J. of Inequal. & Appl., 1999, Vol. 4, pp. 183–213
Reprints available directly from the publisher
Photocopying permitted by license only

A Conjecture of Schoenberg

M.G. DE BRUIN [a,*], K.G. IVANOV [b] and A. SHARMA [c]

[a] *Faculty of Information Technology and Systems, Department of Technical Mathematics and Informatics, Delft University of Technology, P.O. Box 5031, 2600 GA Delft, The Netherlands;* [b] *Institute of Mathematics, Bulgarian Academy of Sciences, P.O. Box 373, 1090 Sofia, Bulgaria;* [c] *Department of Mathematical Sciences, University of Alberta, Edmonton, Alberta, Canada T6G 2G1*

(Received 16 October 1998; Revised 12 November 1998)

For an arbitrary polynomial $P_n(z) = \prod_1^n (z - z_j)$ with the sum of all zeros equal to zero, $\sum_1^n z_j = 0$, the *quadratic mean radius* is defined by

$$R(P_n) := \left(\frac{1}{n}\sum_1^n |z_j|^2\right)^{1/2}.$$

Schoenberg conjectured that the quadratic mean radii of P_n and P_n' satisfy

$$R(P_n') \leq \sqrt{\frac{n-2}{n-1}} R(P_n),$$

where equality holds if and only if the zeros all lie on a straight line through the origin in the complex plane (this includes the simple case when all zeros are real) and proved this conjecture for $n = 3$ and for polynomials of the form $z^n + a_k z^{n-k}$.

It is the purpose of this paper to prove the conjecture for three other classes of polynomials. One of these classes reduces for a special choice of the parameters to a previous extension due to the second and third authors.

Keywords: Geometry of zeros; Weighted sums; Inequalities

AMS/SMC Classification: Primary: 30C15; Secondary: 26C10, 26D15

1 INTRODUCTION

Let $P_n(z) = z^n - a_1 z^{n-1} + a_2 z^{n-2} + \cdots + (-1)^n a_n = \prod_1^n (z - z_j)$ be a given polynomial with real or complex coefficients. In 1986, Schoenberg

* Corresponding author.

[1] considered polynomials $P_n(z)$ with $a_1 = \sum_1^n z_j = 0$ and defined the quadratic mean radius of the polynomial $P_n(z)$ and set

$$R(P_n) := \left(\frac{1}{n}\sum_1^n |z_j|^2\right)^{1/2}.$$

He observed that the quadratic mean radius of P_n and P_n' are related by a simple inequality and offered the

CONJECTURE *For monic polynomials of degree n with the sum of all zeros equal to zero, one has*

$$R(P_n') \leq \sqrt{\frac{n-2}{n-1}}R(P_n),$$

with equality if and only if all zeros z_j of $P_n(z)$ lie on straight line through the origin.

Denoting the zeros of $P_n'(z)$ by w_j $(1 \leq j \leq n-1)$, the conjecture turns after squaring into the equivalent form

$$\frac{n-2}{n}\sum_{j=1}^{n}|z_j|^2 - \sum_{j=1}^{n-1}|w_j|^2 \geq 0. \tag{1}$$

As Schoenberg already noted, the case of a polynomial with **real roots only** is simple: those polynomials satisfy

$$\sum_{j=1}^{n}|z_j|^2 = \sum_{j=1}^{n} z_j^2 = a_1^2 - 2a_2 = -2a_2,$$

while the roots of the derivative satisfy

$$\sum_{j=1}^{n-1}|w_j|^2 = \sum_{j=1}^{n-1} w_j^2 = \left(\frac{n-1}{n}a_1\right)^2 - 2\left(\frac{n-2}{n}a_2\right) = -2\left(\frac{n-2}{n}a_2\right),$$

showing that (1) turns into an equality. Schoenberg proved the conjecture for $n = 3$ and for polynomials of the form $z^n + a_k z^{n-k}$ which he called 'binomial' polynomials. Schoenberg's proof (connected with van den Berg [3], Marsden [6]) is very elegant but does not seem to extend to polynomials of higher degree.

Ivanov and Sharma [2] have shown that the conjecture (1) is true when

$$P_n(z) = (z - z_1)^{m_1}(z - z_2)^{m_2}(z - z_3)^{m_3} \quad \text{with} \sum_{j=1}^{3} m_j z_j = 0. \qquad (2)$$

They also prove the conjecture when $P_n(z)$ is a biquadratic of the form

$$P_4(z) = (z^2 - 2az + b)(z^2 + 2az + c). \qquad (3)$$

We will show that for several other classes of polynomials the conjecture of Schoenberg is true.

The layout of the paper is as follows. In Section 2 the main results will be formulated, including the fact that the conjecture holds for a class of polynomials not having the sum of all zeros equal to zero. This then necessitates the reformulation of the conjecture, or equivalently (1), to that situation; this will be done in Section 3. Finally in Section 4 the proofs will be given.

For general information concerning methods in the realm of inequalities see a.o. Beckenbach and Bellman [4], Kazarinoff [5] and Mitrinovic and Dragoslav [7].

2 MAIN RESULTS

First a theoretical result:

THEOREM 1 *If the conjecture, equivalently formula* (1), *holds for a polynomial $P(z)$ and $m \geq 2$ is an integer, then it also holds for $Q(z) = P(z)^m$.*

Now we give several classes of polynomials for which (1) can actually be proved.

THEOREM 2 *The Schoenberg conjecture holds for the following types of polynomials:*

(A) *For $a, c \in \mathbb{C}$, $k, m_1, m_2 \in \mathbb{N}$ the class*

$$P_n(z) = (z^k - a^k)^{m_1}(z^k - c^k)^{m_2}; \; n = k(m_1 + m_2), \; k \geq 1. \qquad (4)$$

(B) *For $n \in \mathbb{N}$ the class*

$$P_n(z) = (z+1)^{n+1} - z^{n+1}. \tag{5}$$

(C) *For $a, b, c, d \in \mathbb{C}$, $m_1, m_2 \in \mathbb{N}\backslash\{0\}$ the class*

$$P_n(z) = (z^2 + 2az + b)^{m_1}(z^2 + 2cz + d)^{m_2}, \quad n = 2(m_1 + m_2), \tag{6}$$

where a and c are related by

$$(m_1 + 2m_2)a + (2m_1 + m_2)c = 0. \tag{7}$$

For $a = 0$ (then (7) implies $c = 0$ too) the Schoenberg conjecture is true. For $a \neq 0$ (consequently $c \neq 0$ too) the number r is given by

$$r := \frac{m_1 + 2m_2}{2m_1 + m_2}. \tag{8}$$

Then Schoenbergs' conjecture is true under the extra condition

$$\frac{23 - 3\sqrt{5}}{22} \leq r \leq \frac{23 + 3\sqrt{5}}{22}. \tag{9}$$

Remarks

1. The polynomials in (4) for $k = 1$, those in (5) and those in (6) for $a, c \neq 0$, necessitate the reformulation of the conjecture because the sum of the roots of the polynomial is not necessarily equal to zero, this will be done in Section 3.
2. Polynomials in (6) in case of $m_1 = m_2$ satisfy $a = -c$; thus the result by Ivanov and Sharma [2] is found again. Obviously $\frac{1}{2} < r < 2$.
3. The choice for the condition (7) is for sake of convenience: this will become clear from the proof in Section 4.

3 REFORMULATION FOR $\sum z_j \neq 0$

Already Ivanov and Sharma [2] considered this possibility (their Remark 2). Let a polynomial $P_n(z)$ of degree n with roots z_j $(1 \leq j \leq n)$ be given and assume

$$\mathcal{E} := \frac{1}{n}\sum_{j=1}^{n} z_j \neq 0. \tag{10}$$

Introducing $\tilde{P}_n$ by

$$\tilde{P}_n(z) := P_n(z + \mathcal{E}), \tag{11}$$

and the roots $\tilde{z}_j$, resp. $\tilde{w}_j$ of $\tilde{P}_n$, resp. $\tilde{P}_n'$ by

$$\tilde{z}_j = z_j - \mathcal{E} \;\; (1 \le j \le n), \quad \tilde{w}_j = w_j - \mathcal{E} \;\; (1 \le j \le n-1), \tag{12}$$

it is obvious that the sum of the roots of $\tilde{P}_n$ is equal to zero and Schoenbergs' conjecture leads to

$$\frac{n-2}{n-1}\left(\frac{1}{n}\sum_{j=1}^{n}|z_j - \mathcal{E}|^2\right) - \frac{1}{n-1}\sum_{j=1}^{n-1}|w_j - \mathcal{E}|^2 \ge 0. \tag{13}$$

Using $|u - \mathcal{E}|^2 = |u|^2 + |\mathcal{E}|^2 - \bar{u}\mathcal{E} - u\bar{\mathcal{E}}$, (13) can be written out and after collecting the terms with $|\mathcal{E}|^2$ and using $\sum_{j=1}^{n-1} \tilde{w}_j = \sum_{j=1}^{n-1} w_j - (n-1)\mathcal{E}/n$, we find the equivalent of the Schoenberg conjecture in the form

$$\frac{n-2}{n}\sum_{j=1}^{n}|z_j|^2 + |\mathcal{E}|^2 - \sum_{j=1}^{n-1}|w_j|^2 \ge 0, \tag{14}$$

with equality if and only if the zeros of P_n are on a straight line through $\mathcal{E}$.

Remark Note that (14) reduces to (1) when $\mathcal{E} = 0$.

4 PROOFS

In this section full proofs of the main results will be given. Although Theorem 2 makes it possible to deduce several special cases from polynomials of the form (2), this constitutes only a minor simplification.

4.1 Proof of Theorem 1

For the polynomial $P_n(z) = \prod_1^n (z - z_j)$ we have $(1/n)P_n'(z) = \prod_1^{n-1}(z - w_j)$ and formula (1) holds:

$$\frac{n-2}{n}\sum_{j=1}^{n}|z_j|^2 - \sum_{j=1}^{n-1}|w_j|^2 \ge 0. \tag{15}$$

The zeros of $Q(z) = P(z)^m$ are again $z_1, \ldots, z_n$, with multiplicity m each and those of $Q'(z) = mP(z)^{m-1}P'(z)$ are $z_1, \ldots, z_n$, with multiplicity $m-1$ each and $w_1, \ldots, w_{n-1}$.

The expression on the left hand side of (1) can now be calculated for Q:

$$\begin{aligned}
&\frac{mn-2}{mn}\sum_{j=1}^{n} m|z_j|^2 - \left(\sum_{j=1}^{n}(m-1)|z_j|^2 + \sum_{j=1}^{n-1}|w_j|^2\right) \\
&= \left(\frac{mn-2}{n} - (m-1)\right)\sum_{j=1}^{n}|z_j|^2 - \sum_{j=1}^{n-1}|w_j|^2 \\
&= \frac{n-2}{n}\sum_{j=1}^{n}|z_j|^2 - \sum_{j=1}^{n-1}|w_j|^2,
\end{aligned}$$

and this is the same expression as in (15).

4.2 Proof of Theorem 2A

For the class of polynomials given by

$$P_n(z) = (z^k - a^k)^{m_1}(z^k - c^k)^{m_2}, \quad n = k(m_1 + m_2), \ k \geq 1,$$

we have to consider the cases $k = 1$ and $k \geq 2$ seperately. Whether $a = c$ or not is immaterial for the proof.

For $k = 1$ the weighted sum of the roots (compare (10) in Section 3) is

$$\mathcal{E} = \frac{m_1 a + m_2 c}{n},$$

which might well be different from zero. Using the equivalent form (14), we have to prove

$$\begin{aligned}
&\frac{m_1 + m_2 - 2}{m_1 + m_2}\left(m_1|a|^2 + m_2|c|^2\right) + \left|\frac{m_1 a + m_2 c}{m_1 + m_2}\right|^2 \\
&\geq (m_1 - 1)|a|^2 + (m_2 - 1)|c|^2 + \left|\frac{m_2 a + m_1 c}{m_1 + m_2}\right|^2.
\end{aligned}$$

Writing out the absolute values (using $|z + w|^2 = |z|^2 + |w|^2 + \bar{z}w + z\bar{w}$), this turns into an exact equality in accordance with the conjecture as the roots a and c are trivially located on a straight line through $\mathcal{E}$ in the complex plane.

Turning to $k \geq 2$, we can use (1) as the sum of the roots is zero (the coefficient of z^{n-1} is zero!) and we have to show

$$\left(1 - \frac{2m_1}{n}\right)|a|^2 + \left(1 - \frac{2m_2}{n}\right)|c|^2 \geq \left|\frac{m_1 c^k + m_2 a^k}{m_1 + m_2}\right|^{2/k}. \tag{16}$$

For $k = 2$ formula (16) can be simplified to give

$$m_2|a|^2 + m_1|c|^2 \geq |m_1 c^2 + m_2 a^2|^2,$$

and this is true because of the triangle inequality. That same triangle inequality implies that the only possibility to have an equality sign lies in having m_1c^2 and m_2a^2 along the same half line through the origin.

But then $m_1c^2 = tm_2a^2$ for a real t with $t \geq 0$: this shows that the roots of the polynomial can be given by $\pm a, \pm a\sqrt{m_2 t/m_1}$, showing that they are on a straight line through the origin.

Finally we consider the case $k \geq 3$. If $a = 0$, (16) can be written as

$$\left(1 - \frac{2m_2}{n}\right)|c|^2 \geq \left(1 - \frac{m_2}{m_1 + m_2}\right)^{2/k}|c|^2. \tag{17}$$

For $c = 0$ the conjecture is trivially satisfied (then all zeros are located at the origin) and for $c \neq 0$ the inequality (17) turns out to be strict as can be seen from $(1-x)^\alpha = 1 - \alpha x + R$ with $R = \frac{1}{2}\alpha(\alpha - 1)(1-\xi)^{\alpha-2} \lneqq 0$ for $0 < x < 1, 0 < \alpha < 1$.

Now the case $a \neq 0$ remains; let also $c \neq 0$ (otherwise change the roles of a and c and apply the method of proof of (17) again). Put $x = |c/a| \geq 1$ (if $x < 1$, interchange a and c), then (16) is equivalent to

$$f(x) := 1 - \frac{2m_1}{n} + \left(1 - \frac{2m_2}{n}\right)x^2 - \left(\frac{m_2 + m_1 x^k}{m_1 + m_2}\right)^{2/k} \geq 0, \quad x \geq 1.$$

Observe that

$$f(1) = 1 - \frac{2m_1}{n} + 1 - \frac{2m_2}{n} - 1 = 1 - \frac{2}{k} > \frac{1}{3},$$

while $k \geq 3$.

Furthermore

$$f'(x) = 2x\left[1 - \frac{2m_2}{n} - \frac{m_1}{m_1+m_2}\left(\frac{m_2+m_1x^k}{m_1+m_2}\right)^{2/k-1} x^{k-2}\right]$$

or equivalently

$$f'(x) = 2x\left[1 - \frac{2m_2}{n} - \frac{m_1}{m_1+m_2}\left(\frac{m_2+m_1x^k}{(m_1+m_2)x^k}\right)^{2/k-1}\right].$$

Since $(m_2+m_1x^k)/((m_1+m_2)x^k)$ is decreasing as x increases, the function $f'(x)$ is increasing. While

$$f'(1) = \frac{2m_2}{m_1+m_2}\left(1 - \frac{2}{k}\right) > 0,$$

we see that $f'(x) \geq 0$ for $x \geq 1$. So $f(x)$ is an increasing function of x and as $f(1) > \frac{1}{3}$, the inequality (16) is proved.

It is of course obvious that for $k \geq 3$ and $a, c \neq 0$ the zeros can never all be on the same line through the origin.

4.3 Proof of Theorem 2B

We now consider the polynomials

$$P_n(z) = (z+1)^{n+1} - z^{n+1}, \quad n \geq 2,$$

where the weighted sum of the roots follows easily: $\mathcal{E} = -\frac{1}{2}$. As the roots can be given by

$$z_k = \frac{1}{e^{2\pi ik/(n+1)} - 1} \quad (1 \leq k \leq n),$$

and those of P_n' by

$$w_k = \frac{1}{e^{2\pi ik/n} - 1} \quad (1 \leq k \leq n-1),$$

the conjecture takes the form

$$\frac{n-2}{n}\sum_{k=1}^{n}\frac{1}{4\sin^2(k\pi/(n+1))}+\frac{1}{4}-\sum_{k=1}^{n-1}\frac{1}{4\sin^2(k\pi/n)}\geq 0. \qquad (18)$$

Writing $\sin^2(k\pi/(n+1))=(1-\cos(k\pi/(n+1)))(1+\cos(k\pi/(n+1)))$, we find

$$\sum_{k=1}^{n}\frac{1}{\sin^2(k\pi/(n+1))}=\frac{1}{2}(f(1)-f(-1)),$$

where f can be given in terms of Tchebycheff polynomials of the second kind:

$$f(x)=\frac{U_n'(x)}{U_n(x)},\quad U_n(x)=\frac{\sin(n+1)\theta}{\sin\theta}\ \text{ with } x=\cos\theta.$$

Now f is an odd function and using

$$U_n(1)=n+1,\quad U_n'(1)=\frac{n(n+1)(n+2)}{3},$$

we find

$$\sum_{k=1}^{n}\frac{1}{\sin^2(k\pi/(n+1))}=\frac{n(n+2)}{3},$$

and (18) turns into an equality. Moreover, as the zeros z_k of P_n can be seen as the pre-images of the roots of unity $\zeta_k=\mathrm{e}^{2k\pi\mathrm{i}/(n+1)}$, $(1\leq k\leq n)$ under the mapping $\zeta=(z+1)/z$, they are all on the straight line ($\operatorname{Re} z=-\frac{1}{2}$) going through $\mathcal{E}$, proving the conjecture in full.

4.4 Proof of Theorem 2C

The proof for polynomials as in (6) turns out to be rather intricate. First of all the simplest case $a=c=0$ is looked at.

The polynomial reduces to

$$P_n(z)=(z^2+b)^{m_1}(z^2+d)^{m_2},$$

and using any pair of complex numbers β, δ with $\beta^2 = -b$, $\delta^2 = -d$, the conjecture follows from Theorem 2A.

For the sequel we can now assume a, $c \neq 0$; indeed a and c are either both zero or they are both different from zero in view of condition (7).

4.4.1 Reformulation as a Minimizing Problem

In this section the Schoenberg conjecture for class (6) will be reformulated in terms of a problem minimizing a function of two complex variables.

We consider the polynomial $P_{2n}(z)$ of degree $2n$, where

$$P_n(z) = (z^2 + 2az + b)^{m_1}(z^2 + 2cz + d)^{m_2}, \quad n = m_1 + m_2.$$

Then

$$\sum_{j=1}^{2n} |z_j|^2 = 2m_1(|a|^2 + |a^2 - b|) + 2m_2(|c|^2 + |c^2 - d|).$$

Now $P'_{2n}(z) = 2(z^2 + 2az + b)^{m_1-1}(z^2 + 2cz + d)^{m_2-1}Q(z)$, where

$$\begin{aligned} Q(z) = nz^3 &+ \{(m_1 + 2m_2)a + (2m_1 + m_2)c\}z^2 \\ &+ (2nac + m_1 d + m_2 b)z + m_1 ad + m_2 cb. \end{aligned}$$

Here we see the use for the condition (7)

$$(m_1 + 2m_2)a + (2m_1 + m_2)c = 0,$$

leading to a simplification for Q:

$$Q(z) = nz^3 + (2nac + m_1 d + m_2 b)z + m_1 ad + m_2 cb.$$

The zeros $\omega_1, \omega_2, \omega_3$ of $Q(z)$ can then be given explicitly using the primitive root of unity $\alpha = \exp(2\pi i/3)$:

$$\omega_j = u\alpha^j + v\alpha^{2j}, \quad j = 1, 2, 3.$$

Then

$$\sum_{j=1}^{3} |\omega_j|^2 = 3(|u|^2 + |v|^2).$$

Also u, v are related by the relations

$$u^3 + v^3 = -\frac{m_1 ad + m_2 cb}{n}, \quad uv = -\frac{2nac + m_1 d + m_2 b}{3n}. \tag{19}$$

Because we used condition (7), the reformulation (14) of the conjecture has to be used as the sum of the zeros is $2(m_1 a + m_2 c)$. It is directly clear that P_{2n} satisfies (7) and has sum of zeros equal to zero, if and only if $a = c = 0$ (in case $m_1 \neq m_2$) or $m_1 = m_2$ and $c = -a$. Now the first mentioned case has been treated already at the beginning of Section 4.4 and the second case follows from application of Theorem 1 to the polynomial – given in (3) – and treated by Ivanov and Sharma [2].

According to (14) the conjecture of Schoenberg reduces to:

$$\begin{aligned}
&\frac{2n-2}{2n}\left\{2m_1(|a|^2 + |a^2 - b|) + 2m_2(|c|^2 + |c^2 - d|)\right\} + \left|\frac{m_1 a + m_2 c}{n}\right|^2 \\
&\geq [2(m_1 - 1)(|a|^2 + |a^2 - b|) + 2(m_2 - 1)(|c|^2 + |c^2 - d|) \\
&\quad + 3|u|^2 + 3|v|^2].
\end{aligned}$$

On simplifying the above, we get

$$\begin{aligned}
&\frac{2m_2}{n}(|a|^2 + |a^2 - b|) + \frac{2m_1}{n}(|c|^2 + |c^2 - d|) + \left|\frac{m_1 a + m_2 c}{n}\right|^2 \\
&\quad - 3(|u|^2 + |v|^2) \geq 0.
\end{aligned} \tag{20}$$

From (19), we solve for b and d in terms of a, c and u, v: thus we have

$$b = \frac{-n(u^3 + v^3) + 3nuva + 2na^2 c}{m_1(c - a)}, \tag{21}$$

and

$$d = \frac{-n(u^3 + v^3) + 3nuvc + 2nac^2}{m_1(a - c)}. \tag{22}$$

Write $r = (m_1 + 2m_2)/(2m_1 + m_2)$, using (7) we have then $c = -ar$, and this leads to

$$c^2 - d = \frac{(u^3 + v^3 - r^3 a^3 + 3uvra)}{3m_1 a/(2m_1 + m_2)}, \tag{23}$$

$$a^2 - b = -\frac{(u^3 + v^3 + a^3 - 3uva)}{3m_2 a/(2m_1 + m_2)}. \tag{24}$$

Without loss of generality we can take $a = 1$ (this could have been achieved beforehand by scaling the variable z), leading to the following form for the polynomial:

$$P_{2n}(z) = (z^2 + 2z + b)^{m_1}(z^2 - 2rz + d)^{m_2}, \quad r = \frac{m_1 + 2m_2}{2m_1 + m_2}. \tag{25}$$

Now the left hand side of (20) reduces to

$$\begin{aligned}
&\frac{2m_2}{n}\left(1 + \frac{2m_1 + m_2}{3m_2}|u^3 + v^3 + 1 - 3uv|\right) + \left(\frac{2(m_1 - m_2)}{2m_1 + m_2}\right)^2 \\
&\quad + \frac{2m_1}{n}\left\{r^2 + \frac{2m_1 + m_2}{3m_1}|u^3 + v^3 - r^3 + 3ruv|\right\} - 3(|u|^2 + |v|^2) \\
&\quad = \frac{2m_2}{n} + \frac{2m_1}{n}r^2 + \frac{2}{r+1}\left\{|u^3 + v^3 + 1 - 3uv|\right. \\
&\quad + 4\left(\frac{m_1 - m_2}{2m_1 + m_2}\right)^2 + \left.|u^3 + v^3 - r^3 + 3ruv|\right\} - 3(|u|^2 + |v|^2).
\end{aligned}$$

Put $\zeta = u + v$, $W = u - v$ and $\omega = 3W^2$. Then

$$\begin{aligned}
4(u^3 + v^3 + 1 - 3uv) &= 4(u + v + 1)(u^2 + v^2 + 1 - uv - u - v) \\
&= (\zeta + 1)((\zeta - 2)^2 + 3W^2) \\
&= (\zeta + 1)((\zeta - 2)^2 + \omega),
\end{aligned} \tag{26}$$

and

$$\begin{aligned}
4(u^3 + v^3 - r^3 + 3ruv) &= 4(u + v - r)(u^2 + v^2 + r^2 - uv + ru + rv) \\
&= (\zeta - r)((\zeta + 2r)^2 + \omega).
\end{aligned} \tag{27}$$

We now define

$$\begin{aligned}
G(\zeta, \omega) := &\frac{8m_2}{n}(|a|^2 + |a^2 - b|) + \frac{8m_1}{n}(|c|^2 + |c^2 - d|) \\
&+ 4\left|\frac{m_1 a + m_2 c}{n}\right|^2 - 12(|u|^2 + |v|^2).
\end{aligned}$$

Since

$$\frac{m_2}{n}\cdot\frac{2m_1+m_2}{3m_2}=\frac{1}{r+1}=\frac{m_1}{n}\cdot\frac{2m_1+m_2}{3m_1},$$

we can write

$$G(\zeta,\omega)=2\left[4(r^2-r+1)-3|\zeta|^2-|\omega|+\frac{1}{r+1}\right.$$
$$\left.\times\{|\zeta+1|\,|(\zeta-2)^2+\omega|+|\zeta-r|\cdot|(\zeta+2r)^2+\omega|\}\right]. \quad (28)$$

To prove Schoenbergs' conjecture, written in the equivalent form (20), we have to show that

$$G(\zeta,\omega)\geq 0, \quad (29)$$

with equality if and only if all zeros are on a straight line through $\sum z_j$.

4.4.2 The Case $\zeta\in\mathbb{R}$

First we consider the case that $\zeta\in\mathbb{R}$ and introduce some notation. Replace $G(\zeta,\omega)$ by $\tilde{G}(\zeta,\omega)=(r+1)G(\zeta,\omega)/2$, we then have to prove (29) for $\tilde{G}$. Put

$$\zeta:=\xi+\mathrm{i}\eta(\xi,\eta\in\mathbb{R});\quad \omega:=\rho\mathrm{e}^{\mathrm{i}\varphi}\ \ (\rho\geq 0,\ 0\leq\varphi<2\pi). \quad (30)$$

Since we have the following identities:

$$(r-\xi)(\xi+2r)^2+(1+\xi)(\xi-2)^2=(r+1)\{4(r^2-r+1)-3\xi^2\},$$

and

$$(r-\xi)+(1+\xi)=r+1,$$

we can write

$$\begin{aligned}\tilde{G}(\zeta,\omega) &= (r-\xi)(\xi+2r)^2 + (1+\xi)(\xi-2)^2 - ((r-\xi)+(1+\xi))|\omega| \\ &\quad + |r-\xi|\cdot|(\xi+2r)^2+\omega| + |1+\xi|\cdot|(\xi-2)^2+\omega| \\ &= \operatorname{sgn}(r-\xi)|r-\xi|(\xi+2r)^2 + \operatorname{sgn}(1+\xi)|1+\xi|(\xi-2)^2 \\ &\quad - \operatorname{sgn}(r-\xi)|r-\xi|\cdot|\omega| - \operatorname{sgn}(1+\xi)|1+\xi|\cdot|\omega| \\ &\quad + |r-\xi|\cdot|(\xi+2r)^2+\omega| + |1+\xi|\cdot|(\xi-2)^2+\omega| \\ &= |r-\xi|\{|(\xi+2r)^2+\omega| + \operatorname{sgn}(r-\xi)((\xi+2r)^2-|\omega|)\} \\ &\quad + |1+\xi|\{|(\xi-2)^2+\omega| + \operatorname{sgn}(1+\xi)((\xi-2)^2-|\omega|)\}.\end{aligned}$$

As $|(\xi+2r)^2+\omega| \geq |(\xi+2r)^2-|\omega||$ and $|(\xi-2)^2+\omega| \geq |(\xi-2)^2-|\omega||$, this immediately implies $\tilde{G}(\zeta,\omega) \geq 0$.

Introducing

$$C := |(\xi+2r)^2+\omega| + \operatorname{sgn}(r-\xi)((\xi+2r)^2-|\omega|), \tag{31}$$

and

$$D := |(\xi-2)^2+\omega| + \operatorname{sgn}(1+\xi)((\xi-2)^2-|\omega|), \tag{32}$$

it is also clear that $\tilde{G}(\xi,\omega) = 0$ can only happen in the following cases:

a. $\xi = -1$ and $|(2r-1)^2+\omega| = |\omega| - (2r-1)^2$,
b. $\xi = r$ and $|(r-2)^2+\omega| = |\omega| - (r-2)^2$,
c. $\xi \neq -1, r$ and $C = D = 0$.

Writing $\omega = s + \mathrm{i}t$ ($s, t \in \mathbb{R}$), it is a matter of straightforward calculus to prove that the three cases mentioned above lead to the following conditions on s, t (where in each of them ω *turns out to be real too*):

a. $\xi = -1$ and $s \leq -(2r-1)^2$, $t = 0$,
b. $\xi = r$ and $s \leq -(r-2)^2$, $t = 0$,
c. there are different intervals for ξ:
   1. $\xi < -1$ and $-(\xi-2)^2 \leq s \leq -(\xi+2r)^2$, $t = 0$,
   2. $-1 < \xi \leq 1 - r$ and $s \leq (\xi-2)^2$, $t = 0$,
   3. $1 - r < \xi < r$ and $s \leq -(\xi+2r)^2$, $t = 0$,
   4. $\xi > r$ and $-(\xi+2r)^2 \leq s \leq -(\xi-2)^2$, $t = 0$.

The zeros of the polynomial P_{2n}, see formula (25), can be given by

$$-1 \pm \sqrt{1-b}, \quad r \pm \sqrt{r^2-d},$$

where $a=1$, $c=-r$ has been used.

Inserting the explicit expressions for $1-b$ from (24) and r^2-d from (23), using also $\zeta, \omega \in \mathbb{R}$, we find that the numbers under the square root sign are real:

$$r^2-d=\frac{(\zeta-r)\{(\zeta+2r)^2+\omega)\}}{12m_1/(2m_1+m_2)}, \quad 1-b=\frac{(\zeta+1)\{(\zeta-2)^2+\omega)\}}{12m_2/(2m_1+m_2)}.$$

Moreover, the sign of these real numbers is given by

$$\begin{aligned}\operatorname{sgn}(r^2-d) &= \operatorname{sgn}(\zeta-r)((\zeta+2r)^2+\omega),\\ \operatorname{sgn}(1-b) &= -\operatorname{sgn}(\zeta+1)((\zeta-2)^2+\omega).\end{aligned}$$

Carefully checking all possibilities for ζ, ω given above, we conclude

$$r^2-d \geq 0, \quad 1-b \geq 0,$$

i.e. all zeros of P_{2n} are real and thus the conjecture is true.

4.4.3 The Case $\zeta \in \mathbb{C} \backslash \mathbb{R}$

Because of the definition of ζ in (30), we see that $\zeta \in \mathbb{C} \backslash \mathbb{R} \Longleftrightarrow \eta \neq 0$. We introduce some notation:

$$p := |\zeta - r| = \sqrt{(\xi-r)^2+\eta^2}, \tag{33}$$

$$q := |\zeta + 1| = \sqrt{(\xi+1)^2+\eta^2}, \tag{34}$$

$$\alpha + i\beta := (\zeta+2r)^2 \;\Rightarrow\; \alpha = (\xi+2r)^2-\eta^2,\; \beta = 2\eta(\xi+2r), \tag{35}$$

$$\gamma + i\delta := (\zeta-2)^2 \;\Rightarrow\; \gamma = (\xi-2)^2-\eta^2,\; \delta = 2\eta(\xi-2), \tag{36}$$

$$A := |(\zeta+2r)^2+\omega|, \tag{37}$$

$$B := |(\zeta-2)^2+\omega|. \tag{38}$$

Then we have:

$$A^2 = a_1^2 + a_2^2, \quad a_1 = -\alpha \sin \varphi + \beta \cos \varphi, \ a_2 = \rho + \alpha \cos \varphi + \beta \sin \varphi, \tag{39}$$

$$B^2 = b_1^2 + b_2^2, \quad b_1 = -\gamma \sin \varphi + \delta \cos \varphi, \ b_2 = \rho + \gamma \cos \varphi + \delta \sin \varphi, \tag{40}$$

and $G(\zeta, \omega)$ from (28) can be written as

$$G(\zeta, \omega) := 2\left[4(r^2 - r + 1) + \frac{pA + qB}{r+1} - 3(\xi^2 + \eta^2) - \rho\right]. \tag{41}$$

We have to show that $\inf_{\omega \in \mathbb{C}} G(\zeta, \omega) \geq 0$ for fixed $\zeta \in \mathbb{C}$.

The only possible points for which an infimum can occur are the cases:

(i) $\rho = 0$,
(ii) $\rho = \infty$,
(iii) $\partial G / \partial \varphi = \partial G / \partial \rho = 0$ with $0 < \rho < \infty$.

Case (i) When $\rho = |\omega| = 0$, we have $A = |\zeta + 2r|^2$, $B = |\zeta - 2|^2$, thus

$$G(\zeta, 0) = 2\left[4(r^2 - r + 1) + \frac{|\zeta - r| \cdot |\zeta + 2r|^2 + |\zeta + 1| \cdot |\zeta - 2|^2}{r+1} - 3|\zeta|^2\right]. \tag{42}$$

Since

$$\begin{aligned} &|\zeta - r| \cdot |\zeta + 2r|^2 + |\zeta + 1| \cdot |\zeta - 2|^2 \\ &\geq |(\zeta - r)(\zeta + 2r)^2 - (\zeta + 1)(\zeta - 2)^2| \\ &= (r+1)|3\zeta^2 - 4(r^2 - r + 1)| \\ &\geq (r+1)\big|3|\zeta^2| - 4(r^2 - r + 1)\big|, \end{aligned}$$

we see that $G(\zeta, 0) \geq 0$.
In case of equality we write

$$0 = G(\zeta, 0) \geq 2\Big[4(r^2 - r + 1) + \big|3\zeta^2 - 4(r^2 - r + 1)\big| - 3|\zeta|^2\Big] \geq 0,$$

implying

$$\left|3\zeta^2 - 4(r^2 - r + 1)\right| = 3|\zeta|^2 - 4(r^2 - r + 1). \tag{43}$$

With the notation $3\zeta^2 = ge^{\mathrm{i}\tau}$, $g > 0$ (as we are in the situation of $\zeta \notin \mathbb{R}$), formula (43) leads to

$$\{g\cos\tau - 4(r^2 - r + 1)\}^2 + \{g\sin\tau\}^2 = \{g - 4(r^2 - r + 1)\}^2,$$

which on simplification shows

$$8(r^2 - r + 1)g(1 - \cos\tau) = 0.$$

As $g \neq 0$, we must have $\cos\tau = 1$: $3\zeta^2 = g \in \mathbb{R}$. But then

$$3(\xi^2 - \eta^2) + 6\mathrm{i}\xi\eta = g,$$

which is equivalent to the two equations $\xi\eta = 0$ and $3(\xi^2 - \eta^2) = g$. As $\zeta \notin \mathbb{R}$, we find $\eta \neq 0$ and consequently $\xi = 0$ and $g = -3\eta^2 < 0$: a contradiction. Thus we have $G(\zeta, 0) > 0$.

Case (*ii*) As we have $\eta \neq 0$, then

$$|\zeta - r| + |\zeta + 1| > r + 1$$

and thus

$$\frac{G(\zeta,\omega)}{|\omega|} = \frac{8r^2 - 8r + 8}{|\omega|} + \frac{2}{r+1}\left\{|\zeta + 1|\left|\frac{(\zeta - 2)^2 + \omega}{\omega}\right| + |\zeta - r|\left|\frac{(\zeta + 2r)^2 + \omega}{\omega}\right|\right\} - 6\frac{|\zeta|^2}{|\omega|} - 2.$$

Hence

$$\lim_{|\omega|\to\infty}\frac{G(\zeta,\omega)}{|\omega|} = \frac{2}{r+1}\{|\zeta + 1| + |\zeta - r|\} - 2 > 0.$$

and so $\lim_{|\omega|\to\infty} G(\zeta,\omega) = +\infty$, showing that the infimum is not attained for $\rho \to \infty$.

Case (iii) Before the partial derivatives of $G(\zeta, \omega)$ with respect to ρ and φ can be calculated explicitly, we have to consider the cases $A=0$ and $B=0$. First assume $A=0$, then from (37) we see

$$\omega = -(\zeta + 2r)^2,$$

which implies for B from (38):

$$B = \left|(\zeta - 2)^2 - (\zeta + 2r)^2\right| = 4(r+1)|\zeta - 1 + r|.$$

Inserting these into G, we find

$$G(\zeta, \omega) = 8\left[|\zeta + 1| \cdot |\zeta + r - 1| - (\xi^2 + \eta^2 + r\xi + r - 1)\right].$$

For $\xi^2 + \eta^2 + r\xi + r - 1 \leq 0$, we find $G(\zeta, \omega) \geq |\zeta + 1| \cdot |\zeta + r - 1| > 0$, as $\eta \neq 0$.

For $\xi^2 + \eta^2 + r\xi + r - 1 > 0$ it is obvious that the sign of G can be given by

$$\operatorname{sgn} G = \operatorname{sgn}\left[|\zeta + 1|^2 \cdot |\zeta + r - 1|^2 - (\xi^2 + \eta^2 + r\xi + r - 1)^2\right]. \quad (44)$$

Simplification of the right hand side of (44) leads to the form $(2-r)^2\eta^2 > 0$, as $\eta \neq 0$ and $\frac{1}{2} < r < 2$.

The case $B=0$, i.e. $\omega = -(\zeta - 2)^2$ and thus $A = |(\zeta + 2r)^2 - (\zeta - 2)^2|$, can be treated in the same manner:

- for $\xi^2 + \eta^2 - \xi + r - r^2 \leq 0$:

$$G \geq |\zeta - r| \cdot |\zeta + r - 1| > 0;$$

- for $\xi^2 + \eta^2 - \xi + r - r^2 \leq 0$:

$$\operatorname{sgn} G = \operatorname{sgn}[|\zeta - r|^2 \cdot |\zeta + r - 1|^2 - (\xi^2 + \eta^2 - \xi + r - r^2)^2],$$

and the right hand side is equal to $(2r-1)^2\eta^2 > 0$ since $\eta \neq 0$ and $\frac{1}{2} < r < 2$.

Thus we can differentiate with respect to ξ and η to find the stationary points. Putting $\partial G/\partial\rho = \partial G/\partial\varphi = 0$ yields the conditions

$$\left.\begin{aligned} p\frac{a_2}{A} + q\frac{b_2}{B} &= r + 1, \\ p\frac{a_1}{A} + q\frac{b_1}{B} &= 0. \end{aligned}\right\} \tag{45}$$

We shall prove the following:

LEMMA 1 *If $A, B > 0$, then* (45) *is equivalent to*

$$\eta a_2 = \varepsilon(r - \xi)a_1, \tag{46}$$

$$\eta b_2 = -\varepsilon(1 + \xi)b_1, \tag{47}$$

$$\operatorname{sgn} a_2 \operatorname{sgn}(r - \xi) \geq 0, \tag{48}$$

$$\operatorname{sgn} b_2 \operatorname{sgn}(1 + \xi) \geq 0, \tag{49}$$

with $\varepsilon = \pm 1$.

Proof From (46)–(49), we get

$$\eta^2 a_2^2 = (r - \xi)^2 a_1^2,$$
$$\eta^2 b_2^2 = (1 + \xi)^2 b_1^2$$

so that using (33), (34), (39) and (40), we have

$$p^2 a_2^2 = \eta^2 a_2^2 + (r - \xi)^2 a_2^2 = (r - \xi)^2 (a_1^2 + a_2^2) = (r - \xi)^2 A^2,$$
$$q^2 b_2^2 = \eta^2 b_2^2 + (1 + \xi)^2 b_2^2 = (1 + \xi)^2 (b_1^2 + b_2^2) = (1 + \xi)^2 B^2.$$

Thus

$$pa_2 = (r - \xi)A \quad \text{and} \quad qb_2 = (1 + \xi)B, \tag{50}$$

which yields the first equation in (45). From (46) and (50), we get

$$p\frac{a_1}{A} = \frac{a_1}{a_2}(r - \xi) = \frac{\eta}{\varepsilon},$$
$$q\frac{b_1}{B} = \frac{b_1}{b_2}(1 + \xi) = -\frac{\eta}{\varepsilon},$$

which on adding leads to the second equation in (45).

We now show that (45) implies (46)–(49). We rewrite (45) in the following form:

$$p\frac{a_1}{A} = -q\frac{b_1}{B},$$
$$p\frac{a_2}{A} = r + 1 - \frac{qb_2}{B} \quad \left(\text{or } \frac{qb_2}{B} = r + 1 - \frac{pa_2}{A}\right).$$

Squaring and adding two of them, we get

$$p^2\frac{a_1^2}{A^2} + p^2\frac{a_2^2}{A^2} = q^2\frac{b_1^2}{B^2} + (r+1)^2 - 2(r+1)\frac{qb_2}{B} + \frac{q^2b_2^2}{B^2},$$

or

$$p^2 = (r+1)^2 - 2(r+1)q\frac{b_2}{B} + q^2. \tag{51}$$

Similarly we also get

$$q^2 = (r+1)^2 - 2(r+1)p\frac{a_2}{A} + p^2. \tag{52}$$

From the above (51) and (52), we obtain

$$\left.\begin{aligned} q\frac{b_2}{B} &= \frac{(r+1)^2 + q^2 - p^2}{2(r+1)} = \xi + 1, \\ p\frac{a_2}{A} &= \frac{(r+1)^2 + p^2 - q^2}{2(r+1)} = r - \xi, \end{aligned}\right\} \tag{53}$$

since $q^2 - p^2 = (2\xi + 1 - r)(r + 1)$. As $p, q \neq 0$ (while $\eta \neq 0$), (53) implies (48) and (49).

Squaring the equations in (53) and using (34), we obtain

$$q^2b_2^2 = \{(\xi+1)^2 + \eta^2\}b_2^2 = (\xi+1)^2(b_1^2 + b_2^2),$$

which gives

$$\eta^2b_2^2 = (q^2 - (\xi+1)^2)b_2^2 = (\xi+1)^2b_1^2.$$

Similarly using (33):

$$p^2 a_2^2 = \{(\xi - r)^2 + \eta^2\} a_2^2 = (r - \xi)^2 (a_1^2 + a_2^2),$$

which yields

$$\eta^2 a_2^2 = (r - \xi)^2 a_1^2.$$

Thus

$$\eta b_2 = (1 + \xi) b_1 \varepsilon_1, \quad \eta a_2 = (r - \xi) a_1 \varepsilon_2, \quad \text{with } \varepsilon_1, \varepsilon_2 = \pm 1. \tag{54}$$

There are now three possibilities:

1. $a_2 = 0, b_2 \neq 0,$
2. $a_2 \neq 0, b_2 = 0,$
3. $a_2, b_2 \neq 0.$

In the first and second case we have automatically $r - \xi = 0$ resp. $1 + \xi = 0$ as $A, B \neq 0$. This shows that we can choose $\varepsilon_1 = -\varepsilon_2$ without loss of generality. Moreover, this also implies that $a_2 = b_2 = 0$ is not possible as this would imply $r = \xi = -1$.

In the third case finally, we get from (54), (45), (53) and the fact that $r - \xi, 1 + \xi \neq 0$:

$$\begin{aligned} 0 = p\frac{a_1}{A} + q\frac{b_1}{B} &= p\frac{a_1 \eta}{(r - \xi)A}\varepsilon_2 + q\frac{b_2 \eta}{(1 + \xi)B}\varepsilon_1 \\ &= (r - \xi) \cdot \frac{\eta \varepsilon_2}{r - \xi} + (1 + \xi) \cdot \frac{\eta \varepsilon_1}{1 + \xi} \\ &= \eta \varepsilon_2 + \eta \varepsilon_1, \end{aligned}$$

whence $\varepsilon_1 = -\varepsilon_2$.

This completes the proof of the lemma.

Using the above lemma, we can calculate ρ from the following equations, where we have inserted the values a_1, a_2, b_1, b_2:

$$\eta(\rho + \alpha \cos \varphi + \beta \sin \varphi) = \varepsilon(r - \xi)(-\alpha \sin \varphi + \beta \cos \varphi), \tag{55}$$

$$\eta(\rho + \gamma \cos \varphi + \delta \sin \varphi) = -\varepsilon(1 + \xi)(-\gamma \sin \varphi + \delta \cos \varphi), \tag{56}$$

$$\operatorname{sgn}(\rho + \alpha \cos \varphi + \beta \sin \varphi) \operatorname{sgn}(r - \xi) \geq 0, \tag{57}$$

$$\operatorname{sgn}(\rho + \gamma \cos \varphi + \delta \sin \varphi) \operatorname{sgn}(1 + \xi) \geq 0. \tag{58}$$

Subtracting (56) from (55) and simplifying, we obtain

$$\begin{aligned}&[\eta(\beta-\delta)+\varepsilon\{\alpha(r-\xi)+\gamma(1+\xi)\}]\sin\varphi\\&\quad=[\varepsilon\{\beta(r-\xi)+\delta(1+\xi)\}-\eta(\alpha-\gamma)]\cos\varphi.\end{aligned}\tag{59}$$

From the definitions of $\alpha, \beta, \gamma, \delta$ in (35) and (36), we have

$$\left.\begin{aligned}&\alpha-\gamma=4(\xi+r-1)(r+1),\quad \beta-\delta=4\eta(r+1),\\&\alpha+\gamma=2(\xi^2+2(r-1)\xi+2(r^2+1)-\eta^2),\quad \beta+\delta=4\eta(\xi+r-1).\end{aligned}\right\}\tag{60}$$

Also

$$\left.\begin{aligned}&\alpha r+\gamma=(r+1)\{\xi^2-\eta^2+4(r-1)\xi+4(r^2-r+1)\},\\&\beta r+\delta=2\eta(r+1)(\xi+2r-2).\end{aligned}\right\}\tag{61}$$

Using the values in (60) and (61), we get from (59) the following:

$$\varepsilon=+1:\quad [3\xi^2-3\eta^2-4(r^2-r+1)]\sin\varphi=6\xi\eta\cos\varphi,\tag{62}$$

$$\varepsilon=-1:\quad [-3\xi^2-5\eta^2+4(r^2-r+1)]\sin\varphi=2\eta(\xi+4r-4)\cos\varphi.\tag{63}$$

The case $\varepsilon=1$. Put

$$\Phi:=\left[\left\{3\xi^2-3\eta^2-4(r^2-r+1)\right\}^2+(6\xi\eta)^2\right]^{1/2}.\tag{64}$$

Then $\Phi\geq 0$ and $\Phi=0$ if and only if

$$6\xi\eta=0\ \text{ and }\ 3\xi^2-3\eta^2-4(r^2-r+1)=0.$$

This can only happen if $\eta=0$ and $3\xi^2-4(r^2-r+1)=0$, but $\zeta\notin\mathbb{R}$, thus $\eta\neq 0$ and we have $\Phi>0$.

From (62) we have

$$\sin\varphi=\sigma\frac{6\xi\eta}{\Phi},\quad \cos\varphi=\frac{(3\xi^2-3\eta^2-4(r^2-r+1))\sigma}{\Phi},\qquad \sigma=\pm 1.\tag{65}$$

From (55) and the values of $\sin\varphi$, $\cos\varphi$ in (65), we can calculate ρ:

$$\rho+[(\xi+2r)^2-\eta^2]\frac{\sigma}{\Phi}\{3\xi^2-3\eta^2-4(r^2-r+1)\}+2\eta(\xi+2r)\frac{\sigma}{\Phi}6\xi\eta$$
$$=\frac{r-\xi}{\eta}\Big[-\{(\xi+2r)^2-\eta^2\}\frac{\sigma}{\Phi}6\xi\eta$$
$$+2\eta(\xi+2r)\times\frac{\sigma}{\Phi}\{3\xi^2-3\eta^2-4(r^2-r+1)\}\Big] \qquad (66)$$

A tedious calculation leads to

$$\rho=\frac{\sigma}{\Phi}[-3(\xi^2+\eta^2)^2+12(r^2-r+1)\xi^2-4(r^2-r+1)\eta^2-24r(r-1)\xi]. \qquad (67)$$

As $\rho>0$, this implies that the sign of σ is ruled by the sign of the quartic in ξ,η between the square brackets.

To find the left hand side of (55), which actually is the left hand side of (66), rewrite the right hand side of (66):

$$\rho+\alpha\cos\varphi+\beta\sin\varphi$$
$$=4(r-\xi)[-3r\xi^2-(8r^2-2r+2)\xi-3r\eta^2-4(r^3-r^2+r)]\frac{\sigma}{\Phi}. \qquad (68)$$

Because of (57), we see from the above that

$$\sigma[-3r\xi^2-(8r^2-2r+2)\xi-3r\eta^2-4(r^3-r^2+r)]\geq 0. \qquad (69)$$

From (56) we see similarly that

$$\rho+\gamma\cos\varphi+\delta\sin\varphi$$
$$=4(1+\xi)[-3\xi^2+(2r^2-2r+8)\xi-3\eta^2-4(r^2-2r+1)]\frac{\sigma}{\Phi}, \qquad (70)$$

and because of (58), we get

$$\sigma[-3\xi^2+(2r^2-2r+8)\xi-3\eta^2-4(r^2-r+1)]\geq 0. \qquad (71)$$

From (68) and (70), we can now obtain A and B. Indeed, we have

$$A^2=(\rho+\alpha\cos\varphi+\beta\sin\varphi)^2+(-\alpha\sin\varphi+\beta\cos\varphi)^2$$
$$=(\rho+\alpha\cos\varphi+\beta\sin\varphi)^2\left(1+\frac{\eta^2}{(r-\xi)^2}\right),$$

so that, using (68), we get

$$A = \frac{-4\sigma}{\Phi}[3r\xi^2 + (8r^2 - 2r + 2)\xi + 3r\eta^2 + 4r(r^2 - r + 1)]\sqrt{(r-\xi)^2 + \eta^2}.$$

Similarly, using (70)

$$B = \frac{-4\sigma}{\Phi}[3\xi^2 - (2r^2 - 2r + 8)\xi + 3\eta^2 + 4(r^2 - r + 1)]\sqrt{(1+\xi)^2 + \eta^2}.$$

Since $p = \sqrt{(r-\xi)^2 + \eta^2}$, $q = \sqrt{(\zeta + 1)^2 + \eta^2}$, an elementary calculation yields:

$$\begin{aligned}\frac{pA + qB}{r+1} = \frac{-\sigma}{\Phi}[12(\xi^2 + \eta^2)^2 - 36(r^2 - r + 1)\xi^2 + 24r(r-1)\xi \\ + 28(r^2 - r + 1)\eta^2 + 16(r^2 - r + 1)^2].\end{aligned} \qquad (72)$$

From (69)and (71), we see that if $\sigma = 1$, then

$$\xi < -\frac{r\{3\xi^2 + 3\eta^2 + 4(r^2 - r + 1)\}}{(8r^2 - 2r + 2)},$$

and

$$\xi > \frac{3\xi^2 + 3\eta^2 + 4(r^2 - r + 1)}{2r^2 - 2r + 8},$$

which is impossible since $-(1/r)(8r^2 - 2r + 2) < 0 < 2r^2 - 2r + 8$ and also $3\xi^2 + 3\eta^2 + 4(r^2 - r + 1) > 0$. Therefore $\sigma = -1$.

Using (67) and (72) with $\sigma = -1$, we now obtain

$$\begin{aligned}\frac{(pA + qB)}{r+1} - \rho &= \frac{1}{\Phi}[9(\xi^2 + \eta^2)^2 - 24(r^2 - r + 1)\xi^2 + 24(r^2 - r + 1)\eta^2 \\ &\quad + 16(r^2 - r + 1)^2] \\ &= \frac{1}{\Phi}[\{3\xi^2 - 3\eta^2 - 4(r^2 - r + 1)\}^2 + (6\xi\eta)^2] \\ &= \frac{\Phi^2}{\Phi} = \Phi.\end{aligned}$$

Hence we get

$$G(\zeta,\omega) = 2\left[4(r^2-r+1) - 3(\xi^2+\eta^2) + \frac{pA+qB}{r+1} - \rho\right]$$
$$= 2[4(r^2-r+1) - 3(\xi^2+\eta^2) + \Phi] > 0,$$

because

$$\Phi = \sqrt{\{3(\xi^2+\eta^2) - 4(r^2-r+1)\}^2 + 48(r^2-r+1)\eta^2}$$
$$> \left|3(\xi^2+\eta^2) - 4(r^2-r+1)\right|,$$

since $\eta \neq 0$.

The case $\varepsilon = -1$. In this case we have from (63)

$$[3\xi^2 + 5\eta^2 - 4(r^2-r+1)]\sin\varphi = -2\eta(\xi+4r-4)\cos\varphi. \qquad (73)$$

Put

$$\Psi := \left[\left\{3\xi^2 + 5\eta^2 - 4(r^2-r+1)\right\}^2 + \{2\eta(\xi+4r-4)\}^2\right]^{1/2}. \qquad (74)$$

Clearly $\Psi \geq 0$ and $\Psi = 0$ if and only if

$$3\xi^2 + 5\eta^2 - 4(r^2-r+1) = 0, \quad 2\eta(\xi+4r-4) = 0,$$

which implies, because $\eta \neq 0$, that (ξ, η) has to be one of the points satisfying

$$\xi = -4(r-1), \quad 5\eta^2 = -4(11r^2 - 23r + 11). \qquad (75)$$

The case $\Psi \neq 0$. We first assume $\Psi > 0$, then (73) implies

$$\sin\varphi = -\frac{\sigma}{\Psi}2\eta(\xi+4r-4), \quad \cos\varphi = \frac{\sigma}{\Psi}[3\xi^2 + 5\eta^2 - 4(r^2-r+1)], \qquad (76)$$

where $\sigma = \pm 1$. Using the values of $\sin\varphi$ and $\cos\varphi$ from (76), we obtain from (55)

$$\rho + \alpha\cos\varphi + \beta\sin\varphi = -\frac{r-\xi}{\eta}(-\alpha\sin\varphi + \beta\cos\varphi)$$
$$= -\frac{4\sigma(r-\xi)}{\Psi}\mathcal{F}, \qquad (77)$$

where $\mathcal{F}$ is defined by

$$\mathcal{F} = \big[2\xi^3 + 2\eta^2\xi + (7r-2)\xi^2 + (8r^2 - 6r - 2)\xi \\ +(3r+2)\eta^2 + (4r^3 - 4r^2 - 4r)\big], \tag{78}$$

where the values of α, β from (35) have been inserted.

Again using $\sin\varphi$, $\cos\varphi$ from (76), we can calculate ρ from (77):

$$\rho = \frac{\sigma}{\Psi}[5(\xi^2 + \eta^2)^2 - (4r^2 + 20r + 4)\xi^2 - (4r^2 + 36r + 4)\eta^2 \\ + 8(r-1)\xi^3 - 8(r^2 - r)\xi + 8(r-1)\eta^2\xi + 32r^2]. \tag{79}$$

Similarly using (56) and (76), we obtain

$$\begin{aligned} \rho + \gamma\cos\varphi + \delta\sin\varphi &= \frac{1+\xi}{\eta}(-\gamma\sin\varphi + \delta\cos\varphi) \\ &= -\frac{4\sigma(1+\xi)}{\Psi}\mathcal{G}, \end{aligned} \tag{80}$$

with

$$\mathcal{G} = \big[-2\xi^3 - 2\eta^2\xi - (2r-7)\xi^2 + (2r^2 + 6r - 8)\xi \\ +(2r+3)\eta^2 - (4r^2 + 4r - 4)\big]. \tag{81}$$

From (57) and (77) we see

$$\mathcal{F}\sigma \leq 0, \tag{82}$$

and from (58) and (80)

$$\mathcal{G}\sigma \leq 0. \tag{83}$$

Looking more closely at what happens if both inequalities turn into equalities, it is simple to show

$$(\mathcal{F} + \mathcal{G})\sigma < 0. \tag{84}$$

Indeed, assuming $\mathcal{F}\sigma = \mathcal{G}\sigma = 0$, recalling the definition of a_1, a_2, b_1, b_2 in (39) and (40), formula (77) resp. (80) imply

$$a_2 = -\frac{r-\xi}{\eta}\cdot a_1 = 0, \quad \text{resp. } b_2 = \frac{1+\xi}{\eta}\cdot b_1 = 0.$$

As we are still working under the assumptions $A = \sqrt{a_1^2 + a_2^2} > 0$ and $B = \sqrt{b_1^2 + b_2^2} > 0$, we must therefore have $r - \xi = 0$ *and* $1 + \xi = 0$. This, however leads to $r = -1$: a contradiction with $\frac{1}{2} < r < 2$.

From (39) and the values of a_1, a_2 as given in (77), we can calculate

$$A^2 = 16\{(\xi - r)^2 + \eta^2\} \cdot \frac{\mathcal{F}^2}{\Psi^2},$$

and from (40) and the values of b_1, b_2 given in (80) we get

$$B^2 = 16\{(\xi + 1)^2 + \eta^2\} \cdot \frac{\mathcal{G}^2}{\Psi^2}.$$

Using $p = \sqrt{(\xi - r)^2 + \eta^2}$ from (33) and $q = \sqrt{(\xi + 1)^2 + \eta^2}$ from (34) and the inequalities (82), (83), this implies

$$pA = 4\{(\xi - r)^2 + \eta^2\} \cdot \frac{-\sigma\mathcal{F}}{\Psi}, \quad qB = 4\{(\xi + 1)^2 + \eta^2\} \cdot \frac{-\sigma\mathcal{G}}{\Psi}. \tag{85}$$

Thus

$$\begin{aligned}\frac{pA + qB}{r + 1} = \frac{\sigma}{\Psi}[&-4\xi^4 - 24\xi^2\eta^2 - 20\eta^4 + 8(r - 1)\xi^3 - 24(r - 1)\eta^2\xi \\ &+ 4(5r^2 - 11r + 5)\xi^2 - 4(7r^2 - 13r + 7)\eta^2 \\ &- 8r(r - 1)\xi - 16(r^4 - 2r^3 + r^2 - 2r + 1)].\end{aligned} \tag{86}$$

Now (86) leads together with the value for ρ from (79) to the following form of the conjecture

$$\inf_{\omega} G(\zeta, \omega) = 2\{4(r^2 - r + 1) - 3(\xi^2 + \eta^2) - \sigma\Psi\} \geq 0, \tag{87}$$

because $(pA + qB)/(r + 1) - \rho = (\sigma/\Psi) \cdot (-\sigma\Psi^2) = -\sigma\Psi$.

In order to study the difference between $4(r^2 - r + 1) - 3(\xi^2 + \eta^2)$ and $\sigma\Psi$ on the left hand side of (87), we first calculate

$$\begin{aligned}&\Psi^2 - [4(r^2 - r + 1) - 3(\xi^2 + \eta^2)]^2 \\ &\quad= [3\xi^2 + 5\eta^2 - 4(r^2 - r + 1)]^2 + [2\eta(\xi + 4r - 4)]^2 \\ &\qquad - [4(r^2 - r + 1) - 3(\xi^2 + \eta^2)]^2 \\ &\quad= 16\eta^2[\xi^2 + \eta^2 + 2(r - 1)\xi + 3r^2 - 7r + 3] = 16\eta^2 c(\xi, \eta),\end{aligned} \tag{88}$$

where

$$c(\xi, \eta) := \xi^2 + \eta^2 + 2(r-1)\xi + 3r^2 - 7r + 3, \tag{89}$$

which can be written as

$$c(\xi, \eta) = (\xi + r - 1)^2 + \eta^2 - (2r - 1)(2 - r). \tag{90}$$

From the inequalities $\frac{1}{2} < r < 2$ it is immediately clear that the sign of $c(\xi, \eta)$ describes the location of the points (ξ, η) with respect to a circle with center $(1 - r, 0)$ and radius $\sqrt{(2r-1)(2-r)}$.

We now distinguish two cases:

(i) $c(\xi, \eta) \leq 0$,
(ii) $c(\xi, \eta) > 0$.

First we consider case (i):

$$(\xi + r - 1)^2 + \eta^2 \leq (2r - 1)(2 - r). \tag{91}$$

Then (88) shows

$$\Psi \leq |4(r^2 - r + 1) - 3(\xi^2 + \eta^2)|, \tag{92}$$

and the conjecture, i.e. (87), follows *irrespective of the value for* σ if we can only show that the absolute value bars in (92) may be omitted. This means, think of (91), that we have to prove

$$c(\xi, \eta) \leq 0 \ \Rightarrow \ 4(r^2 - r + 1) \geq 3(\xi^2 + \eta^2). \tag{93}$$

An arbitrary point (ξ, η) satisfying (91) can be given as

$$\xi = 1 - r + \lambda \cos\theta, \quad \eta = \lambda \sin\theta, \tag{94}$$

with

$$0 \leq \lambda \leq \sqrt{(2r-1)(2-r)}, \quad 0 \leq \theta < 2\pi. \tag{95}$$

Calculating the value of $3(\xi^2+\eta^2)$, using the values from (94), and replacing λ^2 by its maximal value, we find the upper bound

$$3(\xi^2+\eta^2) \leq 6(1-r)\lambda\cos\theta + 3(-r^2+3r-1). \tag{96}$$

In order to prove (93) in the case that $c(\xi,\eta) \leq 0$, it is sufficient to show

$$6(1-r)\lambda\cos\theta + 3(-r^2+3r-1) \leq 4(r^2-r+1)$$

or

$$6(1-r)\lambda\cos\theta \leq 7r^2-13r+7, \quad (\lambda,\theta) \text{ as in (95).}$$

This is equivalent to

$$6|1-r|\lambda \leq 7r^2-13r+7 \text{ for } 0 \leq \lambda \leq \sqrt{(2r-1)(2-r)}. \tag{97}$$

From

$$(7r^2-3r+7)^2 - 36(1-r)^2 \cdot (2r-1)(r-2) = (11r^2-23r+11)^2 \geq 0$$

(97) follows immediately. *Conclusion*: we can drop the absolute value bars in (92), showing $G(\zeta,\omega) \geq 0$ in the case $c(\xi,\eta) \leq 0$.

If, however, the infimum is *equal to zero* in this case, the fact that $0 < \Psi \leq 4(r^2-r+1) - 3(\xi^2+\eta^2)$ shows that we necessarily have

$$\Psi = 4(r^2-r+1) - 3(\xi^2+\eta^2), \tag{98}$$

$$\sigma = 1 \tag{99}$$

Indeed: $4(r^2-r+1) - 3(\xi^2+\eta^2) \geq 0$ and $\Psi > 0$ rule out the possibility $\sigma = -1$ when (87) is an equality!

But now (98), compare (88), leads to an equality sign in $c(\xi,\eta) \leq 0$, i.e.

$$(\xi+r-1)^2+\eta^2 = (2r-1)(2-r). \tag{100}$$

Furthermore, (99) and (84) imply

$$\mathcal{F}+\mathcal{G} = (r+1)\{5\xi^2+10(r-1)+5\eta^2+4(r^2-r+1)\} < 0,$$

which can be written as

$$5\{(\xi + r - 1)^2 + \eta^2\} < r^2 + 2r + 1. \tag{101}$$

Inserting the value (100) of the left hand side, (101) reduces to

$$-11r^2 + 23r - 11 < 0,$$

which contradicts the condition (9) on r. Thus $G(\zeta, \omega) > 0$.

Finally we consider case (ii) i.e. $c(\xi, \eta) > 0$:

$$(\xi + r - 1)^2 + \eta^2 > (2r - 1)(2 - r). \tag{102}$$

Now (88) implies

$$\Psi > |4(r^2 - r + 1) - 3(\xi^2 + \eta^2)|, \tag{103}$$

and (87) can only be correct if we have $\sigma = -1$! Moreover, (103) shows that we then automatically have $\inf_\omega G(\zeta, \omega) > 0$.

Calculating the sum of $\mathcal{F}$ and $\mathcal{G}$ from (78) and (81), this sum governs the sign of σ, we find

$$\begin{aligned} \frac{\mathcal{F} + \mathcal{G}}{r + 1} = \big[5\{\xi^2 + \eta^2 + 2(r - 1)\xi + 3r^2 - 7r + 3\} \\ - (11r^2 - 23r + 11)\big] > 0, \end{aligned} \tag{104}$$

because of (102) and the condition (9) on r. Thus (84) implies $\sigma = -1$ and the conjecture follows in the form $\inf_\omega G(\zeta, \omega) > 0$ from (103) and (87).

The case $\Psi = 0$. From (75) we know that this case only occurs at two distinct points (ξ, η) satisfying

$$\xi = 4(1 - r), \quad \eta^2 = -\tfrac{4}{5}(11r^2 - 23r + 11). \tag{105}$$

The range for r from (9) has to be written with strict inequalities

$$\frac{23 - 3\sqrt{5}}{22} < r < \frac{23 + 3\sqrt{5}}{22} \Leftrightarrow 11r^2 - 23r + 11 < 0, \tag{106}$$

as $r = (23 \pm 3\sqrt{5})/22$ would lead to $\eta = 0$.

Return to the form of the conjecture as given in (87)

$$\inf_{\omega} G(\zeta, \omega) = 2\{4(r^2 - r + 1) - 3(\xi^2 + \eta^2) - \sigma\Psi\} \geq 0. \tag{107}$$

As G is clearly continuous as a function of (ξ, η), we can take the limit in (107) for $(\xi, \eta) \rightarrow \zeta_0 = (\xi_0, \eta_0)$ with (ξ_0, η_0) one of the solutions of (105). The value of G then turns out to be

$$\inf_{\omega} G(\zeta_0, \omega) = -\tfrac{16}{5}(11r^2 - 23r + 11) = 4\eta^2,$$

which is strictly greater than 0 when r satisfies (106).

This completes the proof for the polynomials (6).

References

[1] I.J. Schoenberg, A conjectured analogue of Rolle's theorem for polynomials with real or complex zeros, *Amer. Math. Monthly*, **93** (1986), 8–13.
[2] K.G. Ivanov and A. Sharma, Quadratic mean radius of a polynomial in $\mathbb{C}(z)$, *Serdica Math. J.*, **22** (1996), 497–514.
[3] F.J. van den Berg, Nogmaals over afgeleide wortelpunten, *Nieuw Archief voor Wiskunde*, **15** (1888), 100–164.
[4] E. Beckenbach and R. Bellman, *An Introduction to Inequalities*, New York: Random House, 1961.
[5] N. Kazarinoff, *Geometric Inequalities*, New York: Random House, 1961.
[6] M. Marsden, *Geometry of Polynomials*, Mathematical Surveys, No. 3, *Amer. Math. Society*, 1966.
[7] D. Mitrinovic and S. Dragoslav, *Recent Advances in Geometric Inequalities*, Boston: Kluwer Academic, 1989.