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Some Opial, Lyapunov, and De la Valée Poussin Inequalities with Nonhomogeneous Boundary Conditions

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We derive Opial-type inequalities for a class of real functions satisfying nonhomogenous boundary conditions and determine the best constant and extremals. The results are then used to obtain generalized Lyapunov and De la Valée Poussin inequalities.

Keywords: Opial inequality; Lyapunov inequality; De la Valée Poussin inequality; Nonhomogeneous boundary conditions; Best constants; Extremals

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1 INTRODUCTION

In 1960 Z. Opial [12] proved a slightly less general form[†] of the following inequality.

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[†] Opial required that y' be continuous and $y > 0$. Also he did not characterize the extremals. The version we give is due to C. Olech [10].

THEOREM A *If y is absolutely continuous on the interval* $[a,b]$, $-\infty < a < b < \infty$, *and* $y(a)=y(b)=0$ *then*

$$\int_a^b |yy'|\,\mathrm{d}x \le \frac{b-a}{4}\int_a^b (y')^2\,\mathrm{d}x. \tag{1.1}$$

Equality holds in (1.1) *if and only if*

$$y = \begin{cases} c(x-a), & \text{if } a \le x \le (a+b)/2, \\ c(b-x), & \text{if } (a+b)/2 \le x \le b. \end{cases}$$

Since 1960 there have been numerous generalizations of Opial's inequality. Many of these extensions contain weights on one or both sides of (1.1), various Lebesgue norms on y or on either or both of the two y' terms, alternative boundary conditions on y, etc. See for example the books of Agarwal and Pang [1] or Mitrinović *et al.* [9, Chapter III]. Both sources contain modern proofs of the inequality and Opial's original proof may also be found in [1]. There have also been recent attempts (see e.g. [5,7]) to prove the inequality and find its best constant when $\int_a^b |y'|^2\mathrm{d}x$ is replaced by $\int_a^b |y^{(n)}|^2\,\mathrm{d}x$ under the boundary conditions

$$y(a) = y'(a) = \cdots = y^{(n-1)}(a) = 0,$$
$$y(b) = y'(b) = \cdots = y^{(n-1)}(b) = 0$$

for $n > 1$.

Opial-type inequalities have many applications. These include the establishment of sufficient conditions for disfocality and disconjugacy, the determination of good lower bounds for the spacing of zeros of solutions of a second order linear differential equation, and proofs of Lyapunov-type or De la Vallée Poussin inequalities.

In this paper we consider a "nonhomogeneous" version of Opial's inequality. Specifically, we ask what can be said about the existence, best constant $C(y_a, y_b)$, and extremals of the inequality

$$\int_a^b |yy'|\,\mathrm{d}x \le C(y_a, y_b)\int_a^b (y')^2\,\mathrm{d}x \tag{1.2}$$

under the boundary conditions

$$y(a) = y_a, \quad y(b) = y_b \tag{1.3}$$

where y_a, y_b are viewed as fixed numbers and at least one of them is nonzero. This setting brings surprising complications and results. Our analysis of it is summarized in Theorem 1 of Section 2. Here we are able to determine the constants $C(y_a, y_b)$ and to characterize extremals for all choices of y_a, y_b. One of our results is unexpected. In the case that $y_a \cdot y_b < 0$ the extremal is *not* the line segment joining (a, y_a) and (b, y_b) but rather a spline with a true interior knot. In Section 3 by using a technique due to Beesack we derive a new Opial-type inequality with nonhomogeneous boundary conditions. This inequality in turn yields an alternative proof of (1.2)–(1.3) when $y_a \cdot y_b \geq 0$. Sections 4 and 5 contain applications: we use the nonhomogeneous Opial inequalities of Theorem 1 to derive some general extensions of the classical Lyapunov and De la Vallée Poussin inequalities. These results may be viewed as giving necessary conditions for interpolation of data by solutions of second order linear differential equations.

Before proceeding further we fix notation: $AC[a, b]$, $\mathcal{L}[a, b]$, $\mathcal{L}^2[a, b]$ or $\mathcal{L}^\infty[a, b]$ stand respectively for the classes of absolutely continuous, Lebesgue integrable, Lebesgue square integrable, or essentially bounded real functions on the finite interval $[a, b]$. We denote the corresponding norm of a function y (or strictly speaking the equivalence class of functions a.e. equal to y) in these Lebesgue spaces by $\|y\|_{1,[a,b]}$, $\|y\|_{2,[a,b]}$, or $\|y\|_{\infty,[a,b]}$. Lastly, in the two point case our generalized Opial inequalities will be defined on the domain $\mathcal{D}(y_a, y_b)$ where

$$\mathcal{D}(y_a, y_b) = \{y \in AC[a, b]\colon\ y' \in L^2[a, b];\ y(a) = y_a \text{ and } y(b) = y_b\}.$$

2 EXISTENCE AND PROPERTIES OF EXTREMALS

It will be convenient to reduce the Opial inequality (1.2)–(1.3) to the solution of a certain minimization problem since the best constant $C(y_a, y_b)$ of (1.2) is just $K(y_a, y_b)^{-1}$ where

$$K(y_a, y_b) := \inf\{J(y)\colon\ y \in \mathcal{D}(y_a, y_b)\} \tag{2.1}$$

and

$$J(y) = \frac{\int_a^b (y')^2 \, dx}{\int_a^b |yy'| \, dx}.$$

Moreover, $y \in \mathcal{D}(y_a, y_b)$ clearly gives equality in (1.2) if and only if it is an extremal of (2.1).

DEFINITION 1 *Given an arbitrary* $\epsilon > 0$, *we call a function* $s \in \mathcal{D}(y_a, y_b)$ *an* ϵ*-approximate extremal if*

$$K(y_a, y_b) \leq J(s) \leq K(y_a, y_b) + \epsilon.$$

It is obvious from the definition of $K(y_a, y_b)$ that ϵ-approximate extremals exist for every $\epsilon > 0$.

DEFINITION 2 *Suppose* $y \in \mathcal{D}(y_a, y_b)$ *is such that there exists a partition* $\mathbb{P}$: $a = x_0 < x_1 < \cdots < x_n = b$ *so that* y *is monotone on each subinterval* $[x_{i-1}, x_i]$. *Let* $|\mathbb{P}|$ *be the number of subintervals of* $\mathbb{P}$ *and* $\mathcal{C}(y)$ *denote the set of all such partitions. Finally, we define*

$$|\mathcal{C}(y)| := \min_{\mathbb{P} \in \mathcal{C}(y)} |\mathbb{P}| - 1.$$

Note that $|\mathcal{C}(y)| = 0$ if and only if y is monotone on $[a, b]$. In general there may be more than one partition in $\mathcal{C}(y)$ which yields $|\mathcal{C}(y)|$. For piecewise strictly monotone functions $|\mathcal{C}(y)|$ may be identified with the number of local maxima or minima of y in (a, b).

We now prove:

LEMMA 1 *For any* $\epsilon > 0$ *there exists an* ϵ*-approximate extremal* $\hat{y}$ *such that* $|\mathcal{C}(\hat{y})|$ *is either zero or one.*

Proof Let y be an ϵ-approximate extremal.

Case (i) Suppose $0 \leq y_a \leq y_b$. Since $y_b - y_a \leq \int_a^b |y'| \, dx$, there exists $c \in (a, b]$ such that

$$y_a + \int_a^c |y'| \, dx = y_b + \int_c^b |y'| \, dx.$$

Define $\hat{y}$ on $[a, b]$ by

$$\hat{y}(x) = \begin{cases} y_a + \int_a^x |y'|\,\mathrm{d}t, & \text{if } a \le x \le c, \\ y_b + \int_x^b |y'|\,\mathrm{d}t, & \text{if } c \le x \le b. \end{cases}$$

Now

$$\hat{y}'(x) = |y'(x)|, \qquad a \le x < c,$$

$$\hat{y}'(x) = -|y'(x)|, \quad c < x \le b,$$

so that $|\hat{y}'(x)| = |y'(x)|$ a.e. Furthermore,

$$|y(x)| = \left| y_a + \int_a^x y'\,\mathrm{d}t \right| \le y_a + \int_a^x |y'|\,\mathrm{d}t = \hat{y}(x), \quad a \le x \le c,$$

$$|y(x)| = \left| y_b - \int_x^b y'\,\mathrm{d}t \right| \le y_b + \int_x^b |y'|\,\mathrm{d}t = \hat{y}(x), \quad c \le x \le b.$$

Hence,

$$J(\hat{y}) = \frac{\int_a^b (\hat{y}')^2\,\mathrm{d}x}{\int_a^b |\hat{y}\hat{y}'|\,\mathrm{d}x} \le \frac{\int_a^b (y')^2\,\mathrm{d}x}{\int_a^b |yy'|\,\mathrm{d}x} = J(y).$$

Thus $\hat{y}$ must be an ϵ-approximate extremal and $|\mathcal{C}(\hat{y})| \le 1$.

Case (ii) Suppose $y_a < 0 < y_b$ and $|y_a| \le y_b$. By the intermediate value theorem there is at least one zero of y which we call z_1 to the right of a. We first apply the construction of Case (i) to y on $[a, z_1]$ and $[z_1, b]$, i.e., define $\hat{y}$ by

$$\hat{y}_1(x) = \begin{cases} y_a - \int_a^x |y'|\,\mathrm{d}t, & a \le x \le c_1, \\ -\int_x^{z_1} |y'|\,\mathrm{d}t, & c_1 \le x \le z_1, \\ \int_{z_1}^x |y'|\,\mathrm{d}t, & z_1 \le x \le c_2, \\ y_b - \int_x^b |y'|\,\mathrm{d}t, & c_2 < x \le b, \end{cases}$$

where $c_1 \in [a, z_1)$ and $c_2 \in (z_1, b]$ are determined by

$$y_a - \int_a^{c_1} |y'| \, dx = -\int_{c_1}^{z_1} |y'| \, dx,$$

$$y_b + \int_{c_2}^{b} |y'| \, dx = \int_{z_1}^{c_2} |y'| \, dx.$$

Then as in Case (i), $|\hat{y}_1(x)| \geq |y(x)|$, $|\hat{y}_1'(x)| = |y'(x)|$, a.e. so that $J(\hat{y}_1) \leq J(y)$. Note also that $|\mathcal{C}(\hat{y}_1)| \leq 2$.

Now define $\hat{y}_2$ by

$$\hat{y}_2 = \begin{cases} \hat{y}_1(a) + \hat{y}_1(c_2) - \hat{y}_1(-x + a + c_2), & a \leq x \leq c_2, \\ \hat{y}_1(x), & c_2 < x \leq b. \end{cases}$$

Then $\hat{y}_2'(x) = \hat{y}_1'(-x + a + c_2)$ on $a \leq x \leq c_2$ so that

$$\int_a^b (\hat{y}_2')^2 \, dt = \int_a^b (\hat{y}_1')^2 \, dt$$

(as a consequence of the transformation $u = -x + a + c_2$). Furthermore

$$\hat{y}_2'(a^+) = \hat{y}_1'(c_2^-) \geq 0, \quad \hat{y}_2'(c_2^-) = \hat{y}_1'(a^+) \leq 0,$$

so that $|\mathcal{C}(\hat{y}_2)| \leq 1$. Let $y_a = \hat{y}_1(a), y_{c_1} = \hat{y}_1(c_1), y_{c_2} = \hat{y}_1(c_2)$. Note that the maximum of $\hat{y}_2$ occurs where

$$-x + a + c_2 = c_1 \Leftrightarrow x = a + c_2 - c_1,$$

and the maximum value

$$\hat{y}_{2,\max} = \hat{y}_1(a) + \hat{y}_2(c_2) - \hat{y}_1(c_1) = y_a + y_{c_2} - y_{c_1}.$$

Then

$$\begin{aligned} \int_a^{c_2} |\hat{y}_2 \hat{y}_2'| \, dt &= (y_a^2 - y_{c_2}^2)/2 + \hat{y}_{2,\max}^2 \\ &= (1/2)\{y_a^2 + 2[y_a + y_{c_2} - y_{c_1}]^2 - y_{c_2}^2\}, \end{aligned}$$

and

$$\int_a^{c_2} |\hat{y}_1\hat{y}_1'|\,\mathrm{d}t = (y_{c_2}^2 - y_a^2)/2 + y_{c_1}^2.$$

It follows that

$$\begin{aligned}
&\int_a^{c_2} |\hat{y}_2\hat{y}_2'|\,\mathrm{d}t - \int_a^{c_2} |\hat{y}_1\hat{y}_1'|\,\mathrm{d}t \\
&\quad = (1/2)\{y_a^2 + 2[y_a + y_{c_2} - y_{c_1}]^2 - y_{c_2}^2 + y_a^2 - 2y_{c_1}^2 - y_{c_2}^2\} \\
&\quad = 2y_a^2 + 2y_a y_{c_2} - 2y_a y_{c_1} - 2y_{c_2}y_{c_1} \\
&\quad = 2(y_a + y_{c_2})(y_a - y_{c_1}) \geq 0,
\end{aligned}$$

since $y_a + y_{c_2} \geq y_a + y_b \geq 0$ and $y(c_1) \leq y(a) \equiv y_a$. We conclude that $J(\hat{y}_2) \leq J(\hat{y}_1)$.

To complete the proof of the lemma we reduce every other case to either Case (i) or Case (ii) by transformations. In particular we can always assume $y_b \geq 0$ and $y_a \leq y_b$. For if y_b is negative we consider the function $\hat{y}_1 = -\hat{y}$. Then $\hat{y}_1$ is an ϵ-approximate extremal such that $|\mathcal{C}(\hat{y}_1)|$ is minimal for the minimization problem (2.1) on $\mathcal{D}(-y_a, -y_b)$. And if $y_a > y_b$ we set $\hat{y}_2(x) = \hat{y}(b + a - x)$. This function is an ϵ-approximate extremal such that $|\mathcal{C}(\hat{y}_2)|$ is minimal for the minimization problem on $\mathcal{D}(y_b, y_a)$. In other words we can find ϵ-approximate extremals for a transformed problem in Case (i) or (ii) and then transform back to the original problem.

We interpose the following lemma which will be required when we prove the uniqueness of extremals to (1.2)–(1.3).

LEMMA 2 *If y is an extremal of (2.1) then $|\mathcal{C}(y)| \leq 1$.*

Proof Let $y \in \mathcal{D}(y_a, y_b)$ be an extremal of (2.1) and $\hat{y}$ be constructed as in Lemma 1. Then we must have that $J(\hat{y}) = J(y)$. Then $\int_a^b |\hat{y}\hat{y}'|\,\mathrm{d}x = \int_a^b |yy'|\,\mathrm{d}x$ since $\int_a^b (\hat{y}')^2\,\mathrm{d}x = \int_a^b (y')^2\,\mathrm{d}x$.

Suppose Case (i) applies and let c be defined as in this case. Then since the definition of $\hat{y}$ implies that $|\hat{y}'| = |y'|$ and $|\hat{y}| \geq |y|$ a.e. so that with $\int_a^b |\hat{y}\hat{y}'|\,\mathrm{d}x = \int_a^b |yy'|\,\mathrm{d}x$ we conclude that

$$\int_a^c |\hat{y}\hat{y}'|\,\mathrm{d}x = \int_a^c |yy'|\,\mathrm{d}x, \quad \int_c^b |\hat{y}\hat{y}'|\,\mathrm{d}x = \int_c^b |yy'|\,\mathrm{d}x.$$

From the first of these equalities and $|\hat{y}'| = |y'|$, we have

$$0 = \int_a^c [|\hat{y}| - |y|]|\hat{y}'|\,\mathrm{d}x \Rightarrow [|\hat{y}| - |y|]|\hat{y}'| = 0, \text{ a.e.} \tag{2.2}$$

Suppose $|\hat{y}(x_0)| > |y(x_0)|$. Then because of the continuity of $|\hat{y}|$ there is a left neighbourhood $\Delta = (x_1, x_0)$ such that $|\hat{y}| > |y|$ on Δ and $y(x_1) = \hat{y}(x_1)$. Therefore from (2.2) $|\hat{y}'| = 0 \Rightarrow |y'| = 0$ a.e. on Δ. This means that $\hat{y}$ and y are constant on Δ so that $|\hat{y}(x_0)| = |y(x_0)|$ contrary to assumption. Hence $|\hat{y}| = |y|$ on $[a, c]$ and similarly on $[c, b]$. Hence $\hat{y} = y$ on $[a, b]$ as both are nonnegative. This shows that $J(\hat{y}) = J(y) \Rightarrow \hat{y} \equiv y$.

In Case (ii) if y is an extremal $J(\hat{y}_1) = J(y)$ the above argument shows that $\hat{y}_1 = y$ on $[a, b]$. Suppose now $J(\hat{y}_2) = J(y)$. Then $J(\hat{y}_2) = J(\hat{y}_1)$ as $J(\hat{y}_2) \le J(\hat{y}_1) \le J(y)$. Returning to the proof in Lemma 1 that $J(\hat{y}_2) \le J(\hat{y}_1)$, we see that either $y_a + y_{c_2} = 0$ or $y_a - y_{c_1} = 0$. However if $\hat{y}(c_1) < y_a$ then $y_{c_2} > y_b \ge -y_a$, so that $\hat{y}(c_1) = y_a$. This means however that $|\mathcal{C}(\hat{y}_2)| = |\mathcal{C}(\hat{y}_1)| \le 1$. (Note that the possibility that y is a line segment ($\Rightarrow y = \hat{y}_1 = \hat{y}_2$) has not been ruled out.)

Thus for both Cases (i) and (ii) of Lemma 1, if y is an extremal then $|\mathcal{C}(y)| \le 1$.

LEMMA 3 *For any $\epsilon > 0$ there exists an ϵ-approximate extremal s_ϵ which is a linear spline with at most one knot.*

Proof By Lemma 1 there is an ϵ-approximate extremal y_ϵ such that $|\mathcal{C}(y)|$ is either zero or one in which case y_ϵ is monotone on the intervals $[a, c_\epsilon]$, $[c_\epsilon, b]$ for some c_ϵ, $a < c_\epsilon < b$. If $|\mathcal{C}(y)|$ is zero let s_ϵ be the straight line joining (a, y_a) and (b, y_b). Otherwise let s_ϵ be the linear spline interpolating y at a, b and c_ϵ. Then s_ϵ and y_ϵ are both monotone on the intervals $[a, c_\epsilon]$ and $[c_\epsilon, b]$. This means that both $\int_a^b |s_\epsilon s_\epsilon'|\,\mathrm{d}x$ and $\int_a^b |y_\epsilon y_\epsilon'|\,\mathrm{d}x$ depend on the values of s_ϵ and y_ϵ at a, b and c_ϵ. For example,

$$\int_a^b |y_\epsilon y_\epsilon'|\,\mathrm{d}x = \begin{cases} y(c)^2 - (y_a^2 + y_b^2)/2, & \text{if } 0 \le y_a \le y_b \le y(c), \\ (y_a^2 + y_b^2)/2 - y(c)^2, & \text{if } 0 \le y(c) \le y_a \le y_b, \\ (y_a^2 + y_b^2)/2 + y(c)^2, & \text{if } y(c) < 0 \le y_a \le y_b. \end{cases}$$

Similar formulas are satisfied if $y_a > y_b \ge 0$ or $y_a \cdot y_b < 0$, etc. Since y_ϵ and s_ϵ agree at a, b and c_ϵ, we conclude that for all configurations of

y_a, y_b and $y(c_\epsilon) = s_\epsilon(c_\epsilon)$

$$\int_a^b |s_\epsilon s_\epsilon'| \,\mathrm{d}x = \int_a^b |y_\epsilon y_\epsilon'| \,\mathrm{d}x.$$

But from the well known fact that s_ϵ is the unique minimizer of the L^2 norm of z' over all interpolants z in $\mathcal{D}(y_a, y_b)$ which agree with s at a, b or at a, b, c_ϵ,

$$\int_a^b (s_\epsilon')^2 \,\mathrm{d}x \leq \int_a^b (y_\epsilon')^2 \,\mathrm{d}x.$$

These two facts imply that $J(s_\epsilon) \leq J(y_\epsilon)$.

Since $\int_a^b |s_\epsilon s_\epsilon'| \,\mathrm{d}x = \int_a^b |y_\epsilon y_\epsilon'| \,\mathrm{d}x$ and s_ϵ is the unique minimizer of $\int_a^b |y|^2 \,\mathrm{d}x$ over $y \in \mathcal{D}(y_a, y_b)$ such that $y(c_\epsilon) = y_\epsilon(c_\epsilon)$, $J(s_\epsilon) < J(y_\epsilon)$ if $s_\epsilon \neq y_\epsilon$. Thus it is sufficient to consider only linear splines with at most one knot to compute $K(y_a, y_b)$. Moreover, if s is an extremal of (2.1), then by Lemma 2 $|\mathcal{C}(s)| \leq 1$. Therefore, repeating the argument of Lemma 3 with s playing the role of y_ϵ and the above remarks will show an extremal if it exists must be a linear spline. We will use these observations in Theorem 1 below to establish the existence, structure, and uniqueness of extremals. First however, we require a simplifying lemma.

LEMMA 4 *If s_ϵ is a linear spline ϵ-approximate extremal with knot $a < c_\epsilon < b$ there is a linear spline ϵ-approximate extremal $\tilde{s}_\epsilon$ with $J(\tilde{s}_\epsilon) \leq J(s_\epsilon)$ and with knot $c^* \equiv c^*(s_\epsilon(c_\epsilon))$ given by*

$$c^* = \frac{b\Delta_a + a\Delta_b}{\Delta_a + \Delta_b} \tag{2.3}$$

where $\Delta_a := |y_a - s_\epsilon(c_\epsilon)|$ and $\Delta_b = |y_b - s_\epsilon(c_\epsilon)|$. Also

$$J(\tilde{s}_\epsilon) = \left(\frac{1}{b-a}\right)\left(\frac{(\Delta_a + \Delta_b)^2}{\int_a^b |\tilde{s}_\epsilon \tilde{s}_\epsilon'| \,\mathrm{d}x}\right). \tag{2.4}$$

Proof Let s_ϵ be a linear spline ϵ-approximate extremal with knot $a < c \equiv c_\epsilon < b$. Set $h_\epsilon = s_\epsilon(c_\epsilon)$. We regard h_ϵ as fixed and consider the family $\mathcal{F}$ of all linear splines $s_{c,\epsilon} \in \mathcal{D}(y_a, y_b)$ with unique knot c somewhere in $[a, b]$ such that $s_{c,\epsilon}(c) = h_\epsilon$. Then a direct calculation shows

$$J(s_{c,\epsilon}) = \frac{((y_a - h_\epsilon)/(c-a))^2(c-a) + ((y_b - h_\epsilon)/(b-c))^2(b-c)}{\int_a^b |s_{c,\epsilon} s'_{c,\epsilon}| \, \mathrm{d}x}. \tag{2.5}$$

As we have seen in the proof of Lemma 3 because of the monotonicity of the $s_{c,\epsilon}$ on the intervals (a, c) and (c, b) the integral $\int_a^b |s_{c,\epsilon} s'_{c,\epsilon}| \, \mathrm{d}x$ will depend only on y_a, y_b, and h_ϵ. Different configurations of y_a, y_b and h_ϵ give different values for the integral, but in all cases they are independent of the location of the knot c.

Since $s_\epsilon \in \mathcal{F}$ it follows that

$$K(y_a, y_b) \leq \min_{a<c<b} J(s_{c,\epsilon}) \leq J(s_\epsilon) \leq K(y_a, y_b) + \epsilon;$$

also because the denominator of (2.5) is independent of c, the minimum is attained at that $c^*(h_\epsilon)$ which is the solution of

$$\min_{a<c<b} \left\{ \left(\frac{y_a - h_\epsilon}{c-a} \right)^2 (c-a) + \left(\frac{y_b - h_\epsilon}{b-c} \right)^2 (b-c) \right\}.$$

The standard calculus argument shows that $c^*(h_\epsilon)$ is given by (2.3) and it is immediate that $\tilde{s}_\epsilon$ is an ϵ-approximate extremal. Equation (2.4) follows by substituting c^* into (2.5).

THEOREM 1 *Set $M = \max\{y_a, y_b\}$ and $m = \min\{y_a, y_b\}$. The values of $K(y_a, y_b)$ together with corresponding extremals are given by the following possibilities*

(i) *If y_a, y_b are not both zero and $y_a \cdot y_b \geq 0$, then the line segment L_{ab} joining (a, y_a) and (b, y_b) is the unique extremal and*

$$K(y_a, y_b) = \left(\frac{2}{b-a} \right) \frac{M - m}{|M + m|}.$$

In the case $y_a = y_b$ there is no extremal and $K(y_a, y_b) = 0$.

(ii) *If* $y_a = y_b = 0$, $K(0,0) = 4/(b-a)$, *and* y *is an extremal if and only if* y *is a multiple of*

$$y_0(x) = \begin{cases} (x-a), & \text{if } a \le x \le (a+b)/2, \\ (b-x), & \text{if } (a+b)/2 \le x \le b. \end{cases}$$

(iii) *If* $y_a \cdot y_b < 0$, *and* $y_a + y_b > 0$, *then*

$$K(y_a, y_b) = \left(\frac{2}{b-a}\right)\left(\frac{M+3|m|}{M+|m|}\right).$$

The corresponding extremal y *which is unique is the linear spline having a unique knot at* (c^*, h^*) *where*

$$\begin{aligned} c^* &= \frac{b(M+2|m|)+a|m|}{M+3|m|}, \\ h^* &= M+|m|. \end{aligned} \tag{2.6}$$

(iv) *If* $y_a \cdot y_b < 0$, *and* $y_a + y_b < 0$, *then*

$$K(y_a, y_b) = \left(\frac{2}{b-a}\right)\left(\frac{3M+|m|}{M+|m|}\right).$$

The corresponding extremal y *which is unique is the linear spline having a unique knot at* (c^*, h^*) *where*

$$\begin{aligned} c^* &= \frac{bM+a(2M+|m|)}{3M+|m|}, \\ h^* &= m-M. \end{aligned} \tag{2.7}$$

(v) *If* $y_a + y_b = 0$ *then* y *is an extremal if and only if it is one of the linear splines in* $\mathcal{D}(y_a, y_b)$ *with a unique interior knot* $c^*(h)$, *for* $h \in \mathbb{R}\backslash[m, M]$ *or the line segment* L_{ab}. *Here*

$$K(y_a, y_b) = \frac{4}{b-a}. \tag{2.8}$$

Proof First we observe that $J(-y) = J(y) = J(\hat{y})$ where $\hat{y}(x) = y(a+b-x)$. Although these transformations do not leave the class

$\mathcal{D}(y_a, y_b)$ invariant, they do allow us when computing $K(y_a, y_b)$, to consider fewer cases. Thus for Case (i) it is sufficient to consider only $0 \le y_a < y_b$ and for Case (iii) it is sufficient to consider only $y_a < 0 < y_b$. (See also Lemma 1.)

In all cases we will begin with $\tilde{s}_\epsilon$ the ϵ-approximate extremal with knot $c^*(h_\epsilon)$ given by Lemma 4. Let $\mathcal{T}$ be the family of all linear splines s^h in $\mathcal{D}(y_a, y_b)$ having at most one knot at $c^*(h)$ where $h \in \mathbb{R}$. We think of h as a parameter and write

$$J(s^h) = \left(\frac{2}{b-a}\right) Q(h)$$

where (cf. (2.4))

$$Q(h) = \frac{(|y_a - h| + |y_b - h|)^2}{2\int_a^b |s^h s^{h\prime}|\,\mathrm{d}x}. \tag{2.9}$$

Clearly since $\tilde{s}_\epsilon \in \mathcal{T}$,

$$K(y_a, y_b) \le \min_{h\in\mathbb{R}} \left(\frac{2}{b-a}\right) Q(h) \equiv \min_{s^h \in \mathcal{T}} J(s^h) \le J(\tilde{s}_\epsilon) \le K(y_a, y_b) + \epsilon.$$

In particular if h^* minimizes $Q(h)$ and s^* is the corresponding spline then

$$K(y_a, y_b) \le J(s^*) \le J(\tilde{s}_\epsilon) \le K(y_a, y_b) + \epsilon,$$

so that s^* is also an ϵ-approximate extremal.

Case (*i*) As we have pointed out it suffices to suppose that $0 \le y_a < y_b$. If $h \ge 0$, then a calculation shows

$$\int_a^b |s^h s^{h\prime}|\,\mathrm{d}x = \tfrac{1}{2}\{|h^2 - y_a^2| + |h^2 - y_b^2|\}; \tag{2.10}$$

while if $h < 0$, we obtain

$$\int_a^b |s^h s^{h\prime}|\,\mathrm{d}x = \tfrac{1}{2}\{2h^2 + y_a^2 + y_b^2\}. \tag{2.11}$$

Substituting (2.10) and (2.11) into (2.9) we find that $Q(h)$ is given by

$$Q(h)=\begin{cases}\dfrac{(2h-y_a-y_b)^2}{2h^2-y_a^2-y_b^2}, & h\geq y_b,\\ \dfrac{y_b-y_a}{y_a+y_b}, & y_a<h<y_b,\\ \dfrac{(y_a+y_b-2h)^2}{y_a^2+y_b^2-2h^2}, & 0\leq h\leq y_a,\\ \dfrac{(y_a+y_b-2h)^2}{y_a^2+y_b^2-2h^2}, & h<0.\end{cases}$$

To compute $K(y_a,y_b)$ we need to minimize the function $Q(h)$. Note that $Q(h)$ is continuous. Examination of this function yields that

(a) $Q'(h)>0$ for $h>y_b$,
(b) $Q'(h)=0$ for $y_a<h<y_b$,
(c) $Q'(h)<0$ for $-[(y_a^2+y_b^2)/(y_a+y_b)]<h<y_a$,
(d) $Q'(h)>0$ for $h<-[(y_a^2+y_b^2)/(y_a+y_b)]$,
(e) $\lim_{h\to\pm\infty}Q(h)=2$.

These facts show that the minimum value of $Q(h)$ is $(y_b-y_a)/(y_a+y_b)$ giving a value

$$K(y_a,y_b)=\left(\frac{2}{b-a}\right)\frac{y_b-y_a}{y_b+y_a}=\left(\frac{2}{b-a}\right)\frac{M-m}{M+m}.$$

Furthermore, the h^* producing the minimum is any element of $[y_a,y_b]$. This seems to correspond to infinitely many ϵ-approximate extremals which we can take for s^*, consisting of the linear splines in $\mathcal{D}(y_a,y_b)$ with an interior knot

$$c^*(h)=\frac{b(h-y_a)+a(y_b-h)}{y_b-y_a}$$

where $h\in[y_a,y_b]$. But this is an illusion since $(c^*(h),h)$ can be easily shown to lie on the line L_{ab}. It follows that $s^*=L_{ab}$ and since this function is independent of ϵ it is an extremal.

In case $y_a = y_b > 0$ we have always that $c^*(h) = (a+b)/2$ and for all $h > y_a$

$$J(s^h) = \left(\frac{4}{b-a}\right)\left(\frac{(h-y_a)^2}{h^2-y_a^2}\right) = \left(\frac{4}{b-a}\right)\left(\frac{h-y_a}{h+y_a}\right).$$

Consequently we can let $h \to y_a$, obtaining that $K(y_a, y_b) = 0$. Therefore if y is an extremal $\int_a^b (y')^2\,\mathrm{d}x = 0$ so that y must be L_{ab}. But then $J(s^*)$ is undefined so there is no extremal.

Case (ii) By Lemmas 3 and 4 there is an ϵ-approximate linear spline extremal $\tilde{s}_\epsilon$ with a knot at

$$c^*(h_\epsilon) = \frac{bh_\epsilon + ah_\epsilon}{2h_\epsilon} = \frac{a+b}{2},$$

where $h_\epsilon = s(c_\epsilon)$. By (2.4) and the fact that formulas (2.10) and (2.11) may be used to evaluate $\int_a^b |\tilde{s}_\epsilon \tilde{s}_\epsilon'|\,\mathrm{d}x$

$$J(\tilde{s}_\epsilon) = \left(\frac{1}{b-a}\right)\left(\frac{(2h_\epsilon)^2}{h_\epsilon^2}\right) = \frac{4}{b-a}.$$

Again since $J(\tilde{s}_\epsilon)$ is independent of ϵ or multiplication of $\tilde{s}_\epsilon$ by a nonzero constant this function or any nontrivial constant multiple of it is an extremal.

Case (iii) As noted above it is enough to take the case $y_a < 0 < y_b$. Again let s^h be the ϵ-approximate spline extremal with knot $(c^*(h), h)$. If $h \geq 0$ we find that

$$\int_a^b |s^h s^{h\prime}|\,\mathrm{d}x = \tfrac{1}{2}(h^2 + y_a^2 + |h^2 - y_b^2|),$$

while if $h < 0$

$$\int_a^b |s^h s^{h\prime}|\,\mathrm{d}x = \tfrac{1}{2}(|h^2 - y_a^2| + h^2 + y_b^2).$$

Again $J(s^h)$ has the form (2.9) but now $Q(h)$ is given by

$$Q(h) = \begin{cases} \dfrac{(2h - y_a - y_b)^2}{2h^2 + y_a^2 - y_b^2}, & h \leq y_b, \\ \dfrac{(y_b - y_a)^2}{y_b^2 + y_a^2}, & y_a < h < y_b, \\ \dfrac{(y_a + y_b - 2h)^2}{2h^2 + y_b^2 - y_a^2}, & h \leq y_a. \end{cases} \tag{2.12}$$

Set $h_1 = y_b - y_a$ and suppose that $y_a + y_b > 0$. We find that

(a) $Q'(h) > 0$ if $h > h_1$,
(b) $Q'(h) < 0$ if $y_b < h < h_1$,
(c) $Q'(h) = 0$ if $y_a < h < y_b$,
(d) $Q'(h) > 0$ if $-h_1 < h < y_a$,
(e) $Q'(h) < 0$ if $h < -h_1$,
(f) $\lim_{h \to \pm\infty} Q(h) = 2$.

It follows that $Q(h)$ has a unique minimum at $h^* = h_1$ and

$$Q(h^*) = \frac{y_b - 3y_a}{y_b - y_a} = \frac{M - 3m}{M - m},$$

so that

$$K(y_a, y_b) = \left(\frac{2}{b-a}\right) \frac{M + 3|m|}{M + |m|}.$$

Using (2.3) the interior knot (c^*, h^*) is given by (2.6).

Case (iv) Again we take $y_a < 0 < y_b$ and assume that $y_a + y_b < 0$. $Q(h)$ is given by (2.12) but the signs of $Q'(h)$ in (a)–(e) above are reversed. Consequently Q has a unique minimum at $h^* = -h_1 = m - M$,

$$Q(h^*) = \frac{3y_b - y_a}{y_b - y_a} = \frac{3M + |m|}{M + |m|},$$

$$K(y_a, y_b) = \left(\frac{2}{b-a}\right) \frac{3M + |m|}{M + |m|},$$

and the interior knot (c^*, h^*) satisfies (2.7). Note that in both this case and Case (iii) that the knot no longer lies on L_{ab} since $h^* > y_b$.

Case (*v*) By examining $Q(h)$ given in Cases (iii) or (iv) with $y_a + y_b = 0$ we see that $Q(h) = 2$ for all h. Consequently $J(s^h) = 4/(b-a)$ for all splines in $\mathcal{T}$. Any of these splines will be an ϵ-appproximate extremal for all $\epsilon > 0$ and consequently an extremal. However, in the case $m < h < M$ $(c^*(h), h)$ lies on L_{ab}.

We now prove uniqueness of the given extremals or classes of extremals in Cases (i)–(v). Suppose s is any extremal. By the remarks following Lemma 3 s must be a linear spline with at most one knot $c' \in [a, b]$. Turning to Lemma 4, we think of s as a member of the family $\mathcal{F}$ of linear splines s_c with unique knot $c \in [a, b]$ such that $s_c = h := s(c')$. Since the functional $J(s_c)$ given by (2.5) is seen to have a *unique* minimum at $c = c^*$, we conclude that $c' = c^*$. Finally, the analysis in the first part of the proof of the present Theorem and the fact that the functional $Q(h)$ has a unique minimum in Cases (i), (iii), and (iv) (and is independent of h in Cases (ii) and (v)) shows that s must be of the form stated in Cases (i)–(v).

We remark that Case (ii) amounts to a new proof of Theorem A.[‡] Also as special case of (i) by assuming that $y_a = 0$ we have a well known "half interval" form of Opial's inequality.

COROLLARY 1 *If y is real and absolutely continuous on $[a, b]$ and $y(a) = 0$ then*

$$\int_a^b |yy'|\,\mathrm{d}x \leq \frac{b-a}{2}\int_a^b (y')^2\,\mathrm{d}x.$$

The extremals are given by $y(x) = k(x-a)$ where k is any constant.

We illustrate Case (iii) of Theorem 1 with a simple example. Let $y_a = -1$, $y_b = 2$, and $[a, b] = [0, 1]$. Then $M = 2$, $m = -1$, and $K(y_a, y_b) = 10/3$. The linear spline extremal has a knot at $(c^*, h^*) = (4/5, 3)$. By way of comparison, the line segment $L_{a,b}$ yields $J(L_{ab}) = 3.6$.

3 A SECOND NONHOMOGENEOUS OPIAL-TYPE INEQUALITY

By modifing a technique due to Beesack (see [1, p. 7]) when $y_0 \cdot y_1 \geq 0$ it is possible to derive another nonhomogeneous Opial-type inequality

[‡] At least six are known. See [1, Chapter 1].

which is apparently new and may be of some independent interest. This inequality may be viewed as a "precursor" to (1.2)–(1.3) since it leads to a different proof of the "nonhomogeneous part" of Case (i) of Theorem 1, i.e., the inequality

$$\int_a^b |yy'|\,\mathrm{d}x \le \left(\frac{b-a}{2}\right)\frac{|M+m|}{M-m}\int_a^b (y')^2\,\mathrm{d}x \tag{3.1}$$

when $y_a \cdot y_b > 0$ together with the uniquenes of the extremal L_{ab}.

THEOREM 2 *Let* $\bar{M} := \max\{|y_a|, |y_b|\}$ *and* $\bar{m} := \min\{|y_a|, |y_b|\}$. *If* $y \in \mathcal{D}(y_a, y_b), y_a \neq y_b$, *and* $y_a \cdot y_b > 0$ *then the inequality*

$$\int_a^b |yy'|\,\mathrm{d}x \le \frac{\bar{m}(\bar{M}-\bar{m})}{2} + \left(\frac{b-a}{2}\right)\frac{\bar{M}}{(\bar{M}-\bar{m})}\int_a^b (y')^2\,\mathrm{d}x \tag{3.2}$$

holds and has a unique extremal L_{ab}.

Proof We assume first that $y_a, y_b > 0$ and $y_a < y_b$ (so that $\bar{M} = y_b$ and $\bar{m} = y_a$). Set $L(x) := ((y_b - y_a)/(b-a))(x-a) + y_a > 0$ and consider the inequality

$$\int_a^b (|y'| - (L'/L)|y|)^2\,\mathrm{d}x \ge 0 \tag{3.3}$$

which is an equality if and only if $y(x) = L(x)$. Equivalently,

$$2\int_a^b |yy'|(L'/L)\,\mathrm{d}x \le \int_a^b (y')^2\,\mathrm{d}x + \int_a^b y^2(L'/L)^2\,\mathrm{d}x. \tag{3.4}$$

However, integration by parts and the identity

$$(L'/L)' = -(L'/L)^2$$

give

$$\begin{aligned} &2\int_a^b |yy'|(L'/L)\,\mathrm{d}x \\ &\quad = (2L'/y_b)\int_a^b |yy'|\,\mathrm{d}x + \int_a^b \left(\int_a^x |yy'|\,\mathrm{d}t\right)(L'/L)^2\,\mathrm{d}x. \end{aligned} \tag{3.5}$$

(Note that the integrals exist for all $y \in \mathcal{D}(y_a, y_b)$ in (3.3)–(3.5) since $L(x) \neq 0$ on $[a, b]$.) Substituting this into (3.4) followed by rearrangement gives

$$\begin{aligned}(2L'/y_b) & \int_a^b |yy'|\,\mathrm{d}x \\ & \leq \int_a^b (y')^2\,\mathrm{d}x + \int_a^b \left(y(x)^2 - 2\int_a^x |yy'|\,\mathrm{d}t\right)(L'/L)^2\,\mathrm{d}x. \qquad (3.6)\end{aligned}$$

Since

$$y(x)^2 \leq 2\int_a^x |yy'|\,\mathrm{d}t + y_a^2$$

with equality if and only if y is monotone, (3.6) becomes

$$\begin{aligned}2\left(\frac{y_b - y_a}{b-a}\right)\left(\frac{1}{y_b}\right)\int_a^b |yy'|\,\mathrm{d}x &\leq \int_a^b (y')^2\,\mathrm{d}x + y_a^2 \int_a^b (L'/L)^2\,\mathrm{d}x \\ &= \int_a^b (y')^2\,\mathrm{d}x + y_a^2\left(\frac{y_b - y_a}{b-a}\right)\left(\frac{1}{y_a} - \frac{1}{y_b}\right)\end{aligned} \qquad (3.7)$$

which is equivalent to (3.2). That L_{ab} is an extremal is obvious. If there is equality in (3.7), then (3.6) must hold with the inequality "$\leq$" reversed and the same is true for (3.4) and (3.3). However, in the case of (3.3) this is impossible unless there is equality, i.e., $y = L(x)$.

Suppose $0 \leq y_b < y_a$ or $y_a, y_b < 0$. Then as in the proof of Lemma 1 or Theorem 1 we use the transformations $\hat{y} = y(a + b - x)$ or $\tilde{y} = -y$ to transform the problem to the previous case. Thus (3.1) is true for $\hat{y} \in \mathcal{D}(y_b, y_a)$ or $\tilde{y} \in \mathcal{D}(|y_a|, |y_b|)$ if and only if it is true for $y \in \mathcal{D}(y_a, y_b)$.

Corollary 2 *If* $y \in \mathcal{D}(y_a, y_b)$, $y_a \neq y_b$, *and* $y_a \cdot y_b > 0$ *then* (3.1) *holds and* L_{ab} *is the unique extremal.*

Proof Since L_{ab} is known to be the unique minimizer of $\|y'\|_{2,[a,b]}$ subject to the interpolation conditions $y(a) = y_a$, $y(b) = y_b$, we have the inequality

$$\int_a^b (y')^2 \geq \frac{(\bar{M} - \bar{m})^2}{b-a}. \qquad (3.8)$$

Hence

$$\bar{m}\frac{\bar{M}-\bar{m}}{2} \leq \left(\frac{b-a}{2}\right)\left(\frac{\bar{m}}{\bar{M}-\bar{m}}\right)\int_a^b (y')^2\,\mathrm{d}x.$$

When this is substituted into (3.2) we obtain

$$\int_a^b |yy'|\,\mathrm{d}x \leq \left(\frac{b-a}{2}\right)\frac{\bar{M}+\bar{m}}{\bar{M}-\bar{m}}\int_a^b (y')^2\,\mathrm{d}x$$

which is equivalent to (3.1) since y_a, y_b have the same sign. Since L_{ab} gives equality in (3.8) and (3.2) which shows that L_{ab} is an extremal for (3.1). Further if there is equality in (3.1) then either (3.8) or (3.2) must hold as reversed inequalities if they are not equal which is impossible. But equality in either (3.8) or (3.2) is attained only by L_{ab}.

4 APPLICATIONS TO LYAPUNOV-TYPE INEQUALITIES

Consider the differential equation

$$y'' + q(x)y = 0, \quad a \leq x \leq b, \tag{4.1}$$

where q is real and $q \in \mathcal{L}(a, b)$. The following well known inequality is commonly attributed to Lyapunov [8].¶

THEOREM B *Suppose y is a nontrivial solution of* (4.1) *such that $y(a) = y(b) = 0$. Then*

$$\frac{4}{b-a} < \int_a^b |q|\,\mathrm{d}x. \tag{4.2}$$

The inequality is sharp in the sense that 4 cannot be replaced by a larger number.

¶ In [8] Lyapunov, however, used the reverse of (4.2) to prove a stablity result for (4.1) with $q \geq 0$ and periodic. The first statement and proof of "Lyapunov's inequality" in more or less its present setting seem due to Borg [2]. Borg attributes the equivalent inequality $\int_a^b |y''y^{-1}|\,\mathrm{d}x > 4/(b-a)$ to Beurling but gives no reference. For the history of this inequality and its relation to [8] see Cheng [6].

Many extensions and variants of this inequality are known (see e.g. [9,13]). For example it is shown in [4] that under the same hypotheses of Theorem B (4.2) can be replaced by the assertion that there exist points $t_1, t_2 \in [a, b]$ such that

$$\frac{4}{b-a} < \left| \int_{t_1}^{t_2} q \,\mathrm{d}x \right|. \tag{4.3}$$

In this section we will use Theorem 1[§] to prove some generalized Lyapunov inequalities. First let $\mathcal{D}^*(y_a, y_b) \subset \mathcal{D}(y_a, y_b)$ consist of all $y \in \mathcal{D}(y_a, y_b)$ such that $(yy')(a) = (yy')(b)$ and if at least one of y_a, y_b is nonzero then $y_a \neq y_b$.

THEOREM 3 *Suppose y is a nontrivial solution of* (4.1) *in* $\mathcal{D}^*(y_a, y_b)$. *Then the following inequality holds*

$$\begin{aligned} 1 \le \max_{x \in [a,b]} &\left| \int_a^x q \,\mathrm{d}x - \int_x^b q \,\mathrm{d}x \right| \\ &\times [\eta(y_a, y_b) L^{-1}(y_a, y_b) + K(y_a, y_b)^{-1}] \end{aligned} \tag{4.4}$$

where

$$\eta(y_a, y_b) := \begin{cases} 1, & \textit{if } y_a, y_b \neq 0, \\ \frac{1}{2}, & \textit{if } y_a = 0 \textit{ or } y_b = 0, \\ 0, & \textit{if } y_a = y_b = 0. \end{cases}$$

and

$$L(y_a, y_b) = \begin{cases} \dfrac{4}{b-a}, & \textit{if } y(a) = y(b) = 0, \\ \dfrac{2}{b-a}, & \textit{if } y \textit{ has a zero in } [a,b], \\ \dfrac{(y_a - y_b)^2}{\max\{y_a^2, y_b^2\}(b-a)}, & \textit{otherwise and } y_a \neq y_b. \end{cases} \tag{4.5}$$

[§] Theorem 2 can also be used, but the results are the same.

Alternatively, there exist points t_1, t_2 *in* $[a, b]$ *such that*

$$1 \leq \left|\int_{t_1}^{t_2} q\,\mathrm{d}x\right| [\eta(y_a, y_b)L^{-1}(y_a, y_b) + K(y_a, y_b)^{-1}]. \tag{4.6}$$

Furthermore, if q *is nonnegative then* $t_1 = a$ *and* $t_2 = b$.

Before proving this, we require a lemma.

LEMMA 5 *If* $y \in \mathcal{D}(y_a, y_b), y_a \neq y_b$ *or* $y_a = y_b = 0$, *and* $y' \in \mathcal{L}^2[a, b]$, *then*

$$\| y\|^2_{\infty,[a,b]} \leq L(y_a, y_b)^{-1} \int_a^b (y')^2\,\mathrm{d}x. \tag{4.7}$$

Furthermore, in all cases the constant $L(y_a, y_b)^{-1}$ *is sharp.*

Proof Suppose $y(c) = \|y\|_{\infty,[a,b]}$ and $y_a = y_b = 0$. Since

$$\begin{aligned}\int_a^b |yy'|\,\mathrm{d}x &\geq \left|\int_a^c yy'\,\mathrm{d}x\right| + \left|\int_c^b yy'\,\mathrm{d}x\right| \\ &= \|y\|^2_{\infty,[a,b]},\end{aligned}$$

we have using Opial's inequality that

$$L(y_a, y_b) \equiv \frac{4}{b-a} = \min_{y \in \mathcal{D}(y_a, y_b)} \frac{\int_a^b (y')^2\,\mathrm{d}x}{\int_a^b |yy'|\,\mathrm{d}x} \leq \frac{\int_a^b (y')^2\,\mathrm{d}x}{\| y\|^2_{\infty,[a,b]}}.$$

If $\tilde{y}$ is an extremal of Opial's inequality, for example, a linear spline with a knot at $(a+b)/2$ such that $\tilde{y}(a) = \tilde{y}(b) = 0$ then there is equality in (4.7) which shows that $L(y_a, y_b)^{-1}$ is sharp in this case. If $y(t) = 0$ for some $t \in [a, b]$, then

$$\int_a^b |yy'|\,\mathrm{d}x \geq \int_t^c |yy'|\,\mathrm{d}x = \frac{\| y\|^2_{\infty,[a,b]}}{2\| y\|_{\infty,[a,b]}},$$

from which the result follows by the argument of the previous case. Here too $L(y_a, y_b)^{-1}$ is the best constant; for if say $t = a$, then the extremal of Corollary 1 gives equality in (4.7).

Suppose that $y_a \neq y_b$ and there is no zero of y in $[a, b]$. For $\epsilon < 0$ let y_ϵ be an ϵ-approximate extremal of

$$L(y_a, y_b) = \inf_{y \in \mathcal{D}(y_a, y_b)} J_\infty(y) := \frac{\int_a^b (y')^2\,\mathrm{d}x}{\|y\|_{\infty,[a,b]}}$$

in the sense of Definition 1. If $y_\epsilon(c) = \|y_\epsilon\|_{\infty,[a,b]}$ and s_ϵ is the linear spline with a knot at c interpolating y_a, y_c and y_b, then $\|s_\epsilon\|_{\infty,[a,b]} = \|y\|_{\infty,[a,b]}$ and $\int_a^b (s_\epsilon')^2\,\mathrm{d}x \le \int_a^b (y_\epsilon')^2\,\mathrm{d}x$ so s_ϵ is also an ϵ-approximate extremal of J_∞. By the argument of Lemma 4, there is another linear spline ϵ-approximate extremal $\tilde{s}_\epsilon$ with "optimal knot" c^* given by (2.3) such that $\tilde{s}_\epsilon(c^*) = \|s_\epsilon\|_{\infty,[a,b]} \equiv h$ and

$$J_\infty(\tilde{s}_\epsilon) = \left(\frac{1}{b-a}\right)\left(\frac{(\Delta_a + \Delta_b)^2}{\|s_\epsilon\|^2_{\infty,[a,b]}}\right).$$

If $h \ge \max\{|y_a|, |y_b|\}$ then

$$J_\infty(\tilde{s}_\epsilon) = \left(\frac{1}{b-a}\right)\left(\frac{(2h - (y_a + y_b))^2}{h^2}\right).$$

A calculus argument shows that the minimum occurs at the boundary $\max\{y_a, y_b\}$ or $\min\{y_a, y_b\}$ in which case $\tilde{s}_\epsilon = L_{ab}$ and

$$J_\infty(\tilde{s}_\epsilon) = \frac{(y_a - y_b)^2}{\max\{y_a^2, y_b^2\}(b-a)}. \tag{4.8}$$

Calculations similar to those of Theorem 1 show that for any of the $\tilde{s}_\epsilon$ such that $\min\{y_a, y_b\} < h < \max\{y_a, y_b\}$ that $J_\infty(\tilde{s}_\epsilon)$ is also given by (4.8). Since L_{ab} is an extremal of (4.7), $L(y_a, y_b)^{-1}$ is sharp in this final case as well.

Proof of Theorem 3 If y is a nontrivial solution of (4.1) with the property $yy'(a) = yy'(b)$, then multiplying (4.1) by y and integrating by parts twice yields that

$$
\begin{aligned}
0 &= \int_a^b [y'' + q(x)y]\, y\, \mathrm{d}x = -\int_a^b (y')^2\, \mathrm{d}x + \int_a^b q(x)y^2\, \mathrm{d}x \\
&= -\int_a^b (y')^2 + Q(b)y^2(b) - Q(a)y^2(a) - 2\int_a^b Q(x)yy'\, \mathrm{d}x \\
&\le -\int_a^b (y')^2 + \max_{x\in[a,b]} |Q(x)| \left\{ (y^2(b) + y^2(a)) + 2\int_a^b |yy'|\, \mathrm{d}x \right\}, \quad (4.9)
\end{aligned}
$$

where Q is any antiderivative of q. Next rearrangement of (4.9) followed by application of Lemma 5 and Theorem 1 gives

$$
\begin{aligned}
\int_a^b (y')^2 \le 2\|Q(x)\|_{\infty,[a,b]} &[\eta(y_a, y_b)L(y_a, y_b)^{-1} \\
&+ K(y_a, y_b)^{-1}] \int_a^b (y')^2\, \mathrm{d}x.
\end{aligned}
$$

After canceling $\int_a^b (y')^2\, \mathrm{d}x$, taking

$$
Q(x) = \left(\frac{1}{2}\right)\left(\int_a^x q\, \mathrm{d}x - \int_x^b \mathrm{d}x\right),
$$

and rearranging we obtain (4.4). To prove (4.6), we choose $Q(x) = \int_a^x q\, \mathrm{d}x + \mu$ where μ is some constant. Set

$$
\begin{aligned}
M &= \max_{x\in[a,b]} \int_a^x q\, \mathrm{d}x = \int_a^{t_2} q\, \mathrm{d}x, \\
m &= \min_{x\in[a,b]} \int_a^x q\, \mathrm{d}x = \int_a^{t_1} q\, \mathrm{d}x,
\end{aligned}
$$

and choose $\mu = -(M+m)/2$. Then since $\int_a^x q\, \mathrm{d}x \in [m, M]$ we have that

$$
\begin{aligned}
\left| \int_a^x q\, \mathrm{d}x - \frac{M+m}{2} \right| &\le \frac{M-m}{2} \\
&= \frac{1}{2}\left(\int_a^{t_2} q\, \mathrm{d}x - \int_a^{t_1} q\, \mathrm{d}x\right) \\
&= \frac{1}{2}\left(\int_{t_1}^{t_2} q\, \mathrm{d}x\right);
\end{aligned}
$$

(4.5) follows at once.

If $y(a)=y(b)=0$, $\eta(y_a, y_b)=0$ and $K(0,0)=4/(b-a)$ by Theorem A. Hence the classical Lyapunov inequality or (4.3) are immediate corollaries of our theorem. Note also that the strict inequality in (4.2) or (4.3) is also implied although it is not present in the general inequalities (4.4) and (4.6). This is because the extremal of Opial's inequality has a discontinuity at $(a+b)/2$ and therefore cannot be a solution of the equation (4.1).

COROLLARY 3 *If there is a solution of* (4.1) *such that*

$$y(a)+y(b)=0, \quad y'(a)+y'(b)=0, \quad y(a)\neq 0.$$

Then we have the inequalities

$$\frac{2}{b-a}\leq\begin{cases}\max_{x\in[a,b]}\left|\int_a^x q\,\mathrm{d}x-\int_x^b q\,\mathrm{d}x\right|,\\ \left|\int_{t_1}^{t_2} q\,\mathrm{d}x\right|\end{cases}$$

where t_1, t_2 *are as in Theorem* 3.

Proof Theorem 3 applies since $yy'(b)=yy'(a)$ and $\eta(y_a, y_b)=1$. By (2.8) of Theorem 1 and (4.5) $K(y_a, y_b)=4/(b-a)=L(y_a, y_b)$. The inequalities follow if we substitute this into (4.4) or (4.6).

COROLLARY 4 *If there is a solution of* (4.1) *such that* $y(a)=y'(b)=0$,

$$\frac{4}{3(b-a)}<\begin{cases}\max_{x\in[a,b]}\left|\int_a^x q\,\mathrm{d}x-\int_x^b q\,\mathrm{d}x\right|,\\ \left|\int_{t_1}^{t_2} q\,\mathrm{d}x\right|\end{cases}$$

where t_1, t_2 *are as in Theorem* 3.

Proof Again this is an application of Theorem 3 using the value $K(0, y_b)=2/(b-a)=L(0, y_b)$ in Theorem 1 (i) and (4.5). Strict inequality holds because the extremals of the Opial-type inequality in (i) are straight lines which cannot satisty the boundary conditions $y(a)=y'(b)=0$.

COROLLARY 5 *If there is a solution y of* (4.1) *such that* $y'(a) = y'(b) = 0$ *while* $y(a), y(b) \neq 0$ *and are of different signs, then*

$$\frac{1}{b-a} < \begin{cases} \max_{x\in[a,b]} \left|\int_a^x q\,\mathrm{d}x - \int_x^b q\,\mathrm{d}x\right|, \\ \left|\int_{t_1}^{t_2} q\,\mathrm{d}x\right| \end{cases}$$

where t_1, t_2 *are as in Theorem* 3.

Proof We can suppose that $y(a) > 0$. Examination of either (iii) or (iv) of Theorem 1 and (4.5) shows that

$$K(y_a, y_b)^{-1} < \frac{b-a}{2} = L(y_a, y_b)^{-1}.$$

Also since there is a zero of y in (a, b) we then substitute these bounds into (4.4) or (4.6) as in the two previous Corollaries.

5 APPLICATIONS TO DE LA VALÉE POUSSIN INEQUALITIES

De la Valée Poussin proved the following result.

THEOREM C *Let y be a nontrivial solution of the second order two point boundary value problem with zero endpoint conditions*

$$\begin{aligned} & y'' + g(t)y' + f(t)y = 0, \\ & y(a) = y(b) = 0. \end{aligned} \tag{5.1}$$

Then

$$1 < 2\|g\|_{\infty,[a,b]}(b-a) + \|f\|_{\infty,[a,b]}\frac{(b-a)^2}{2}.$$

This inequality has been improved by many writers in various ways. In particular Z. Opial [11] showed that

$$\pi^2 \leq 4\|g\|_{\infty,[a,b]}(b-a) + \|f\|_{\infty,[a,b]}(b-a)^2. \tag{5.2}$$

For additional discussion of De la Valée Poussin inequalities see [9, Chapter VI]. Some variant inequalities of this type are also given in [3].

Just as for Lyapunov inequalities Theorem 1 and Lemma 5 allow us to derive De la Valée Poussin inequalities for other than zero boundary conditions.

THEOREM 4 *Let $y \in \mathcal{D}(y_a, y_b)^*$ be a nontrivial solution of* (5.1). *Then*

$$1 < \|g\|_{\infty,[a,b]}K(y_a, y_b)^{-1} + \|f\|_{1,[a,b]}L(y_a, y_b)^{-1} \tag{5.3}$$

or

$$1 < \|g\|_{\infty,[a,b]}K(y_a, y_b)^{-1} + \|f\|_{\infty,[a,b]}(b-a)L(y_a, y_b)^{-1}. \tag{5.4}$$

Proof Multiplication of (5.1) by y, followed by integration by parts, obvious Hölder estimates, Theorem 1, and Lemma 5 gives successively

$$\begin{aligned}\int_a^b (y')^2\,\mathrm{d}x &\le \|g\|_{\infty,[a,b]}\int_a^b |yy'|\,\mathrm{d}x + \int_a^b |f|\,\mathrm{d}x\|y\|^2_{\infty,[a,b]}\\ &< \|g\|_{\infty,[a,b]}K(y_a, y_b)^{-1}\int_a^b (y')^2\,\mathrm{d}x\\ &\quad + \int_a^b |f|\,\mathrm{d}x\,L(y_a, y_b)^{-1}\int_a^b (y')^2\,\mathrm{d}x\end{aligned}$$

or

$$1 < |g\|_{\infty,[a,b]}K(y_a, y_b)^{-1} + \left(\int_a^b |f|\,\mathrm{d}x\right)L(y_a, y_b)^{-1}.$$

From which (5.3) and (5.4) follow by cancelation and Hölder's inequality.

In particular if $y(a) = y(b) = 0$ we have

$$4 < \|g\|_{\infty,[a,b]}(b-a) + \left(\int_a^b |f|\,\mathrm{d}x\right)(b-a).$$

This may be improved using the Sobolev/Wirtinger-type inequality (see [3])

$$\int_a^b y^2\,\mathrm{d}x \le \frac{(b-a)^2}{\pi^2}\int_a^b (y')^2\,\mathrm{d}x$$

to bound $\int_a^b fy^2\,\mathrm{d}x \le \|f\|_{\infty,[a,b]} \int_a^b (y')^2\,\mathrm{d}x$ after multiplying (5.1) by y and integrating by parts, etc. We get

$$1 < \frac{\|g\|_{\infty,[a,b]}}{4}(b-a) + \frac{\|f\|_{\infty,[a,b]}}{\pi^2}(b-a)^2$$

which is a better inequality than either the original De la Valée Poussin inequality or (5.2).

The inequality corresponding to the boundary conditions of either Corollary 3 or 4 is

$$2 < \|g\|_{\infty,[a,b]}(b-a) + \|f\|_{\infty,[a,b]}(b-a)^2.$$

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