\Phi(x) + \lambda\Psi(x) + h(\lambda)),$$

therefore

$$\sup_{\lambda\in]0,+\infty[} \inf_{x\in X} (\Phi(x)+ \lambda\Psi(x)+ h(\lambda)) = \inf_{x\in X} \sup_{\lambda\in[0,+\infty[} (\Phi(x) + \lambda\Psi(x)+\lambda(\lambda)),$$

that implies the thesis.

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