

Another Approach to the Theory of Differential Inequalities Relative to Changes in the Initial Times

V. LAKSHMIKANTHAM^{a,*}, S. LEELA^b and J. VASUNDHARA DEVI^c

^aFlorida Institute of Technology, Department of Mathematical Sciences, Melbourne, FL 32901, USA; ^bSUNY at Geneseo, Department of Mathematics, Geneseo, NY 14454, USA; ^cGayathri Vidya Parishad College, Department of Mathematics, Visakhapatnam 530 016, India

(Received 20 October 1998; Revised 12 November 1998)

In Lakshmikantam *et al.*, *J. Nonlinear Analysis Appl.* (to appear), Lakshmikantam and Vatsala, *J. Inequalities Appl.* (to appear) and Shaw and Yakur (to appear), the investigation of the initial value problems of differential equations relative to the changes in the initial time is initiated and an approach is discussed. In this paper, another direction is taken to study the same situation obtaining different set of conditions.

Keywords: Changes in initial time for solutions; Initial and terminal value problems

AMS Subject Classification: 34A40; 34D20

1. INTRODUCTION

An investigation of initial value problems (IVPs) of differential equations where the initial time differs with each solution is initiated recently in [3,4,7]. When we deal with the real world phenomena, it is impossible not to make errors in the starting time and therefore it is important to study the variance in initial time. When we do consider such a change of initial time for each solution, then we are faced with the

* Corresponding author.

problem of comparing any two solutions which differ in the initial starting time. There may be several ways of comparing and to each choice of measuring the difference of two solutions, we may obtain a different result. In [3,4,7], an attempt was made to discuss such situations in one direction. In this paper, we shall consider another approach of comparing so that we can utilize the existing results.

2. PRELIMINARIES

Consider the differential equation

$$x' = f(t, x), \quad x(0) = x_0, \quad (2.1)$$

where $f \in C[I \times R, R]$, $I = [0, T]$. Let us list the following results for our later use.

THEOREM 2.1 *Let*

- (i) $\alpha_0, \beta_0 \in C^1[I, R]$, $f \in C[I \times R, R]$ and $\alpha'_0 \leq f(t, \alpha_0)$, $\beta'_0 \geq f(t, \beta_0)$ on I ;
- (ii) $\alpha_0(t) \leq \beta_0(t)$ on I and $\alpha_0(0) \leq x_0 \leq \beta_0(0)$;
- (iii) $f(t, x) - f(t, y) \geq -M(x - y)$, $\alpha_0(t) \leq y \leq x \leq \beta_0(t)$, $t \in I$.

Then there exist monotone sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ such that

$$\alpha_n(t) \rightarrow \rho(t), \quad \beta_n(t) \rightarrow r(t) \quad \text{as } n \rightarrow \infty,$$

uniformly on I , (ρ, r) are the minimal and maximal solutions of (2.1) satisfying $\alpha_0(t) \leq \rho(t) \leq r(t) \leq \beta_0(t)$ on I .

For a proof, see [5]. We need the following lemma, to prove a corresponding result of Theorem 2.1 for terminal value problems (TVPs).

LEMMA 2.1 *Let $p \in C^1[I, R]$ and $p'(t) \leq -Mp(t)$, $M \geq 0$ with $p(T) \geq 0$. Then $p(t) \geq 0$ on I .*

The proof is immediate since $p(t) \geq p(T)e^{M(T-t)}$ and $p(T) \geq 0$.

We also need the following existence result.

THEOREM 2.2 *Assume that condition (i) of Theorem 2.1 holds. Suppose that $\alpha_0(0) \leq \beta_0(0)$. Then there exists a solution $x(t)$ of IVP (2.1) satisfying*

$$m(t) \leq x(t) \leq M(t) \text{ on } I,$$

provided $\alpha_0(0) \leq x_0 \leq \beta_0(0)$, where $m(t) = \min[\alpha_0(t), \beta_0(t)]$ and $M(t) = \max[\alpha_0(t), \beta(t)]$, $t \in I$.

This result is proved in [6] under the setting of Caratheodory condition. Note that the usual condition $\alpha_0(t) \leq \beta_0(t)$ on I is not assumed in Theorem 2.2. See [5,6] for details.

We can now indicate the proof of the following result on TVP

$$y' = f(t, y), \quad y(T) = y_0, \tag{2.2}$$

which corresponds to Theorem 2.1.

THEOREM 2.3 *Let condition (i) of Theorem 2.1 hold. Suppose further*

- (ii)* $\beta_0(T) \leq y_0 \leq \alpha_0(T)$ and $\beta_0(t) \leq \alpha_0(t)$ on I ;
- (iii)* $f(t, x) - f(t, y) \geq -M(x - y)$, $\beta_0(t) \leq y \leq x \leq \alpha_0(t)$, $t \in I$, $M \geq 0$.

Then there exist monotone sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ such that

$$\alpha_n(t) \rightarrow \rho(t), \quad \beta_n(t) \rightarrow r(t) \quad \text{as } n \rightarrow \infty,$$

uniformly and $y(t) = \rho(t) = r(t)$ is the unique solution of (2.2) satisfying $\beta_0(t) \leq y(t) \leq \alpha_0(t)$, $t \in I$.

Proof We shall merely indicate the proof. Consider the linear differential equations with TVPs given by

$$\begin{aligned} \alpha'_n &= f(t, \alpha_{n-1}) - M(\alpha_n - \alpha_{n-1}), & \alpha_n(T) &= y_0, \\ \beta'_n &= f(t, \beta_{n-1}) - M(\beta_n - \beta_{n-1}), & \beta_n(T) &= y_0. \end{aligned}$$

It is easy to see that for each n , α_n, β_n are the unique solutions of the TVPs on I . Also, using Lemma 2.1 and following the proof of Theorem 2.1 in [5], one can show that

$$\beta_0 \leq \beta_1 \leq \dots \leq \beta_n \leq \alpha_n \leq \dots \leq \alpha_1 \leq \alpha_0, \quad t \in I,$$

which yields using standard argument that $\alpha_n \rightarrow \rho$, $\beta_n \rightarrow r$ as $n \rightarrow \infty$ uniformly on I , (ρ, r) are solutions of TVP (2.2) and

$$\beta_0(t) \leq r(t) \leq \rho(t) \leq \alpha_0(t), \quad t \in I.$$

Since condition (ii)* is left uniqueness condition, it follows that $r(t) = \rho(t) = y(t)$ is the unique solution of TVP (2.2) and the proof is complete.

3. MAIN RESULTS

Let us begin with the basic results on differential inequalities.

THEOREM 3.1 *Assume that*

(i) $\alpha, \beta \in C^1[R_+, R], f \in C[R_+ \times R, R]$ and

$$\alpha' \leq f(t, \alpha) \quad \alpha(t_0) \leq x_0, \quad t \geq t_0 \geq 0,$$

$$\beta' \geq f(t, \beta), \quad \beta(\tau_0) \geq x_0, \quad t \geq \tau_0 \geq 0;$$

(ii) $f(t, x) - f(t, y) \leq L(x - y), x \geq y, t \geq \tau_0;$

(iii) $\tau_0 > t_0$ and $\int_{t_0}^{\tau_0} f(s, \alpha(s)) ds \leq 0.$

Then $\alpha(t) \leq \beta(t), t \geq \tau_0.$

Proof We have, using (i) and (iii),

$$\alpha(\tau_0) \leq \alpha(t_0) + \int_{t_0}^{\tau_0} f(s, \alpha(s)) ds \leq \alpha(t_0),$$

which implies $\alpha(\tau_0) \leq x_0 \leq \beta(\tau_0).$ The standard result [2,5] now yields the conclusion. Based on (2.2), the following existence result can be proved immediately.

THEOREM 3.2 *Assume that*

(i) $\alpha, \beta \in C^1[R_+, R], f \in C[R_+ \times R, R]$ and $\alpha' \leq f(t, \alpha), t_0 \leq t \leq \tau_0 + T, \beta' \geq f(t, \beta), \tau_0 \leq t \leq \tau_0 + T, t_0, \tau_0 \geq 0,$ and $\alpha(t_0) \leq x_0 \leq \beta(\tau_0);$

(ii) $t_0 < \tau_0$ and $\int_{t_0}^{\tau_0} f(s, \alpha(s)) ds \leq 0.$

Then there exists a solution $x(t)$ of IVP

$$x' = f(t, x), \quad x(\tau_0) = x_0, \quad (3.1)$$

satisfying

$$m(t) \leq x(t) \leq M(t), \quad \tau_0 \leq t \leq \tau_0 + T, \quad (3.2)$$

where m and M are the functions as defined in Theorem 2.2.

Proof As in Theorem 3.1, we get

$$\alpha(\tau_0) \leq \alpha(t_0) \leq x_0 \leq \beta(\tau_0).$$

We then apply Theorem 2.2 to get the desired result.

We shall next discuss monotone iterative technique in the present framework.

THEOREM 3.3 *Assume that conditions (i) and (ii) of Theorem 3.2 hold. Suppose further that*

$$f(t, x) - f(t, y) \geq -M(x - y), \quad x \geq y, \quad M \geq 0. \quad (3.3)$$

Then there exist maximal and minimal solutions (r, ρ) of IVP (3.1) on $\tau_0 \leq t \leq \tau_0 + T$ such that

$$m(t) \leq \rho(t) \leq r(t) \leq M(t), \quad \tau_0 \leq t \leq \tau_0 + T,$$

where m, M are the same functions defined in Theorem 2.2.

Proof We note that conditions (i) and (ii) of Theorem 3.2 show that $\alpha' \leq f(t, \alpha), \beta' \geq f(t, \beta)$ for $\tau_0 \leq t \leq \tau_0 + T$ and $\alpha(\tau_0) \leq x_0 \leq \beta(\tau_0)$. Let

$$A = [t \in J: \alpha(t) < \beta(t)],$$

$$B = [t \in J: \beta(t) < \alpha(t)] \quad \text{and} \quad C = J \setminus A \cup B,$$

where $J = [\tau_0, \tau_0 + T]$. The sets A and B are countable union of disjoint open intervals in J , namely,

$$A = \bigcup_{i=1}^{\infty} (a_i, b_i), \quad B = \bigcup_{j=1}^{\infty} (\bar{a}_j, \bar{b}_j).$$

As an application of Theorem 2.1, we get for each i , there exist minimal and maximal solutions (ρ_i, r_i) of IVP

$$x'_i = f(t, x_i), \quad x(a_i) = m(a_i) \quad \text{if } a_i \neq \tau_0, \quad x(\tau_0) = x_0,$$

such that

$$\alpha(t) \leq \rho_i(t) \leq r_i(t) \leq \beta(t) \quad \text{in } [a_i, b_i].$$

Similarly, applying Theorem 2.3, there exist, for each j , a unique solution $y_j(t)$ of TVP

$$y'_j = f(t, y_j), \quad y_j(\bar{b}_j) = m(\bar{b}_j)$$

such that

$$\beta(t) \leq y_j(t) \leq \alpha(t), \quad \text{for } t \in [\bar{a}_j, \bar{b}_j].$$

We now define (ρ, r) on J by

$$\rho(t) = \begin{cases} \rho_i(t) & \text{in } [a_i, b_i], \\ y_j(t) & \text{in } [\bar{a}_j, \bar{b}_j], \\ m(t) & \text{otherwise,} \end{cases} \quad r(t) = \begin{cases} r_i(t) & \text{in } [a_i, b_i], \\ y_j(t) & \text{in } [\bar{a}_j, \bar{b}_j], \\ m(t) & \text{otherwise,} \end{cases}$$

from which it is easy to see that $\rho' = f(t, \rho)$, $r' = f(t, r)$, $t \in A \cup B$. Since for any $t_1 \in C$, $\alpha'(t_1)$, $\beta'(t_1)$, $m'(t_1)$, $M'(t_1)$ exist and $\alpha'(t_1) = \beta'(t_1)$, it follows that $\alpha'(t_1) = \beta'(t_1) = m'(t_1) = M'(t_1)$. Moreover, we have

$$\frac{m(t) - m(t_1)}{t - t_1} \leq \frac{\rho(t) - \rho(t_1)}{t - t_1} \leq \frac{M(t) - M(t_1)}{t - t_1} \quad \text{for } t > t_1,$$

and as a result, we obtain $\rho'_+(t_1) = \alpha'_+(t_1)$. Similarly, for $t < t_1$, we can get $\rho'_-(t_1) = \alpha'_-(t_1)$ and therefore, $\rho'(t) = \alpha'(t) = f(t, \alpha(t)) = f(t, \rho(t))$ for $t \in C$. Similar proof holds for $r(t)$. We thus have

$$m(t) \leq \rho(t) \leq r(t) \leq M(t), \quad t \in J,$$

and the proof is complete.

4. VARIATION OF PARAMETERS

Consider the differential systems

$$y' = F(t, y), \tag{4.1}$$

$$x' = f(t, x), \tag{4.2}$$

where $f, F \in C[R_+ \times R^n, R^n]$ and $F(t, 0) \equiv 0$. If we assume that $\partial F / \partial y(t, y)$ exists and is continuous on $R_+ \times R^n$, then we know [2] that $\partial y / \partial t_0(t, t_0, x_0)$ and $\partial y / \partial x_0(t, t_0, x_0)$ are the solutions of the variational system

$$z' = F_y(t, y(t, t_0, x_0))z, \tag{4.3}$$

satisfying the initial conditions

$$\frac{\partial y}{\partial t_0}(t_0, t_0, x_0) = -F(t_0, x_0), \quad \frac{\partial y}{\partial x_0}(t_0, t_0, x_0) = \text{identity matrix}$$

and the identity

$$\frac{\partial y}{\partial t_0}(t, t_0, x_0) + \frac{\partial y}{\partial x_0}(t, t_0, x_0)F(t_0, x_0) \equiv 0, \quad \text{for } (t, t_0, x_0), \quad (4.4)$$

holds, where $y(t, t_0, x_0)$ is the solution of IVP (4.1) existing on $[t_0, \infty)$. Then the relation between the solutions $x(t, t_0, x_0)$ of (4.2) and $y(t, t_0, x_0)$ of (4.1), which start at the same initial date (t_0, x_0) , can be obtained by the method of variation of parameters. For, setting $p(s) = y(t, s, x(s))$, $x(s) = x(s, t_0, x_0)$, $t_0 \leq s \leq t$, we have

$$\frac{dp(s)}{ds} = \frac{\partial y}{\partial t_0}(t, s, x(s)) + \frac{\partial y}{\partial x_0}(t, s, x(s))f(s, x(s)) \equiv g(t, s, x(s)). \quad (4.5)$$

Integrating (4.5) from t to t_0 , we get

$$x(t, t_0, x_0) = y(t, t_0, x_0) + \int_{t_0}^t g(t, s, x(s)) ds, \quad t \geq t_0. \quad (4.6)$$

If $f(t, x) = F(t, x) + R(t, x)$, then using (4.4) one obtains the well known Alekseev's formula [2]

$$x(t, t_0, x_0) = y(t, t_0, x_0) + \int_{t_0}^t \frac{\partial y}{\partial x_0}(t, s, x(s))R(s, x(s)) ds, \quad t \geq t_0. \quad (4.7)$$

The foregoing method does not work when the solutions of (4.1) and (4.2) start with different initial data. We need a different strategy.

Let $x(t, \tau_0, x_0)$ and $y(t, t_0, z_0)$ be the solutions of (4.2) and (4.1) through (τ_0, x_0) and (t_0, z_0) respectively existing on $[\tau_0, \infty)$, $[t_0, \infty)$, and $\tau_0 > t_0$. To obtain a variation of parameters formula connecting these two solutions, we proceed as follows. Let $y(t, t_0, y_0)$ be the solution of (4.1) through (t_0, y_0) existing on $[t_0, \infty)$. Set $q(s) = y(t, t_0, y_0s)$, $0 \leq s \leq 1$, $t_0 \leq t \leq \tau_0$.

Then

$$\frac{dq(s)}{ds} = \frac{\partial y}{\partial x_0}(t, t_0, y_0 s) y_0,$$

which on integration yields

$$y(t, t_0, y_0) = y(t, t_0, 0) + \int_0^1 \frac{\partial y}{\partial x_0}(t, t_0, y_0 s) ds y_0. \quad (4.8)$$

Letting

$$z_0 = y(\tau_0, t_0, y_0), \quad (4.9)$$

we have, in view of uniqueness of solutions of (4.1),

$$y(t, \tau_0, z_0) = y(t, t_0, y_0), \quad t \geq \tau_0. \quad (4.10)$$

Since $y(t, t_0, 0) \equiv 0$, (4.8) reduce to

$$y(t, t_0, y_0) = \int_0^1 \frac{\partial y}{\partial x_0}(t, t_0, y_0 s) y_0 ds. \quad (4.11)$$

Employing the same argument as before, to the function $p(s) = y(t, \tau_0, x_0 s + (1-s)z_0)$, it follows that

$$y(t, \tau_0, x_0) = y(t, \tau_0, z_0) + \int_0^1 \frac{\partial y}{\partial x_0}(t, \tau_0, x_0 s + (1-s)z_0)(x_0 - z_0) ds. \quad (4.12)$$

As a result, we obtain from (4.6), (4.10), (4.11) and (4.12) the desired relation between $x(t, \tau_0, x_0)$ of (4.2) and $y(t, t_0, y_0)$ of (4.1), namely,

$$\begin{aligned} x(t, \tau_0, x_0) &= y(t, t_0, y_0) + \int_0^1 \frac{\partial y}{\partial x_0}(t, \tau_0, x_0 s + (1-s)z_0)(x_0 - z_0) ds \\ &\quad + \int_{\tau_0}^t g(t, s, x(s, \tau_0, x_0)) ds, \quad t \geq \tau_0. \end{aligned} \quad (4.13)$$

This reduces to

$$\begin{aligned}
 x(t, \tau_0, x_0) &= y(t, t_0, y_0) + \int_0^1 \frac{\partial y}{\partial x_0}(t, \tau_0, x_0 s + (1-s)z_0)(x_0 - z_0) \, ds \\
 &\quad + \int_{\tau_0}^t \frac{\partial y}{\partial x_0}(t, s, x(s))R(s, x(s, \tau_0, x_0)) \, ds, \quad t \geq \tau_0, \quad (4.14)
 \end{aligned}$$

if $f(t, x) = F(t, x) + R(t, x)$, in view of (4.4), which is analogous to (4.7) in the present setup.

Remark The interesting special case is when (τ_0, y_0) are such that

$$x_0 = y(\tau_0, t_0, y_0). \tag{4.15}$$

In this case, (4.13) and (4.14) reduce to (4.6) and (4.7) which are the usual relations.

5. STABILITY CRITERIA

We shall discuss a typical result concerning the stability behavior of solutions of (4.2) in our present framework. We need the following lemma [2].

LEMMA 5.1 *Let $f \in C[R_+ \times R^n, R^n]$ and $G(t, r) = \max_{|x-x_0| \leq r} |f(t, x)|$ for some x_0 and r . Suppose that $r^*(t, t_0, w_0)$ is the maximal solution of*

$$w' = G(t, w), \quad w(t_0) = w_0 \geq 0. \tag{5.1}$$

Let $x(t, t_0, y_0)$ be the solution of (4.2) through (t_0, y_0) . Then

$$|x(t, t_0, y_0) - x_0| \leq r^*(t, t_0, |x_0 - y_0|), \quad t_0 \leq t \leq \tau_0.$$

Proof Setting $m(t) = |x(t, t_0, y_0) - x_0|$, we see that

$$D^+ m(t) \leq |f(t, x(t, t_0, y_0))|, \quad m(t_0) = |x_0 - y_0|.$$

It then follows that $D^+ m(t) \leq \max_{|x-x_0| \leq m(t)} |f(t, x)| = G(t, m(t))$ for $t_0 \leq t \leq \tau_0$. The conclusion is immediate from the Comparison Theorem 1.4.1 in [2].

We need the following definitions of stability.

DEFINITION *The given solution $x_0(t, t_0, y_0)$ of (4.2) is said to uniformly stable, if given $\epsilon > 0$, $t_0, \tau_0 \in \mathbb{R}_+$, $\tau_0 > t_0$, there exist $\delta_1, \delta_2 > 0$ such that*

$$\begin{aligned} |x_0 - y_0| < \delta_1, \quad |t_0 - \tau_0| < \delta_2 \quad \text{implies} \\ |x(t, \tau_0, x_0) - x_0(t, t_0, y_0)| < \epsilon, \quad t \geq \tau_0. \end{aligned}$$

As a typical result, we prove the following result on stability criteria.

THEOREM 5.1 *Assume that*

(i) $V \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+]$, $V(t, x)$ is locally Lipschitzian in x and

$$\begin{aligned} D^+ V(t, x - y) \\ \equiv \limsup_{h \rightarrow 0^+} \frac{1}{h} [V(t + h, x - y + h(f(t, x) - f(t, y))) - V(t, x - y)] \\ \leq g(t, V(t, x - y)), \end{aligned}$$

where $g \in C[\mathbb{R}_+^2, \mathbb{R}]$;

(ii) $b(|x|) \leq V(t, x) \leq a(|x|)$, $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$ and $a, b \in \mathbb{K}$.

Then the stability properties of the trivial solution of

$$w' = g(t, w), \quad w(\tau_0) = w_0 \geq 0, \quad (5.2)$$

imply the corresponding stability properties of the solution $x_0(t, t_0, x_0)$ of (5.1).

Proof We shall only prove stability. Let $\epsilon > 0$, $t_0, \tau_0 \in \mathbb{R}_+$ be given. Assume that the trivial solution of (5.2) is uniformly stable. Then given $b(\epsilon) > 0$, $\tau_0 \geq 0$, there exists a $\delta_1 > 0$ such that

$$w_0 < \delta, \quad \text{implies} \quad w(t, \tau_0, w_0) < b(\epsilon), \quad t \geq \tau_0,$$

where $w(t, \tau_0, w_0)$ is any solution of (5.2). Choose $\delta_2 = a^{-1}(\delta_1)$. From Lemma 5.1, we have for $t = \tau_0$,

$$|x_0(\tau_0, t_0, y_0) - x_0| \leq r^*(\tau_0, t_0, |x_0 - y_0|).$$

By uniqueness of solutions of (5.1), we see that

$$x_0(t, t_0, y_0) = x_0(t, \tau_0, z_0),$$

where $z_0 = x_0(\tau_0, t_0, y_0)$. Since $\lim_{\tau_0 \rightarrow t_0, y_0 \rightarrow x_0} r^*(\tau_0, t_0, |x_0 - y_0|) = 0$, it follows that given $\delta_2 > 0$ there exist $\delta, \delta^0 > 0$ such that

$|x_0(\tau_0, t_0, y_0) - x_0| < \delta_2$; whenever $|t_0 - \tau_0| < \delta^0$ and $|x_0 - y_0| < \delta$. We claim that $|x_0 - y_0| < \delta$ and $|t_0 - \tau_0| < \delta^0$ implies $|x(t, \tau_0, x_0) - x_0(t, t_0, y_0)| < \epsilon$, $t \geq \tau_0$. If not, there would exist a $t_1 > \tau_0$ and a solution $x(t, \tau_0, x_0)$ with $|x_0 - y_0| < \delta$ and $|t_0 - \tau_0| < \delta^0$ satisfying

$$|x(t_1, \tau_0, x_0) - x_0(t_1, t_0, y_0)| = \epsilon \quad \text{and} \quad |x(t, \tau_0, x_0) - x_0(t, t_0, y_0)| \leq \epsilon$$

for $\tau_0 \leq t \leq t_1$. Hence using (i) and (ii), we get, with $w_0 = a(|x_0 - z_0|)$, and applying standard arguments [2],

$$\begin{aligned} b(\epsilon) &= b(|x(t_1, \tau_0, x_0) - x(t_1, t_0, y_0)|) \\ &\leq V(t_1, x(t_1, \tau_0, x_0) - x_0(t_1, t_0, y_0)) \\ &\leq r(t_1, \tau_0, V(\tau_0, x_0 - x_0(\tau_0, t_0, y_0))) \\ &\leq r(t_1, \tau_0, a(|x_0 - x_0(\tau_0, t_0, y_0)|)) \\ &= r(t_1, \tau_0, a(|x_0 - z_0|)) \\ &\leq r(t_1, \tau_0, a(\delta_2)) = r(t_1, \tau_0, \delta_1) < b(\epsilon). \end{aligned}$$

This contradiction proves uniform stability of the solution $x_0(t, t_0, y_0)$ of (5.1). Other stability results can be proved based on usual stability results and the foregoing proof [2].

Remark In Theorem 3.1, if conditions (ii) and (iii) are replaced by

- (i) $f(t, x) - f(t, y) \leq L(x - y)$, $x \geq y$ and $t \geq t_0$; and
- (ii) $t_0 > \tau_0 \geq 0$ and $\int_{\tau_0}^{t_0} f(s, \beta(s)) \, ds \geq 0$, then we have

$$\beta(t_0) \geq \beta(\tau_0) + \int_{\tau_0}^{t_0} f(s, \beta(s)) \, ds \geq \beta(\tau_0),$$

which gives $\beta(t_0) \geq \beta(\tau_0) \geq x_0 \geq \alpha(t_0)$ so that one can use the standard result, as before, to conclude $\alpha(t) \leq \beta(t)$, $t \geq t_0$. This change would provide the corresponding dual results, which are not discussed to avoid monotony.

References

- [1] V. Lakshmikantham, Advances in stability theory: Old and new, *Proc. of CESA-98 and IMACS Multiconference*, Tunisia 1998 (CD Rom). Also, *System Analysis, Modeling, Simulation* (to appear).
- [2] V. Lakshmikantham and S. Leela, *Differential and Integral Inequalities*, Vol. I, Academic Press, New York 1969.

- [3] V. Lakshmikantham, S. Leela and J. Vasundhara Devi, Stability criteria for solutions of differential equations relative to initial time difference, *J. Nonlinear Analysis and Appl.* (to appear).
- [4] V. Lakshmikantham and A.S. Vatsala, Differential inequalities with initial time difference and applications, *J. Inequalities and Appl.* (to appear).
- [5] G.S. Ladde, V. Lakshmikantham and A.S. Vatsala, *Monotone Iterative Technique for Nonlinear Differential Equations*, Pitman publishing, Inc. 1985.
- [6] C. Marcelli and P. Rubbioni, A new extension of classical Muller's theorem, *Nonlinear Analysis*, **28** (1997), 1759–1767.
- [7] M.D. Shaw and C. Yakur, Nonlinear variation of parameters with initial time difference (to appear).