

# Markov and Bernstein Type Inequalities for Polynomials

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In an answer to a question raised by chemist Mendeleev, A. Markov proved that if  $p(z) = \sum_{\nu=0}^n a_{\nu} z^{\nu}$  is a real polynomial of degree  $n$ , then

$$\max_{-1 \leq x \leq 1} |p'(x)| \leq n^2 \max_{-1 \leq x \leq 1} |p(x)|.$$

The above inequality which is known as Markov's Inequality is best possible and becomes equality for the Chebyshev polynomial  $T_n(x) = \cos n \cos^{-1} x$ .

Few years later, Serge Bernstein needed the analogue of this result for the unit disk in the complex plane instead of the interval  $[-1, 1]$  and the following is known as Bernstein's Inequality.

If  $p(z) = \sum_{\nu=0}^n a_{\nu} z^{\nu}$  is a polynomial of degree  $n$  then

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|.$$

This inequality is also best possible and is attained for  $p(z) = \lambda z^n$ ,  $\lambda$  being a complex number.

The above two inequalities have been the starting point of a considerable literature in Mathematics and in this article we discuss some of the research centered around these inequalities.

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## 1. INTRODUCTION

Some years after the chemist Mendeleev invented the periodic table of the elements he made a study of the specific gravity of a solution as a function of the percentage of the dissolved substance [81]. This function is of some practical importance: for example it is used in testing beer and wine for alcoholic content, and in testing the cooling system of an automobile for concentration of anti-freeze; but present-day physical chemists do not seem to find it as interesting as Mendeleev did. Nevertheless Mendeleev's study led to mathematical problems of great interest, some of which are even today inspiring research in Mathematics.

An example of the kind of curve that Mendeleev obtained is in Fig. 1 (alcohol in water, percentage by weight). He noticed that the curves could be closely approximated by successions of quadratic arcs and he wanted to know whether the corners where the arcs joined were really there, or just caused by errors of measurement. In mathematical terms, this amounts to considering a quadratic polynomial  $P(x) = px^2 + qx + r$  with  $|P(x)| \leq 1$  for  $-1 \leq x \leq 1$ , and estimating how large can  $|P'(x)|$  be on  $-1 \leq x \leq 1$  (for details, how the Mendeleev's problem in Chemistry amounts to this mathematical problem in polynomials, see ([11]). Surprisingly Mendeleev himself was able to solve this mathematical

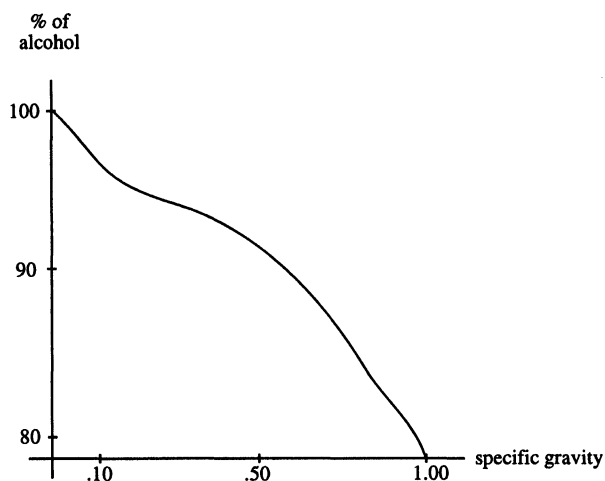


FIGURE 1

problem and proved that  $|P'(x)| \leq 4$ ; and this is the most that can be said, since when  $P(x) = 1 - 2x^2$  we have  $|P(x)| \leq 1$  for  $-1 \leq x \leq 1$  and  $|P'(\pm 1)| = 4$ . By using this result Mendelev was able to convince himself that the corners in his curve were genuine; and he was presumably right, since his measurements were quite accurate (they agree with modern tables to three or more significant figures).

Mendelev told his result to a Russian mathematician A.A. Markov, who naturally investigated the corresponding problem in a more general setup, that is, for polynomials of arbitrary degree  $n$ . He [77] proved the following result which is known as Markov's Theorem.

**THEOREM 1.1** *If  $p(x) = \sum_{\nu=0}^n a_{\nu}x^{\nu}$  is a real polynomial of degree  $n$  and  $|p(x)| \leq 1$  on  $[-1, 1]$  then*

$$|p'(x)| \leq n^2 \quad \text{for } -1 \leq x \leq 1. \tag{1.1}$$

*The inequality is best possible and is attained at only  $x = \pm 1$  only when  $p(x) = \pm T_n(x)$ , where  $T_n(x)$  (the so-called Chebyshev polynomial of the first kind) is  $\cos(n \cos^{-1} x)$  (which actually is a polynomial, since  $\cos n\theta$  is a polynomial in  $\cos \theta$ ). In fact*

$$T_n(x) = \cos(n \cos^{-1} x) = 2^{n-1} \prod_{\nu=1}^n \left\{ x - \cos \left( \left( \nu - \frac{1}{2} \right) \pi / n \right) \right\}.$$

It was several years later around 1926 when a Russian mathematician Serge Bernstein needed the analogue of Theorem 1.1 for the unit disk in the complex plane instead of the interval  $[-1, 1]$ . He wanted to know if  $p(z)$  is a polynomial of degree at most  $n$  (by a polynomial of degree at most  $n$  we mean an expression of the form  $\sum_{\nu=0}^n a_{\nu}z^{\nu}$ ,  $a_{\nu}$  being complex and  $z$  a complex variable) with  $|p(z)| \leq 1$  for  $|z| \leq 1$ , then what is  $\max |p'(z)|$  for  $|z| \leq 1$ ? The answer to this is given by the following which is known as Bernstein's inequality [9].

**THEOREM 1.2** *If  $p(z) = \sum_{\nu=0}^n a_{\nu}z^{\nu}$  is a polynomial of degree at most  $n$ , then*

$$\max_{|z| \leq 1} |p'(z)| \leq n \max_{|z| \leq 1} |p(z)|. \tag{1.2}$$

*The result is best possible and the equality holds for  $p(z) = \lambda z^n$ ,  $\lambda$  being a complex number.*

The above Bernstein's inequality has an analogue for trigonometric polynomials which states that if  $t(\theta) = \sum_{\nu=-n}^n a_{\nu} e^{i\nu\theta}$  is a trigonometric polynomial (possibly with complex coefficients) of degree  $n$ ,  $|t(\theta)| \leq 1$  for  $0 \leq \theta < 2\pi$  then for  $0 \leq \theta < 2\pi$ ,

$$|t'(\theta)| \leq n. \quad (1.3)$$

In (1.3) equality holds if and only if  $t(\theta) = e^{i\gamma} \cos(n\theta - \alpha)$ , where  $\gamma$  and  $\alpha$  are arbitrary real numbers.

Note that a trigonometric polynomial  $t(\theta) = \sum_{\nu=-n}^n c_{\nu} e^{i\nu\theta}$  (possibly with complex coefficients) is said to be real if  $c_{\nu} = \bar{c}_{-\nu}$ .

Inequality (1.3) is also known as Bernstein's inequality although Bernstein [9] proved (1.3) with  $2n$ , in place of  $n$ . His proof was based on a variational method. Inequality (1.3) in the present form first appeared in print in a paper of Feketé [38] who attributes the proof to Fejer [36]. Bernstein [10] attributes the proof to E. Landau (see [37,99]). Alternative proofs of the inequality (1.3) have been supplied by F. Riesz [94], M. Riesz [95], de la Vallée Poussin [106], Rogosinski [96] and others, and each of these methods has led to interesting extensions of the inequality (1.3).

If  $p(z) = \sum_{\nu=0}^n a_{\nu} z^{\nu}$  is a polynomial of degree at most  $n$ , then  $t(\theta) = p(e^{i\theta})$  is a trigonometric polynomial of degree  $n$  with  $|t(\theta)| \leq 1$ , real  $\theta$ , hence applying (1.3) to  $t(\theta) = p(e^{i\theta})$  one can get Theorem 1.2.

Bernstein needed the above inequalities in order to answer the following question of best approximation raised by de la Vallée Poussin in the early part of this century; Is it possible to approximate every polygonal line by polynomials of degree  $n$  with an error of  $o(1/n)$  as  $n$  becomes large? (The result that the approximation can be carried out with an error of  $o(1/n)$  was proved by de la Vallée Poussin himself).

This problem has played an important role in the development of the theory of approximation and was answered in the negative by Bernstein [9]. He in fact showed that the best approximation of the function  $|x|$  in the interval  $[-1, 1]$  by a polynomial of degree  $2n > 0$ , lies between  $(\sqrt{2} - 1)/4(2n - 1)$  and  $2/\pi(2n + 1)$ .

Inequalities of Markov and Bernstein type are fundamental for the proof of many inverse theorems in polynomial approximation theory (see [31,68,74,80,103]). For instance, Telyakovskii (see [82]) writes: "Among those that are fundamental in approximation theory are the

extremal problems connected with the inequalities for the derivatives of polynomials . . . . The use of inequalities of this kind is a fundamental method in proofs of inverse problems of approximation theory. Frequently further progress in inverse theorems has depended on first obtaining a corresponding generalization or analogue of Markov's and Bernstein's inequalities".

Several monographs and papers have been published in this area (see [11,30,82,91,93,104]), and it is not possible to include all of them here. The papers that we have referred to have many references on these topics.

This paper has seven sections, including Section 1, which is an introduction. In Section 2, we discuss some of the generalizations and refinements of the Bernstein and Markov inequalities while in Section 3, Bernstein type inequalities for polynomials having no zeros in a circle have been studied. Section 4 deals with the Bernstein type inequalities for polynomials having all their zeros in a circle. Section 5 is devoted to the Bernstein type inequalities for polynomials satisfying  $p(z) \equiv z^n p(1/z)$  and for the polynomials satisfying  $p(z) \equiv z^n \{p(1/\bar{z})\}$ . In Section 6, we give some Bernstein type inequalities in the  $L^p$ -norm which generalize some of the inequalities discussed in Sections 1–5. Lastly, in Section 7, we mention Bernstein type inequalities for wavelets, Bernstein–Markov type inequalities in normed spaces and other related results.

**2. SOME GENERALIZATIONS AND REFINEMENTS OF MARKOV'S THEOREM AND BERNSTEIN'S INEQUALITY**

We begin with the following theorem of A.A. Markov [77] which has already been stated in Section 1 (see Theorem 1.1).

**THEOREM 2.1** *If  $p(x) = \sum_{\nu=0}^n a_\nu x^\nu$  is a real polynomial of degree at most  $n$ ,  $|p(x)| \leq 1$  for  $-1 \leq x \leq 1$ , then for  $-1 \leq x \leq 1$ ,*

$$|p'(x)| \leq n^2. \tag{2.1}$$

*The result is best possible and the equality is attained for  $p(x) = \pm T_n(x)$ , where  $T_n(x)$  is the Chebyshev polynomial of the first kind.*

Clearly we can also assert that if  $|p(x)| \leq M$  on  $[-1, 1]$  then  $|p'(x)| \leq Mn^2$  on  $[-1, 1]$ . Having now found an upper bound for  $|p'(x)|$ , it would be natural to go on and ask for an upper bound for  $|p^{(k)}(x)|$

where  $k \leq n$ . Iterating Markov's theorem yields that if  $|p(x)| \leq M$  then  $|p^{(k)}(x)| \leq n^{2k}M$ , but this inequality is not sharp. V.A. Markov (the brother of A.A. Markov) considered the problem of determining exact bounds for  $|p^{(k)}(x)|$  on  $[-1, 1]$ . His results appeared in a Russian journal in the year 1892; a German version of this remarkable paper was later published in *Mathematische Annalen*. Among other things he [78] proved

**THEOREM 2.2** *Under the hypothesis of Theorem 2.1, we have for  $-1 \leq x \leq 1$ ,*

$$|p^{(k)}(x)| \leq \frac{n^2(n^2 - 1^2) \cdots (n^2 - (k - 1)^2)}{1 \cdot 3 \cdot 5 \cdots (2k - 1)}, \quad (2.2)$$

for every  $k = 1, 2, \dots, n$ .

The right hand side of this inequality is exactly equal to  $T_n^{(k)}(1)$  where  $T_n(x) = \cos(n \cos^{-1} x)$  is the Chebyshev polynomial of the first kind and thus this inequality is sharp.

It was shown by Duffin and Schaeffer [29] that for (2.2), it is enough to assume that  $|p(x)| \leq 1$  at the  $(n + 1)$  points  $x = \cos(k\pi/n)$ ;  $k = 0, 1, \dots, n$ . In particular they [29] showed

**THEOREM 2.3** *If  $p(x)$  is a polynomial of degree  $n$  with real coefficients such that*

$$|p(\cos \nu\pi/n)| \leq 1 \quad (\nu = 0, 1, 2, \dots, n) \quad (2.3)$$

then also (2.2) holds.

Obviously it is interesting to ask if there are  $(n + 1)$  other points in the interval  $(-1, 1)$  such that if  $|p(x)| \leq 1$  at these points then also (2.2) holds. Duffin and Schaeffer [29] gave a negative answer to this question. In fact they showed that if  $E$  is any closed subset of  $(-1, 1)$  which does not contain all the points  $\cos \nu\pi/n$ ;  $\nu = 0, 1, 2, \dots, n$ , then there is a polynomial of degree  $n$  which is bounded by 1 on  $E$  but for which (2.2) does not hold.

The above refined Theorem 2.2 of Duffin and Schaeffer [29] is known as Duffin–Schaeffer's inequality and has interesting applications in *Numerical Analysis* (see [8]). Bojanov and Nikolov [13] proved Duffin–Schaeffer type inequality for ultraspherical polynomials and Bojanov

and Rahman [14] established some related extremal problems for algebraic polynomials.

The paper of Markov [78] contains several interesting results besides Theorem 2.2. Among many other things, it contains

**THEOREM 2.4** *Let  $p(x) = \sum_{\nu=0}^n a_\nu z^\nu$  be as in Theorem 2.1. Then for  $1 \leq \nu \leq n$ , we have*

$$|a_\nu| = \frac{1}{\nu!} |p^{(\nu)}(0)| \leq \begin{cases} \frac{1}{\nu!} T_n^{(\nu)}(0), & \text{if } (n - \nu) \text{ is even,} \\ \frac{1}{\nu!} T_{n-1}^{(\nu)}(0), & \text{if } (n - \nu) \text{ is odd,} \end{cases}$$

and

$$\max_{-1 \leq x \leq 1} |p^{(\nu)}(x)| \leq T_n^{(\nu)}(1).$$

Erdős [35] showed that it is possible to improve upon Theorem 2.1 if the zeros of  $p(x)$  lie in  $\mathbb{R} \setminus (-1, 1)$ . In this connection he [35] proved

**THEOREM 2.5** *Let  $p(x)$  be as in Theorem 2.1 and let all the zeros of  $p$  be real and lie in  $\mathbb{R} \setminus (-1, 1)$ . Then*

$$|p'(x)| \leq \frac{n}{2} \left(1 - \frac{1}{n}\right)^{-n+1}, \quad \text{for } -1 \leq x \leq 1.$$

The above inequality becomes equality only at  $\pm 1$  for

$$p(x) = e^{i\gamma} \frac{n^n}{2^n(n-1)^{n-1}} (1+x)(1-x)^{n-1}, \quad \gamma \in \mathbb{R},$$

$$p(x) = e^{i\gamma} \frac{n^n}{2^n(n-1)^{n-1}} (1+x)^{n-1}(1-x), \quad \gamma \in \mathbb{R},$$

respectively.

Erdős [35] also proved

**THEOREM 2.6** *Let  $p(x)$  be a polynomial of degree at most  $n$ , and  $|p(x)| \leq 1$  for  $-1 \leq x \leq 1$ . If  $p(x)$  is real for real  $x$  and  $p(z) \neq 0$  for  $|z| < 1, z \in \mathbb{C}$ , then*

$$|p'(x)| \leq 4\sqrt{n}/(1 - |x|)^2, \quad \text{for } x \in [-1, 1].$$

It has been shown that in the above theorem,  $\sqrt{n}$  cannot be replaced by a function of  $n$  tending to infinity more slowly.

Markov type inequalities for constrained polynomials were obtained by Erdélyi [32,33]. Inequalities for derivatives of polynomials with real zeros were obtained by Szabados [101] and Szabados and Varma [102]. A conjecture of Szabados was later proved by Borwein [15] where he showed that for every polynomial of degree less than or equal to  $n$ , with  $k$  zeros in  $(-1, 1)$  and  $(n - k)$  zeros in  $\mathbb{R} \setminus (-1, 1)$ ,

$$\max_{-1 \leq x \leq 1} |p'(x)| \leq 9n(k + 1) \max_{-1 \leq x \leq 1} |p(x)|.$$

To formulate a general Markov–Bernstein inequality, Erdélyi and Szabados [34] conjectured an inequality which was later established by Borwein and Erdélyi [19] (see the paper for other interesting accounts). Some related extremal problems for polynomials were solved by Bojanov and Rahman [14].

Markov type inequalities for curved majorants were obtained by Varma [107,108]. Extremal polynomials for weighted Markov inequalities have been obtained by Mohapatra *et al.* [83] and extremal polynomial for Markov inequality on  $\mathbb{R}$  for the Hermite weight has been obtained by Li *et al.* [73].

Recently there has appeared a paper of Arsenault and Rahman [2] which contains many interesting results and historical development of the estimates of Markov type. (For further discussion and literature related to Markov and Bernstein type inequalities, see also [16,91,62])

As mentioned in Section 1, in order to answer a question of de la Vallée Poussin on best approximation which he raised in the early part of this century, Bernstein [9] proved and made considerable use of the following

**THEOREM 2.7** *If  $t(\theta) = \sum_{\nu=-n}^n a_\nu e^{i\nu\theta}$  is a trigonometric polynomial (possibly with complex coefficients) of degree  $n$ ,  $|t(\theta)| \leq 1$  for  $0 \leq \theta < 2\pi$ , then*

$$|t'(\theta)| \leq n, \tag{2.4}$$

*with equality holding if and only if  $t(\theta) = e^{i\gamma} \cos(n\theta - \alpha)$ , where  $\gamma$  and  $\alpha$  are arbitrary real numbers.*

As mentioned in Section 1, Bernstein [9] in fact proved (2.4) with  $2n$ , in place of  $n$  and the inequality (2.4) in the present form first appeared



in print in a paper of Feketé [38] who attributes the proof to Fejer [36]. Inequality (2.4) is also known as Bernstein's inequality.

If  $p(x) = \sum_{\nu=0}^n a_{\nu}x^{\nu}$  is a polynomial of degree  $n$  (possibly with complex coefficients) on  $(-1, 1)$  then  $p(\cos \theta)$  is a trigonometric polynomial of degree  $n$  and so by Bernstein's inequality (2.4) we have  $|p'(\cos \theta)\sin \theta| \leq n$ , which is equivalent to  $|p'(x)| \leq n(1 - x^2)^{-1/2}$  for  $-1 \leq x \leq 1$ . We thus have

**THEOREM 2.8** *If  $p(x) = \sum_{\nu=0}^n a_{\nu}x^{\nu}$  is a polynomial of degree  $n$  (possibly with complex coefficients) and  $|p(x)| \leq 1$  for  $-1 \leq x \leq 1$ , then*

$$|p'(x)| \leq \frac{n}{\sqrt{1 - x^2}}, \quad -1 \leq x \leq 1. \tag{2.5}$$

*The equality is attained at the points  $x = x_{\nu} = \cos[(2\nu - 1)\pi/2n]$ ,  $1 \leq \nu \leq n$ , if and only if  $p(x) = \gamma T_n(x)$ , where  $|\gamma| = 1$  and so (2.5) is best possible.*

The above theorem which was also proved by Bernstein [9] at the same time as Theorem 2.7 (with  $2n$  instead of  $n$ ) gives an estimate for  $|p'(x)|$  that is much better than Markov's when  $x$  is not near  $\pm 1$ , but it does not yield Markov's theorem directly since it tells us nothing about  $|p'(x)|$  when  $x$  is near  $\pm 1$ . However Markov's Theorem 2.1 can nevertheless be deduced from Bernstein's Theorem 2.7, and for this, note that if we apply Bernstein's inequality (2.4) to the polynomial  $q(\theta) = p(\cos \theta)\sin \theta$ , we get

$$|p(1)| \leq (n + 1) \max_{|x| \leq 1} |p(x)|\sqrt{1 - x^2}.$$

Now replacing the polynomial  $p$  by the polynomial  $p_u(x) = p(ux)$ , for a fixed  $u \in [-1, 1]$ , we obtain

$$\begin{aligned} |p(u)| &= |p_u(1)| \leq (n + 1) \max_{|x| \leq 1} |p(ux)|\sqrt{1 - x^2} \\ &\leq (n + 1) \max_{|x| \leq 1} |p(ux)|\sqrt{1 - (ux)^2} \\ &= (n + 1) \max_{|x| \leq |u|} |p(x)|\sqrt{1 - x^2} \\ &\leq (n + 1) \max_{|x| \leq 1} |p(x)|\sqrt{1 - x^2}, \end{aligned}$$

that is,

$$\max_{-1 \leq x \leq 1} |p(x)| \leq (n+1) \max_{-1 \leq x \leq 1} |p(x)| \sqrt{1-x^2},$$

which is known as Schur's inequality, and, if we combine Bernstein's inequality (Theorem 2.8) with this Schur's inequality (when applied to the polynomial  $p'(x)$ ), the Markov's inequality follows. For further generalizations of Schur's inequality, see [7].

As mentioned in Section 1, if  $p(z) = \sum_{\nu=0}^n a_\nu z^\nu$  is a polynomial of degree  $n$  where  $z$  is a complex variable, then  $t(\theta) = p(e^{i\theta})$  is a trigonometric polynomial of degree  $n$  with  $|t(\theta)| \leq 1$  for  $0 \leq \theta < 2\pi$ ; hence applying (2.4) to  $t(\theta) = p(e^{i\theta})$  we get the following, also known as Bernstein's inequality (see Theorem 1.2).

**THEOREM 2.9** *If  $p(z) = \sum_{\nu=0}^n a_\nu z^\nu$  is a polynomial of degree at most  $n$ , then*

$$\max_{|z| \leq 1} |p'(z)| \leq n \max_{|z| \leq 1} |p(z)|. \quad (2.6)$$

*The equality here holds for  $p(z) = \lambda z^n$ ,  $\lambda$  being a complex number.*

Note that by maximum modulus principle,  $\max_{|z| \leq 1} |p(z)| = \max_{|z|=1} |p(z)|$  and so if we denote  $\|p\| = \max_{|z|=1} |p(z)|$ , the inequality (2.6) can be written equivalently as

$$\|p'\| \leq n\|p\|. \quad (2.7)$$

Now we present proofs of Theorems 2.7 and 2.9. Note that Theorem 2.7 clearly yields Theorems 2.8 and 2.9.

The first proof that we present here is due to de Bruijn [26] (see [88]). It requires the following lemmas.

**LEMMA 2.1** *If  $p(z)$  is a polynomial of degree  $n$  with all its zeros in  $|z| \leq 1$ , and if  $q(z) = z^n \overline{p(1/\bar{z})}$ , then for  $|z| \geq 1$ ,*

$$|q'(z)| \leq |p'(z)|. \quad (2.8)$$

*Proof* Since  $p(z)$  has all its zeros in  $|z| \leq 1$ , the polynomial  $q(z)$  will have all its zeros in  $|z| \geq 1$ , and because  $|p(z)| = |q(z)|$  on  $|z| = 1$ , applying

maximum modulus principle to  $q(z)/p(z)$  which is analytic in  $|z| \geq 1$  we get that the polynomial  $q(z) - \lambda p(z)$  has no zeros outside the unit circle for every  $\lambda$  with  $|\lambda| > 1$ . By Gauss–Lucas theorem the polynomial  $q'(z) - \lambda p'(z)$  also does not vanish outside the unit circle for every  $\lambda$  with  $|\lambda| > 1$ , which implies that  $|q'(z)| \leq |p'(z)|$  for  $|z| \geq 1$ , and the Lemma 2.1 is established.

LEMMA 2.2 *If  $p(z)$  is a polynomial of degree  $n$  such that  $|p(z)| \leq 1$  for  $|z| \leq 1$  and if  $q(z) = z^n \overline{p(1/\bar{z})}$ , then for  $|z| \geq 1$ ,*

$$|p'(z)| + |q'(z)| \leq n|z|^{n-1}. \tag{2.9}$$

*Proof* Consider the polynomial  $p_1(z) = p(z) - \lambda z^n$ ,  $|\lambda| \geq 1$ . Rouché’s theorem implies that the zeros of  $p_1(z)$  lie inside the unit circle, and so by Gauss–Lucas theorem,  $p'_1(z) = p'(z) - \lambda n z^{n-1}$  also has all its zeros inside the unit circle and hence  $|p'(z)| \leq n|z|^{n-1}$  for  $|z| \geq 1$ . Therefore it is possible to choose  $\arg \lambda$  such that for  $|\lambda| > 1$ ,

$$|p'_1(z)| = |p'(z) - \lambda z^{n-1} n| = |\lambda| n |z|^{n-1} - |p'(z)|. \tag{2.10}$$

If  $q_1(z) = z^n \overline{p_1(1/\bar{z})}$ , then clearly

$$|q'_1(z)| = |q'(z)|, \tag{2.11}$$

and therefore on applying Lemma 2.1 to the polynomial  $p_1(z)$  which has all its zeros inside the unit circle, we get

$$|q'(z)| = |q'_1(z)| \leq |p'_1(z)| = |\lambda| n |z|^{n-1} - |p'(z)|,$$

which implies

$$|p'(z)| + |q'(z)| \leq |\lambda| n |z|^{n-1},$$

and letting  $|\lambda| \rightarrow 1$ , this yields for  $|z| \geq 1$ ,

$$|p'(z)| + |q'(z)| \leq n|z|^{n-1},$$

which is (2.9).

*Proof of Theorem 2.9* This follows trivially from Lemma 2.2.

To prove Theorem 2.7, first note that it is sufficient to prove the theorem for real trigonometric polynomials. For if  $t(\theta)$  is a trigonometric polynomial (not necessarily real) of degree  $n$ , and if  $\theta_0$  is a real number such that  $|t'(\theta_0)| = \max_{0 \leq \theta < 2\pi} |t'(\theta)|$  we may choose a complex constant  $\lambda$ ,  $|\lambda| = 1$ , such that  $\lambda t'(\theta_0)$  is positive. Clearly  $\text{Re}\{\lambda t(\theta)\}$  is a real trigonometric polynomial, so assuming that Theorem 2.7 has been proved for real trigonometric polynomials, we get

$$\left| \frac{d}{d\theta} \text{Re}\{\lambda t(\theta)\} \right| \leq n \max_{0 \leq \theta < 2\pi} |\text{Re}\{\lambda t(\theta)\}| \leq n \max_{0 \leq \theta < 2\pi} |t(\theta)|.$$

In particular,

$$|\text{Re}\{\lambda t'(\theta_0)\}| \leq n \max_{0 \leq \theta < 2\pi} |t(\theta)|,$$

and since  $\lambda t'(\theta_0) > 0$ , the above inequality is equivalent to

$$\{\lambda t'(\theta_0)\} \leq n \max_{0 \leq \theta < 2\pi} |t(\theta)|,$$

which implies

$$\max_{0 \leq \theta < 2\pi} |t'(\theta)| \leq n \max_{0 \leq \theta < 2\pi} |t(\theta)|,$$

and thus we have proved that if the Bernstein inequality (Theorem 2.7) holds for the real trigonometric polynomials it holds for the complex trigonometric polynomials as well.

To prove Theorem 2.7 for real trigonometric polynomials, let  $t(\theta) = \sum_{\nu=-n}^n c_\nu e^{i\nu\theta}$ ,  $c_\nu = \bar{c}_{-\nu}$  be a trigonometric polynomial of degree  $n$ ,  $|t(\theta)| \leq 1$  for real  $\theta$ , and let  $e^{in\theta} t(\theta) = p(e^{i\theta})$ ,  $\theta$  being real. Then  $p(z) = z^n \sum_{\nu=-n}^n c_\nu z^\nu$  is a polynomial of degree  $2n$ . If  $q(z) = z^{2n} \overline{p(1/\bar{z})}$ , then  $q(e^{i\theta}) = e^{2in\theta} \overline{p(e^{i\theta})} = e^{2in\theta} e^{-in\theta} \overline{t(\theta)} = e^{in\theta} t(\theta) = p(e^{i\theta})$ , for all real  $\theta$ . Also for  $|z| = 1$ ,  $|p(z)| = |p(e^{i\theta})| = |e^{in\theta} t(\theta)| = |t(\theta)| \leq 1$ , and so by maximum modulus principle,  $|p(z)| \leq 1$  for  $|z| \leq 1$ . By Lemma 2.2, we therefore get

$$\left| \frac{d}{d\theta} p(e^{i\theta}) \right| + \left| \frac{d}{d\theta} q(e^{i\theta}) \right| = 2 \left| \frac{d}{d\theta} p(e^{i\theta}) \right| \leq 2n,$$

which implies

$$|\operatorname{Im}t(\theta) + t'(\theta)| \leq n. \tag{2.12}$$

Since  $t(\theta)$  is real, (2.12) in fact gives

$$n^2 t^2(\theta) + \{t'(\theta)\}^2 \leq n^2, \tag{2.13}$$

an inequality sharper than (2.4) for real trigonometric polynomials. In the general case in which the trigonometric polynomial  $t(\theta)$  is complex, we cannot say that the sum of the absolute magnitudes of the two terms on the left is at most  $n^2$ . This can be seen from the example  $t(\theta) = e^{in\theta}$ . In-equality (2.13) was first explicitly stated by van der Corput and Schaake [23] although it is implicit in an earlier inequality due to Szegő [103].

We now present another proof of Theorem 2.9, which is due to O'Hara [85]. His proof depends on the following lemma.

LEMMA 2.3 *If  $P(z)$  is any complex polynomial of degree at most  $n$ , and  $z_1, z_2, \dots, z_n$  are the zeros of  $z^n + 1$ , then for every complex number  $t$ ,*

$$tP'(t) = \frac{n}{2}P(t) + \frac{1}{n} \sum_{\nu=1}^n P(tz_\nu) \frac{2z_\nu}{(z_\nu - 1)^2}. \tag{2.14}$$

*Proof* For a complex number  $t$ , define the function

$$\begin{aligned} Q_t(z) &= (P(tz) - P(t))/(z - 1), \quad z \neq 1 \\ Q_t(1) &= tP'(t). \end{aligned}$$

Then  $Q_t(z)$  is a polynomial of degree at most  $(n - 1)$ . Applying Lagrange's interpolation formula to  $Q_t(z)$ , with  $z_1, z_2, \dots, z_n$  as interpolation nodes, one gets

$$Q_t(z) = \sum_{\nu=1}^n \frac{Q_t(z_\nu)(z^n + 1)}{(z - z_\nu)nz_\nu^{n-1}} = \frac{1}{n} \sum_{\nu=1}^n \frac{Q_t(z_\nu)(z^n + 1)z_\nu}{(z_\nu - z)}.$$

Also, since  $Q_t(1) = tP'(t)$ , this gives

$$\begin{aligned} tP'(t) &= \frac{1}{n} \sum_{\nu=1}^n \frac{Q_t(z_\nu)2z_\nu}{(z_\nu - 1)} \\ &= \frac{1}{n} \sum_{\nu=1}^n \frac{2z_\nu P(tz_\nu)}{(z_\nu - 1)^2} - \frac{1}{n} \sum_{\nu=1}^n \frac{2z_\nu P(t)}{(z_\nu - 1)^2}. \end{aligned} \tag{2.15}$$

To complete the proof, we must show that

$$\frac{n}{2}P(t) = -\frac{P(t)}{n} \sum_{\nu=1}^n \frac{2z_{\nu}}{(z_{\nu}-1)^2},$$

or equivalently,

$$-\frac{n}{2} = \frac{1}{n} \sum_{\nu=1}^n \frac{2z_{\nu}}{(z_{\nu}-1)^2}, \quad (2.16)$$

which follows easily on applying (2.15) to  $P(t) = t^n$ , and noting that since  $z_{\nu}$  is a zero of  $z^n + 1$ , therefore  $z_{\nu}^n = -1$  for  $\nu = 1, 2, \dots, n$  and the proof of Lemma 2.3 thus follows.

*Another Proof of Theorem 2.9* By Lemma 2.3, we have for any complex  $z$ ,

$$\begin{aligned} |zp'(z)| &= \left| \frac{n}{2}p(z) + \frac{1}{n} \sum_{\nu=1}^n p(zz_{\nu}) \frac{2z_{\nu}}{(z_{\nu}-1)^2} \right| \\ &\leq \frac{n}{2}|p(z)| + \frac{1}{n} \sum_{\nu=1}^n |p(zz_{\nu})| \left| \frac{2z_{\nu}}{(z_{\nu}-1)^2} \right|. \end{aligned}$$

If  $z_0$  is a point on  $|z| = 1$  such that  $|p'(z_0)| = \max_{|z|=1} |p'(z)|$ , then

$$\begin{aligned} |p'(z_0)| &\leq \frac{n}{2}|p(z_0)| + \frac{1}{n} \sum_{\nu=1}^n |p(z_0z_{\nu})| \left| \frac{2z_{\nu}}{(z_{\nu}-1)^2} \right| \\ &\leq \frac{n}{2} \max_{|z|=1} |p(z)| + \frac{1}{n} \max_{|z|=1} |p(z)| \sum_{\nu=1}^n \left| \frac{2z_{\nu}}{(z_{\nu}-1)^2} \right| \\ &= \frac{n}{2} \max_{|z|=1} |p(z)| + \max_{|z|=1} |p(z)| \frac{1}{n} \sum_{\nu=1}^n \left( \frac{-2z_{\nu}}{(z_{\nu}-1)^2} \right), \end{aligned}$$

since as is easy to verify that

$$\frac{2z_{\nu}}{(z_{\nu}-1)^2} = -\frac{1}{2 \sin^2 \theta/2},$$

if  $z_\nu = e^{i\theta}$  (Note that  $z_\nu$  for  $\nu = 1, 2, \dots, n$  being zeros of  $z^n + 1$ , lie on  $|z| = 1$ ), we get

$$\begin{aligned} |p'(z_0)| &\leq \frac{n}{2} \max_{|z|=1} |p(z)| + \frac{n}{2} \max_{|z|=1} |p(z)|, \quad \text{by (2.16)} \\ &= n \max_{|z|=1} |p(z)| \end{aligned}$$

which proves Theorem 2.9.

For another proof of Theorem 2.7, see [82, p. 532].

In 1928, Szegő [103] proved Theorem 2.9 under a much weaker hypothesis, namely  $\max_{|z|=1} |\operatorname{Re} p(z)| \leq 1$  for  $|z| = 1$ . He in fact proved

**THEOREM 2.10** *If  $p(z) = \sum_{\nu=0}^n a_\nu z^\nu$  is a polynomial of degree  $n$  with  $|\operatorname{Re} p(z)| \leq 1$  for  $|z| \leq 1$ , then  $|p'(z)| \leq n$  for  $|z| \leq 1$ . Equality holds for  $|p(z)| = \lambda z^n$ ,  $|\lambda| = 1$ .*

Here we present a proof due to Malik [76] which is based on the application of the following result of de Bruijn [26], whose proof we omit.

**LEMMA 2.4** *Let  $C$  be a circular domain in the  $z$ -plane and  $\Delta$  an arbitrary point set in the  $\omega$ -plane. If the polynomial  $p(z)$  of degree  $n$  satisfies  $p(z) = \omega \in \Delta$  for every  $z \in C$ , then for every  $z$  and  $t$  in  $C$ ,*

$$\frac{t}{n} p'(z) + p(z) - \frac{z p'(z)}{n} \in \Delta. \tag{2.17}$$

*Here by a circular domain we mean the interior (or exterior) of a circle or a half-plane.*

*Proof of Theorem 2.10* Let  $C$  be the unit disk  $|z| \leq 1$  and let  $\Delta$  be the strip  $-1 \leq \operatorname{Re} \omega \leq 1$ . Since by hypothesis  $|\operatorname{Re} p(z)| \leq 1$  for  $|z| \leq 1$ , the image of the unit disk  $|z| \leq 1$  under the mapping  $\omega = p(z)$  is contained in the strip  $\Delta$ , and therefore by Lemma 2.4, if  $|z| \leq 1$ ,  $|t| \leq 1$ , then  $(t/n)p'(z) + p(z) - z(p'(z))/n \in \Delta$ . Setting  $t = 1$ , we see that a disk of radius  $|p'(z)|/n$  and center  $p(z) - z p'(z)/n$  (which also belongs to  $\Delta$ ; for this, take  $t = 0$ ), must be contained in  $-1 \leq \operatorname{Re} \omega \leq 1$  for any  $z$  with  $|z| \leq 1$ . Since the maximum modulus of such a disk cannot exceed 1, we get for any  $z$  with  $|z| \leq 1$ ,  $|p'(z)/n| \leq 1$ , which is equivalent to  $|p'(z)| \leq n$  for  $|z| \leq 1$  and so the Theorem 2.10 is established.

Theorem 2.10 was also proved by Mohapatra *et al.* [86] for which they first proved

**THEOREM 2.11** *Suppose  $\lambda$  is any complex number with  $|\lambda| = 1$  and let  $\tau_1, \tau_2, \dots, \tau_n$  be the  $n$ th roots of  $\lambda$ . If  $p(z)$  is a polynomial of degree at most  $n$ , then for all  $z$  on the unit circle  $|z| = 1$ ,*

$$np(z) - zp'(z) + \frac{\lambda}{z^{n-1}}p'(z) = \frac{1}{n} \sum_{\nu=1}^n p(\tau_\nu) \left| \frac{z^n - \lambda}{z - \tau_\nu} \right|^2 \quad (2.18)$$

and

$$\frac{1}{n} \sum_{\nu=1}^n \left| \frac{z^n - \lambda}{z - \tau_\nu} \right|^2 = n. \quad (2.19)$$

We omit the proof of this theorem. Note that (2.19) is a special case of (2.18) when  $p(z) = z^n$ .

If we replace  $\lambda$  by  $-\lambda$  in (2.18) and subtract the resulting inequality from (2.18) we obtain the identity that if  $p(z)$  is a polynomial of degree at most  $n$  then for all  $z$  on the unit circle  $|z| = 1$ ,

$$\frac{2\lambda p'(z)}{z^{n-1}} = \frac{1}{n} \sum_{\nu=1}^n p(\tau_\nu) \left| \frac{z^n - \lambda}{z - \tau_\nu} \right|^2 - \frac{1}{n} \sum_{\nu=1}^n p(\sigma_\nu) \left| \frac{z^n + \lambda}{z - \sigma_\nu} \right|^2, \quad (2.20)$$

where  $\sigma_1, \sigma_2, \dots, \sigma_n$  are the  $n$ th roots of  $-\lambda$ .

Using (2.18), (2.19) and (2.20), they gave simple proofs of Theorem 2.10 and of some other results.

*Proof of Theorem 2.10* Equating real parts in (2.20) and using triangle inequality along with (2.19) we get

$$2 \left| \operatorname{Re} \left[ \left( \frac{\lambda}{z^{n-1}} \right) p'(z) \right] \right| \leq n \max_{|z|=1} |\operatorname{Re} p(z)| + n \max_{|z|=1} |\operatorname{Re} p(z)|. \quad (2.21)$$

Let  $z_0$  be such that  $|p'(z_0)| = \max_{|z|=1} |p'(z)|$ . Then for some  $\epsilon$  with  $|\epsilon| = 1$ ,  $p'(z_0) = |p'(z_0)|\epsilon = \epsilon \max_{|z|=1} |p'(z)|$ . Theorem 2.10 now follows from (2.21) by putting  $z = z_0$  and  $\lambda = z_0^{n-1}/\epsilon$ .



By using Theorem 2.11, they also proved

**THEOREM 2.12** *If  $p(z)$  is a polynomial of degree at most  $n$ ,  $q(z) = z^n \overline{p(1/\bar{z})}$ , then*

$$\max_{|z|=1} (|p'(z)| + |q'(z)|) = n \max_{|z|=1} |p(z)|. \tag{2.22}$$

We omit the proof of this theorem. Note that (2.22) is an improved form of Lemma 2.2 which states that for  $|z| = 1$ ,

$$|p'(z)| + |q'(z)| \leq n \max_{|z|=1} |p(z)|.$$

The following theorem which is an improvement of Theorem 2.9 is also due to Mohapatra *et al.* [86].

**THEOREM 2.13** *Let  $z_1, z_2, \dots, z_{2n}$  be any  $2n$  equally spaced points on the unit circle, say  $z_\nu = ue^{i\nu\pi/n}$ ,  $|u| = 1$ ,  $1 \leq \nu \leq 2n$ . Then for any polynomial  $p(z)$  of degree at most  $n$ , we have*

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \left[ \max_{\nu \text{ odd}} |p(z_\nu)| + \max_{\nu \text{ even}} |p(z_\nu)| \right]. \tag{2.23}$$

As an application of Theorem 2.11, one can also obtain the following refinement of Bernstein's inequality given by Frappier *et al.* [43, Theorem 8].

**THEOREM 2.14** *If  $p(z)$  is a polynomial of degree at most  $n$ , then*

$$\max_{|z|=1} |p'(z)| \leq n \max_{1 \leq \nu \leq 2n} |p(e^{i\nu\pi/n})|, \tag{2.24}$$

*i.e., in (1.2) or in (2.6),  $\|p\| = \max_{|z|=1} |p(z)|$  can be replaced by the maximum of  $|p(z)|$  in the  $(2n)$ th roots of unity.*

In both Theorems 2.9 and 2.10, equality holds if and only if  $p(z) = \lambda z^n$ ,  $\lambda$  being a complex number, i.e. if  $p(z) = \sum_{\nu=0}^n a_\nu z^\nu$ , then we have equality if and only if  $a_0 = a_1 = a_2 = \dots = a_{n-1} = 0$ . Therefore if any of the  $a_i$ ,  $i = 0, 1, 2, \dots, n - 1$  is non-zero, then it should be possible to improve on

the bound in Bernstein's inequality. This fact was observed by Frappier *et al.* [43], who proved

**THEOREM 2.15** *Let  $p(z)$  be a polynomial of degree at most  $n$ , then for  $R > 1$ ,*

$$\|p(Rz) - p(z)\| + \psi_n(R)|p(0)| \leq (R^n - 1)\|p\|, \quad (2.25)$$

where

$$\psi_n(R) = \frac{(R-1)(R^{n-1} + R^{n-2})\{R^{n+1} + R^n - (n+1)R + (n-1)\}}{R^{n+1} + R^n - (n-1)R + (n-3)}, \quad (2.26)$$

if  $n \geq 2$ , and  $\psi_1(R) = R - 1$ . The coefficient of  $|p(0)|$  is the best possible for each  $R$ .

Dividing both sides of (2.25) by  $(R - 1)$  and letting  $R \rightarrow 1$ , we obtain

**COROLLARY 2.1** *If  $p(z)$  is a polynomial of degree at most  $n$ , then*

$$\|p'\| + \epsilon_n|p(0)| \leq n\|p\|, \quad (2.27)$$

where  $\epsilon_n = 2n/(n+2)$ , if  $n \geq 2$ , whereas  $\epsilon_1 = 1$ . The coefficient of  $|p(0)|$  is the best possible for each  $n$ .

For more results in this direction, see [100]

In order to prove the above inequalities, Frappier *et al.* [43] developed a method based on convolutions of analytic functions (see also [97]). Their method also gives the dependence of  $\|p'\|$  on the coefficient  $|a_1|$  and for this, they proved

**THEOREM 2.16** *For a polynomial  $p(z)$  of degree at most  $n$ , we have*

$$\|p'\| + c_n|p'(0)| \leq n\|p\|, \quad (2.28)$$

where  $c_1 = 0$ ,  $c_2 = \sqrt{2} - 1$ ,  $c_3 = 1/\sqrt{2}$ , whereas for  $n \geq 4$ ,  $c_n$  is the unique root of the equation

$$f(x) = 16n - 8(3n + 2)x^2 - 16x^3 + (n + 4)x^4 = 0,$$

lying in  $(0, 1)$ . The coefficient of  $|p(0)|$  is the best possible for each  $n$ .

Since  $f(x) = (4 + 4x - x^2)(4n - 4nx - (n + 4)x^2)$ , Frappier [40] found that the coefficient  $c_n$  appearing in Theorem 2.16 is in fact

$$c_n = \frac{2n}{n + 4} \left( \sqrt{\frac{2(n + 2)}{n}} - 1 \right), \quad n \geq 4. \tag{2.29}$$

Also, he proved the following more general

**THEOREM 2.17** *Let  $p(z) = \sum_{\nu=0}^n a_\nu z^\nu$ ,  $n \geq 4$  be a polynomial of degree at most  $n$ . If  $0 \leq B \leq c_n$ , then*

$$\|p'\| + \frac{4n - 4nB - (n + 4)B^2}{2(n + 2)} |a_0| + B|a_1| \leq n\|p\|. \tag{2.30}$$

For  $B=0$ , the above inequality reduces to (2.27) while for  $B = c_n$ , it reduces to (2.28).

Frappier [41] also proved the following, where  $\|p'\|$  depends on  $a_2$ .

**THEOREM 2.18** *If  $p(z) = \sum_{\nu=0}^n a_\nu z^\nu$  is a polynomial of degree at most  $n$ ,  $n \geq 6$ , then*

$$\|p'\| + d_n |a_2| \leq n\|p\|,$$

where  $d_n$  is the unique root of the equation

$$4n - (12n + 4)x^2 - x^3 + (5n + 7)x^4 - \frac{5}{2}x^5 - \frac{(n + 6)}{16}x^6 = 0.$$

### 3. BERNSTEIN TYPE INEQUALITIES FOR POLYNOMIALS WITH NO ZEROS IN A CIRCLE

We begin with Bernstein’s inequality mentioned in Sections 1 and 2 which states that if  $p(z)$  is a polynomial of degree at most  $n$ , then

$$\|p'\| \leq n\|p\|, \tag{3.1}$$

where as earlier,  $\|p\| = \max_{|z|=1} |p(z)|$ .

Since the equality in (3.1) holds if and only if  $p(z) = \lambda z^n$  ( $\lambda$  being a complex number) which has all its zeros at the origin, one would expect a relationship between the bound  $n$  and the distance of the zeros of the polynomial from the origin. This fact was observed by Erdős [35] who conjectured that if the polynomial  $p(z)$  has no zeros in  $|z| < 1$ , then  $\|p'\| \leq (n/2)\|p\|$ . This conjecture was proved in the special case when  $p(z)$  has all its zeros on  $|z| = 1$  independently by Polya and by Szegő (see [72]). In the general case the conjecture was proved for the first time by Lax [72], who proved

**THEOREM 3.1** *If  $p(z)$  is a polynomial of degree  $n$ ,  $p(z) \neq 0$  for  $|z| < 1$ , then*

$$\|p'\| \leq \frac{n}{2} \|p\|. \quad (3.2)$$

*The result is best possible and the equality in (3.2) holds for any polynomial which has all its zeros on  $|z| = 1$ .*

Simpler proofs of this result were later given by de Bruijn [26] and Aziz and Mohammed [6]. For some generalizations of Lax's result, Theorem 3.1 for entire functions of exponential type, see [12,44,45,63,89].

It was proposed by Professor R.P. Boas to obtain inequalities analogous to (3.1) for polynomials having no zeros in  $|z| < K$ ,  $K > 0$  and the following partial result in this connection was proved by Malik [76].

**THEOREM 3.2** *If  $p(z)$  is a polynomial of degree at most  $n$  having no zeros in  $|z| < K$ ,  $K \geq 1$ , then*

$$\|p'\| \leq \left( \frac{n}{1+K} \right) \|p\|. \quad (3.3)$$

*The result is best possible with equality holding for  $p(z) = (z+K)^n$ .*

For quite some time it was believed that if  $p(z) \neq 0$  in  $|z| < K$ ,  $K \leq 1$ , then the inequality analogous to (3.3) should be

$$\|p'\| \leq \frac{n}{1+K^n} \|p\|, \quad (3.4)$$

till Professor E.B. Saff gave the example  $p(z) = (z - \frac{1}{2})(z + \frac{1}{3})$  to counter this belief. As can be easily verified for this polynomial, the left hand side of (3.4) is approximately 2.1666 while the right hand side is  $[2/(1 + (\frac{1}{3})^2)]\|p\| \approx 2.144 < 2.1666$  and so (3.4) in general does not hold true.

For polynomials of degree  $n$  and having no zeros in  $|z| < K, K \leq 1$ , Govil [51] proved that

$$\|p'\| \leq \frac{n}{K^n + K^{n-1}} \|p\|. \tag{3.5}$$

Obviously the above bound is of interest only if  $K^n + K^{n-1} > 1$ . For another result in this direction see [52].

Govil and Rahman [63] generalized Theorem 3.2 of Malik [76] for any order derivative of the polynomial  $p(z)$  and proved

**THEOREM 3.3** *If  $p(z)$  is a polynomial of degree at most  $n$ , having no zeros in  $|z| < K, K \geq 1$ , then*

$$\|p^{(s)}\| \leq \frac{n(n-1)\cdots(n-s+1)}{1 + K^s} \|p\|. \tag{3.6}$$

For  $s = 1$ , (3.6) obviously reduces to (3.3).

Another generalization of (3.3) was later given by Chan and Malik [22] who proved

**THEOREM 3.4** *If  $p(z) = a_0 + \sum_{\nu=\mu}^n a_\nu z^\nu$  is a polynomial of degree at most  $n, p(z) \neq 0$  in  $|z| < K, K \geq 1$ , then*

$$\|p'\| \leq \frac{n}{1 + K^\mu} \|p\|. \tag{3.7}$$

*The equality in (3.7) is attained for  $p(z) = (z^\mu + K^\mu)^{n/\mu}$ ,  $n$  being a multiple of  $\mu$ .*

The inequality (3.7) in the case  $\mu = 2$  can also be found in [53, Lemma 4].

It can be shown that if  $p(z) \neq 0$  in  $|z| < K, K \geq 1$  then the equality in (3.3) can hold if and only if  $|a_1/a_0| = n/K$  and hence it should be possible to improve upon (3.3) if  $|a_1/a_0| \leq cn/K$  where  $0 \leq c \leq 1$ . This fact was

observed by Govil *et al.* [64] who obtained a bound in terms of the coefficients  $a_0$ ,  $a_1$  and  $a_2$ . They proved

**THEOREM 3.5** *If  $p(z) = \sum_{\nu=0}^n a_\nu z^\nu$  is a polynomial of degree at most  $n$ ,  $p(z) \neq 0$  in  $|z| < K$ ,  $K \geq 1$ , then*

$$\|p'\| \leq \frac{n|a_0| + K^2|a_1|}{(1 + K^2)n|a_0| + 2K^2|a_1|} \|p\|; \quad (3.8)$$

furthermore

$$\|p'\| \leq \left( \frac{n}{1 + K} \right) \frac{(1 - |\lambda|)(1 + K^2|\lambda|) + K(n - 1)|\mu - \lambda^2|}{(1 - |\lambda|)(1 - K + K^2 + K|\lambda|) + K(n - 1)|\mu - \lambda^2|} \|p\|, \quad (3.9)$$

where

$$\lambda = \frac{Ka_1}{na_0}, \quad \mu = \frac{2K^2}{n(n - 1)} \frac{a_2}{a_0}.$$

Both the above inequalities are best possible. For even  $n$ , the equality in (3.8) holds for

$$p(z) = \frac{a_0}{K^n} (ze^{i\gamma} + Ke^{i\alpha})^{n/2} (ze^{i\gamma} + Ke^{-i\alpha})^{n/2},$$

where  $\gamma$  and  $\alpha$  are arbitrary real numbers. Whether  $n$  is even or odd, equality holds in (3.9) for

$$p(z) = \frac{a_0}{K^n} (z + K)^{n_1} \left( z^2 + 2Kz \frac{na - n_1}{n - n_1} + K^2 \right)^{(n - n_1)/2},$$

if  $n_1$  is an integer such that  $n/3 \leq n_1 \leq n$ ,  $(n - n_1)$  is even, and  $(3n_1 - n)/(n + n_1) \leq a \leq 1$ .

Although Malik's inequality (3.3) is best possible, the drawback of this result is that the bound depends only upon the modulus of the zero of smallest modulus and not on the moduli of other zeros. For example, for both the polynomials  $p(z) = (z + K)^n$  and  $p(z) = (z + K)(z + K + \ell)^{n-1}$ ,  $K \geq 1$ ,  $\ell > 0$ , Malik's inequality (3.3) will give the same bound and for this reason it will obviously be of interest to obtain a bound which depends upon the location of all the zeros rather than just on the location of the

the zero of smallest modulus. This was done by Govil and Labelle [61] who proved

**THEOREM 3.6** *Let  $p(z) = a_n \prod_{\nu=1}^n (z - z_\nu)$ ,  $a_n \neq 0$ , be a polynomial of degree  $n$ . If  $|z_\nu| \geq K_\nu \geq 1$ ,  $1 \leq \nu \leq n$ , then*

$$\|p'\| \leq n \left\{ \left( \sum_{\nu=1}^n \frac{1}{K_\nu - 1} \right) / \left( \sum_{\nu=1}^n \frac{K_\nu + 1}{K_\nu - 1} \right) \right\} \|p\|. \tag{3.10}$$

In (3.10) the equality holds for  $p(z) = (z + K)^n$ ,  $K \geq 1$ .

As can be easily verified, inequality (3.10) is in fact equivalent to

$$\|p'\| \leq \frac{n}{2} \left\{ 1 - \frac{1}{1 + (2/n) \sum_{\nu=1}^n [1/K_\nu - 1]} \right\} \|p\|. \tag{3.11}$$

If  $K_\nu \geq K$ ,  $K \geq 1$  for  $1 \leq \nu \leq n$ , then clearly

$$\sum_{\nu=1}^n \frac{1}{K_\nu - 1} / \sum_{\nu=1}^n \frac{K_\nu + 1}{K_\nu - 1} \leq \frac{1}{1 + K},$$

so that the bound in (3.10) (or in (3.11)) is in general at least as sharp as in (3.3). In fact, excepting the case when the polynomial  $p(z)$  has all its zeros on  $|z| = K$ ,  $K > 1$ , the bound obtained by (3.10) (or by (3.11)) is always sharper than the bound obtainable from Malik's inequality (3.3). If  $K_\nu = 1$  for some  $\nu$ ,  $1 \leq \nu \leq n$ , then the inequality (3.10) (or (3.11)) reduces to Lax's inequality (3.2). The statement of the Theorem 3.6 might suggest that one needs to know all the zeros of the polynomial but it is not so. No doubt the usefulness of the theorem will be heightened if the polynomial is given in terms of the zeros. If in particular, the polynomial  $p(z)$  is the product of two or more polynomials having zeros in  $|z| \geq K_1 > 1$ ,  $|z| \geq K_2 > 1$ , etc., each of norm  $\leq 1$ , then  $p(z)$  would be of norm  $\leq 1$ , and one would have a better estimate for  $\|p'\|$  by (3.10) (or (3.11)) than from (3.3).

Another refinement of Lax's result (Theorem 3.1) was given by Aziz and Dawood [5].

**THEOREM 3.7** *If  $p(z)$  is a polynomial of degree at most  $n$  having no zeros in  $|z| < 1$ , then*

$$\|p'\| \leq \frac{n}{2} \left\{ \|p\| - \min_{|z|=1} |p(z)| \right\}. \tag{3.12}$$

The result is best possible and equality holds for  $p(z) = \alpha z^n + \beta$  where  $|\alpha| = |\beta|$ .

The above result of Aziz and Dawood [5] was generalized by Govil [57] who proved

**THEOREM 3.8** *If  $p(z)$  is a polynomial of degree at most  $n$  having no zeros in  $|z| < K$ ,  $K \geq 1$ , then*

$$\|p^{(s)}\| \leq \frac{n(n-1) \cdots (n-s+1)}{1+K^s} \left\{ \|p\| - \min_{|z|=K} |p(z)| \right\}. \quad (3.13)$$

The above theorem sharpens Theorem 3.4 due to Govil and Rahman [63]. For  $s = 1$ , Theorem 3.8 clearly reduces to

**THEOREM 3.9** *If  $p(z)$  is a polynomial of degree at most  $n$ , having no zeros in  $|z| < K$ ,  $K \geq 1$ , then*

$$\|p'\| \leq \left( \frac{n}{1+K} \right) \left( \|p\| - \min_{|z|=K} |p(z)| \right). \quad (3.14)$$

Here the equality is attained for  $p(z) = (z+K)^n$ .

It is clear that the above theorem sharpens Theorem 3.2 due to Malik [76] and for  $K = 1$  it reduces to Theorem 3.7 due to Aziz and Dawood [5].

#### 4. BERNSTEIN TYPE INEQUALITIES FOR POLYNOMIALS WITH ALL THEIR ZEROS IN A CIRCLE

We again begin with Bernstein's inequality (Theorem 1.2) that if  $p(z)$  is a polynomial of degree at most  $n$ ,  $\|p\| = \max_{|z|=1} |p(z)|$ , then

$$\|p'\| \leq n\|p\|, \quad (4.1)$$

with equality holding for the polynomials  $p(z) = \lambda z^n$ ,  $\lambda$  being a complex number.

In case the polynomial  $p(z)$  has all its zeros in  $|z| \leq 1$ , then as is evident from  $p(z) = \lambda z^n$  ( $\lambda$  a complex number) it is not possible to improve upon the bound in (4.1). Hence if  $p(z)$  has all its zeros in  $|z| \leq 1$ , it would be of



interest to obtain an inequality in the reverse direction and this was done by Turán [105], who proved

**THEOREM 4.1** *If  $p(z)$  is a polynomial of degree  $n$  having all its zeros in  $|z| \leq 1$ , then*

$$\|p'\| \geq \frac{n}{2} \|p\|. \tag{4.2}$$

*The result is best possible and the equality holds for all polynomials of degree  $n$  which have all their zeros on  $|z| = 1$ .*

It will obviously be of interest to obtain an inequality analogous to (4.2) for polynomials having all their zeros in  $|z| < K$ ,  $K > 0$ . Malik [76] considered the case when  $K \leq 1$ , and using Theorem 3.2, he obtained

**THEOREM 4.2** *If  $p(z)$  is a polynomial of degree  $n$ , having all its zeros in  $|z| \leq K \leq 1$ ,  $K > 0$ , then*

$$\|p'\| \geq \left(\frac{n}{1+K}\right) \|p\|. \tag{4.3}$$

*Here the equality holds for the polynomial  $p(z) = (z + K)^n$ .*

A simple and direct proof of this result was later given by Govil [54] which is as follows.

If  $p(z) = a_n \prod_{\nu=1}^n (z - z_\nu)$  is a polynomial of degree  $n$  having all its zeros in  $|z| \leq K \leq 1$ , then

$$\left| \frac{p'e^{i\theta}}{p(e^{i\theta})} \right| \geq \operatorname{Re} \left( e^{i\theta} \frac{p'(e^{i\theta})}{p(e^{i\theta})} \right) = \sum_{\nu=1}^n \operatorname{Re} \left( \frac{e^{i\theta}}{e^{i\theta} - z_\nu} \right) \geq \sum_{\nu=1}^n \frac{1}{1+K},$$

that is,

$$|p'(e^{i\theta})| \geq \left(\frac{n}{1+K}\right) |p(e^{i\theta})|,$$

where  $\theta$  is real. Choosing  $\theta_0$  such that  $|p(e^{i\theta_0})| = \max_{0 \leq \theta < 2\pi} |p(e^{i\theta})|$ , we get

$$|p'(e^{i\theta_0})| \geq \left(\frac{n}{1+K}\right) \max_{0 \leq \theta < 2\pi} |p(e^{i\theta})|,$$

from which (4.3) follows.

The above argument does not hold for  $K > 1$ , for then  $\operatorname{Re}(e^{i\theta}/(e^{i\theta} - z_\nu))$  may not be greater than or equal to  $1/(1 + K)$ .

Govil [54] also settled the case when  $K > 1$ , by proving

**THEOREM 4.3** *If  $p(z)$  is a polynomial of degree  $n$  having all its zeros in  $|z| \leq K$ ,  $K \geq 1$ , then*

$$\|p'\| \geq \frac{n}{1 + K^n} \|p\|. \quad (4.4)$$

*The result is best possible and the equality holds for the polynomial  $p(z) = (z^n + K^n)$ .*

A simpler proof of this result was later given by Datt [24].

Note that for  $K > 1$ , the extremal polynomial turns out to be of the form  $(z^n + K^n)$  while for  $K < 1$ , it has the form  $(z + K)^n$ . Thus 1 is a critical value of this parameter for the problem under consideration and one should not expect the same kind of reasoning to work for both  $K < 1$  and  $K > 1$ .

Govil [55] later generalized Theorem 4.3 for functions of exponential type. The generalization of Turán's result (Theorem 4.1) for functions of exponential type was done by Rahman [90].

The following refinement of Theorem 4.2 was done by Giroux *et al.* [50].

**THEOREM 4.4** *Let  $p(z) = a_n \prod_{\nu=1}^n (z - z_\nu)$  be of degree  $n$ . If  $|z_\nu| \leq 1$ ,  $1 \leq \nu \leq n$ , then*

$$\|p'\| \geq \sum_{\nu=1}^n \left( \frac{1}{1 + |z_\nu|} \right) \|p\|. \quad (4.5)$$

*There is equality in the above inequality if the zeros are all positive.*

A generalization of the above Theorem 4.4 was obtained by Aziz [3].

**THEOREM 4.5** *If all the zeros of the polynomial  $p(z) = a_n \prod_{\nu=1}^n (z - z_\nu)$  of degree  $n$  lie in  $|z| \leq K$ , where  $K \geq 1$ , then*

$$\|p'\| \geq \frac{2}{1 + K^n} \sum_{\nu=1}^n \frac{K}{K + |z_\nu|} \|p\|. \quad (4.6)$$

*Equality in the above holds again for  $p(z) = z^n + K^n$ .*

Note that the inequality (4.6) is also a refinement of the inequality (4.4). For some refinements of Theorems 4.5 and 4.3, see [56].

The following result which is a refinement of Turán’s result (Theorem 4.1) was given by Aziz and Dawood [5].

**THEOREM 4.6** *If  $p(z)$  is a polynomial of degree  $n$  having all its zeros in  $|z| \leq 1$ , then*

$$\|p'\| \geq \frac{n}{2} \left\{ \|p\| + \min_{|z|=1} |p(z)| \right\}. \tag{4.7}$$

*The equality here holds for  $p(z) = \alpha z^n + \beta$ ,  $|\beta| \leq |\alpha|$ .*

We conclude this section by stating the following theorem due to Govil [57], which generalizes the above Theorem 4.6 of Aziz and Dawood [5].

**THEOREM 4.7** *If  $p(z)$  is a polynomial of degree  $n$ , having all its zeros in  $|z| \leq K$ , then for  $K \leq 1$ ,*

$$\|p'\| \leq \left( \frac{n}{1+K} \right) \|p\| + \frac{n}{K^{n-1}(1+K)} \min_{|z|=K} |p(z)|. \tag{4.8}$$

*If  $K \geq 1$ , then*

$$\|p'\| \geq \left( \frac{n}{1+K^n} \right) \left\{ \|p\| + \min_{|z|=K} |p(z)| \right\}. \tag{4.9}$$

*Both the above inequalities are best possible. In the first case, the equality is attained for  $p(z) = (z + K)^n$  and in the second case for  $p(z) = z^n + K^n$ .*

**5. BERNSTEIN TYPE INEQUALITIES FOR POLYNOMIALS**

**SATISFYING  $p(z) \equiv z^n p(1/z)$  OR  $p(z) \equiv z^n \overline{p(1/\bar{z})}$**

If  $p(z) = \sum_{\nu=0}^n a_\nu z^\nu$  is a polynomial of degree  $n$ , it is obviously of interest to obtain an inequality analogous to Bernstein’s inequality

$$\|p'\| \leq n\|p\|, \tag{5.1}$$

for polynomials satisfying  $p(z) \equiv z^n p(1/z)$  or  $p(z) \equiv z^n \overline{p(1/\bar{z})}$ , and in this direction the following result (see [54, Lemma 4, 86,98]), which is easy to obtain, is well known.

**THEOREM 5.1** *If  $p(z) = \sum_{\nu=0}^n a_\nu z^\nu$  is a polynomial of degree  $n$  satisfying  $p(z) \equiv z^n \overline{\{p(1/\bar{z})\}}$ , then*

$$\|p'\| = \frac{n}{2} \|p\|. \tag{5.2}$$

Let  $\Pi_n$  denote the class of polynomials of degree  $n$  satisfying  $p(z) \equiv z^n \overline{p(1/\bar{z})}$ . The class  $\Pi_n$  is interesting because for any polynomial  $p(z)$  of degree  $n$ , the polynomial  $P(z) = z^n p(z + 1/z)$  is always in  $\Pi_{2n}$ . It was proposed by Professor Q.I. Rahman to obtain inequality analogous to Bernstein’s inequality (5.1) for polynomials belonging to  $\Pi_n$ , and in an attempt to answer this question perhaps the first result in this direction was proved by Govil *et al.* [60] who established the following partial result.

**THEOREM 5.2** *If  $p(z)$  is a polynomial belonging to  $\Pi_n$  and having all its zeros in the left half-plane or in the right half-plane, then*

$$\|p'\| \leq \frac{n}{\sqrt{2}} \|p\|. \tag{5.3}$$

It is not known if (5.3) is best possible, however by considering  $p(z) = z^n + 2iz^{n/2} + 1$ ,  $n$  being even, they showed that if  $p(z)$  simply belongs to  $\Pi_n$ , then the bound in (5.3) cannot in general be smaller than  $n/\sqrt{2}$ .

Later Frappier *et al.* [43] considered the polynomial  $p(z) = (1 + iz)^2 + z^{n-2}(z + i)^2$ . Note that this polynomial belongs to  $\Pi_n$  and on  $|z| = 1$ ,  $|p(z)| \leq |1 + iz|^2 + |z + i|^2 = |-i + z|^2 + |z + i|^2 = 4$ , while  $|p'(i)| = 4(n - 1)$  and so  $\|p'\| \geq 4(n - 1)$ . Thus  $\|p'\|/\|p\| \geq 4(n - 1)/4 = (n - 1) > n/\sqrt{2}$ , that is, if  $p(z)$  only belongs to  $\Pi_n$ , the bound in (5.3) should be something  $\geq (n - 1) > n/\sqrt{2}$ , implying that by just assuming  $p \in \Pi_n$  there is nearly no improvement in the derivative estimate in (5.1). In the same paper [43] by rather deep method they showed that if  $p \in \Pi_n$ , then

$$\|p'\| \leq (n - \delta_n) \|p\|, \quad \text{where } \delta_n \rightarrow 2/5 \text{ as } n \rightarrow \infty. \tag{5.4}$$

Aziz [3] considered another subclass of  $\Pi_n$  and proved

**THEOREM 5.3** *Let  $p(z) = \sum_{\nu=0}^n (\alpha_\nu + i\beta_\nu)z^\nu$ ,  $\alpha_\nu \geq 0$ ,  $\beta_\nu \geq 0$ ,  $\nu = 0, 1, 2, \dots, n$  be a polynomial belonging to  $\Pi_n$ . Then*

$$\|p'\| \leq \frac{n}{\sqrt{2}} \|p\|. \tag{5.5}$$

The equality in (5.5) again holds for the polynomial  $p(z) = z^n + 2iz^{n/2} + 1$ ,  $n$  being even.

As is easy to observe, the hypothesis of Theorem 5.3 is equivalent to that  $p(z)$  belongs to  $\Pi_n$  and that all the coefficients of  $p(z) = \sum_{\nu=0}^n a_\nu z^\nu$  lie in the first quadrant of the complex plane. In fact, if all the coefficients of a polynomial  $p(z)$  belonging to  $\Pi_n$  lie in a sector of opening  $\pi/2$ , say in,  $\psi \leq \arg z \leq \psi + \pi/2$ , for some real  $\psi$ , then the polynomial  $P(z) = e^{-i\psi} p(z)$  belongs to  $\Pi_n$  and has all its coefficients lying in the first quadrant of the complex plane. Since  $\|P\| = \|p\|$  and  $\|P'\| = \|p'\|$ , we may apply Theorem 5.3 to  $P(z)$  to get that if  $p(z) \in \Pi_n$  and has all its coefficients lying in a sector of opening at most  $\pi/2$ , then also (5.5) holds. The following result that is equivalent to this statement appears in [69].

**THEOREM 5.4** *Let  $p(z) = \sum_{\nu=0}^n a_\nu z^\nu$  where  $a_\nu = \alpha_\nu e^{i\phi} + \beta_\nu e^{i\psi}$ ,  $\alpha_\nu \geq 0$ ,  $\beta_\nu \geq 0$ ;  $\nu = 0, 1, 2, \dots, n$ ,  $0 \leq |\phi - \psi| \leq \pi/2$ , be a polynomial of degree  $n$ . If further  $p(z) \in \Pi_n$ , then*

$$\|p'\| \leq \frac{n}{\sqrt{2}} \|p\|. \tag{5.6}$$

The result is best possible with equality holding for the polynomial  $p(z) = z^n + 2iz^{n/2} + 1$ ,  $n$  being an even integer.

Following result of Datt and Govil [25] generalizes Theorem 5.4 and so also Theorem 5.3.

**THEOREM 5.5** *Let  $p(z) = \sum_{\nu=0}^n (\alpha_\nu + i\beta_\nu) z^\nu$  be a polynomial of degree  $n$  belonging to  $\Pi_n$ . If on  $|z|=1$ , the maximum of  $|\sum_{\nu=0}^n \alpha_\nu z^\nu|$  and  $|\sum_{\nu=0}^n \beta_\nu z^\nu|$  is attained at the same point, then*

$$\|p'\| \leq \frac{n}{\sqrt{2}} \|p\|. \tag{5.7}$$

The equality here holds again for  $p(z) = z^n + 2iz^{n/2} + 1$ ,  $n$  being an even integer.

Another generalization of Theorem 5.4 has recently been given by Govil and Vetterlein [65]. They in fact proved

**THEOREM 5.6** *If  $p(z)$  is a polynomial belonging to  $\Pi_n$  with all its coefficients lying in a sector of opening at most  $\gamma$ , where  $0 \leq \gamma \leq 2\pi/3$ , then*

$$\|p'\| \leq \frac{n}{2 \cos \gamma/2} \|p\|. \quad (5.8)$$

*The result is best possible for  $0 \leq \gamma \leq \pi/2$ , with equality holding for the polynomial  $p(z) = z^n + 2e^{i\gamma} z^{n/2} + 1$ ,  $n$  being even.*

Although the class  $\Pi_n$  of polynomials has been extensively studied among others by Frappier and Rahman [42] and Frappier *et al.* [43], the problem of obtaining a sharp inequality analogous to Bernstein's inequality (5.1) is still open. However, the following sharp inequality in the reverse direction, which is easy to obtain, is due to Dewan and Govil [27].

**THEOREM 5.7** *If  $p(z)$  is a polynomial belonging to  $\Pi_n$ , then*

$$\|p'\| \geq \frac{n}{2} \|p\|. \quad (5.9)$$

*The result is best possible and the equality holds for  $p(z) = (z^n + 1)$ .*

## 6. BERNSTEIN TYPE INEQUALITIES IN THE $L^p$ -NORM

We again recall Bernstein's inequality, which states that if  $p(z) = \sum_{\nu=0}^n a_\nu z^\nu$  is a polynomial of degree  $n$ , and  $\|p\| = \max_{|z|=1} |p(z)|$ , then

$$\|p'\| \leq n \|p\|. \quad (6.1)$$

In this section, we consider generalizations of this and of some other inequalities discussed in earlier sections under the  $L^p$ -norm.

Let  $\mathcal{P}_n$  denote the set of all polynomials (over the complex field) of degree at most  $n$ . For  $p \in \mathcal{P}_n$ , define the norm of  $p$  by

$$\|p\|_\delta = \left( \frac{1}{2\pi} \int_0^{2\pi} |p(e^{i\theta})|^\delta d\theta \right)^{1/\delta}, \quad 0 < \delta < \infty.$$

The limiting cases are  $\|p\|_\infty$ , the supremum norm, and

$$\|p\|_0 = \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \log |p(e^{i\theta})| \, d\theta\right).$$

If  $p \in \mathcal{P}_n$ , then it is well known that

$$\|p'\|_\delta \leq n\|p\|_\delta, \quad 0 \leq \delta \leq \infty. \tag{6.2}$$

The above inequality is best possible with equality holding for  $p(z) = \lambda z^n$ ,  $\lambda$  being a complex number, and as is easy to see, the Bernstein’s inequality (6.1) is the case  $\delta = \infty$  of this inequality. For  $1 \leq \delta \leq \infty$ , the inequality was obtained by Zygmund [111] by using an interpolation formula of M. Riesz. Lorentz [75] has derived this case of (6.2) from Hardy–Littlewood–Polya order relation. For  $0 < \delta < 1$ , the inequality (6.2) was obtained by Máté and Nevai [79] with an extra factor of  $(4e)^{1/\delta}$  on its right hand side. The inequality in the form as mentioned in (6.2) was obtained by Arestov [1], who used subharmonic functions and Jensen’s formula in order to arrive at this. Golitschek and Lorentz [49] (also see [87, 108]) gave a simpler proof of this inequality and also obtained its generalization.

For polynomials not vanishing in  $|z| < 1$ , de Bruijn [26] proved the following generalization of Lax’s result, Theorem 3.1.

**THEOREM 6.1** *If  $p \in \mathcal{P}_n, p(z) \neq 0$  in  $|z| < 1$ , then for  $\delta \geq 1$ ,*

$$\|p'\|_\delta \leq nc_\delta^{1/\delta} \|p\|_\delta, \tag{6.3}$$

where  $c_\delta = 2^{-\delta} \sqrt{\pi} \Gamma(\frac{1}{2}\delta + 1) / \Gamma(\frac{1}{2}\delta + \frac{1}{2})$ . The result is sharp and the equality holds for  $p(z) = (\alpha + \beta z^n), |\alpha| = |\beta|$ .

To obtain Lax’s inequality (3.2) from (6.3), simply make  $\delta \rightarrow \infty$  and note that  $\lim_{\delta \rightarrow \infty} c_\delta^{1/\delta} = \frac{1}{2}$ . Theorem 6.1 in the case  $\delta = 2$  was proved by Lax [72] himself. For an alternate proof of Theorem 6.1, see [89]. The inequality (6.3) in fact holds for  $\delta \geq 0$  and this was proved by Rahman and Schmeisser [92]. A simpler proof and a generalization of Theorem 6.1 was later given by Aziz [4].

Dewan and Govil [28] considered the class of polynomials  $p(z)$  satisfying  $p(z) \equiv z^n \{p(1/\bar{z})\}$ , and proved

**THEOREM 6.2** *If  $p \in \mathcal{P}_n$ , and satisfies  $p(z) \equiv z^n \{p(1/\bar{z})\}$ , then for  $\delta \geq 1$ , the inequality (6.3) holds. The result is best possible and the equality holds again for  $p(z) = (\alpha + \beta z^n)$ ,  $|\alpha| = |\beta|$ .*

Govil and Jain [59] proved the following more complete result.

**THEOREM 6.3** *If  $p \in \mathcal{P}_n$ , and satisfies  $p(z) \equiv z^n \{p(1/\bar{z})\}$ , then for  $\delta \geq 1$ ,*

$$\frac{n}{2} \|p\|_\delta \leq \|p'\| \leq n c_\delta^{1/\delta} \|p\|_\delta, \quad (6.4)$$

where  $c_\delta$  is the same as defined in Theorem 6.1.

*Both the above inequalities are best possible and they both reduce to equality for the polynomial  $p(z) = (z^n + 1)$ .*

The above result of Govil and Jain [59] has recently been extended by Govil [58] by showing that the inequality (6.4) in fact holds for  $\delta \geq 0$ . Since  $\lim_{\delta \rightarrow \infty} c_\delta^{1/\delta} = 1/2$ , we get on making  $\delta \rightarrow \infty$  in (6.4) that if the polynomial satisfies  $p(z) \equiv z^n \{p(1/\bar{z})\}$ , then

$$\|p'\| = \frac{n}{2} \|p\|,$$

which is Theorem 5.1.

For polynomials not vanishing in  $|z| < K$ ,  $K \geq 1$ , Govil and Rahman [63] proved

**THEOREM 6.4** *If  $p \in \mathcal{P}_n$ ,  $p(z) \neq 0$  in  $|z| < K$ ,  $K \geq 1$ , then*

$$\|p'\|_\delta \leq n E_\delta^{1/\delta} \|p\|_\delta, \quad (6.5)$$

where  $E_\delta = 2\pi / \int_0^{2\pi} |K + e^{i\theta}|^\delta d\theta$ .

Since  $\lim_{\delta \rightarrow \infty} E_\delta^{1/\delta} = 1/(1 + K)$ , on making  $\delta \rightarrow \infty$ , the inequality (6.5) reduces to Malik's inequality (3.3). For  $K = 1$ , Theorem 6.4 reduces to Theorem 6.1 of de Bruijn [26]. Theorem 6.4 is not sharp and the sharp inequality does not seem to be obtainable even for  $\delta = 2$ .



Gardner and Govil [47] have generalized the above result of Govil and Rahman [63] by proving

**THEOREM 6.5** *Let  $p(z) = a_n \prod_{\nu=1}^n (z - z_\nu)$ ,  $a_n \neq 0$ , be a polynomial of degree  $n$ . If  $|z_\nu| \geq K_\nu \geq 1$ ,  $1 \leq \nu \leq n$ , then for  $\delta \geq 0$ ,*

$$\|p'\|_\delta \leq nF_\delta^{1/\delta} \|p\|_\delta, \tag{6.6}$$

where  $F_\delta = \{2\pi / \int_0^{2\pi} |t_0 + e^{i\theta}|^\delta d\theta\}$ , and  $t_0 = \{1 + n / \sum_{\nu=1}^n 1 / (K_\nu - 1)\}$ . The result is best possible in the case  $K_\nu = 1$ ,  $1 \leq \nu \leq n$ , and the equality holds for  $p(z) = (1 + z)^n$ .

The above result in the case  $\delta \geq 1$  was also proved by Gardner and Govil [46].

If  $K_\nu = 1$  for some  $\nu$ ,  $1 \leq \nu \leq n$  then  $t_0 = 1$  and (6.6) reduces to the inequality (6.3) due to de Bruijn [26]. If  $K_\nu \geq K$  for some  $K \geq 1$ ,  $1 \leq \nu \leq n$ , then as is easy to verify  $F_\delta \leq \{2\pi / \int_0^{2\pi} |K + e^{i\theta}|^\delta d\theta\}^{1/\delta}$ , and so the above inequality reduces to the inequality (6.5) due to Govil and Rahman [63]. If in Theorem 6.5, we make  $\delta \rightarrow \infty$ , we get Theorem 3.7 due to Govil and Labelle [61].

Closely related to the results considered above is the following theorem of Gardner and Weems [48].

**THEOREM 6.6** *If  $p(z) = a_0 + \sum_{\nu=m}^n a_\nu z^\nu$  and  $p(z) \neq 0$  for  $|z| < K$ , where  $K \geq 1$ , then for  $m \geq 1$ ,  $0 \leq \delta \leq \infty$ ,*

$$\|p'\|_\delta \leq \frac{n}{\|s_0 + z\|_\delta} \|p\|_\delta, \tag{6.7}$$

where

$$s_0 = K^{m+1} \left( \frac{m|a_m|K^{m-1} + n|a_0|}{n|a_0| + m|a_m|K^{m+1}} \right).$$

### 7. BERNSTEIN TYPE INEQUALITIES IN OTHER ENVIRONMENTS

In this section, we mention some of the areas where Bernstein type inequalities are used and their applications. We begin with relation between Bernstein and Nikolisky inequalities. Nikolisky proved that

if  $1 \leq p \leq q \leq \infty$ , then for any trigonometric polynomial  $t_n$  of order  $n$ ,

$$\|t_n\|_{L^q} \leq c(p, q)n^{1/p-1/q}\|t_n\|_{L^p},$$

where the norms are taken over  $[0, 2\pi]$ . Bernstein and Nikolisky inequalities play an important role in Fourier analysis and approximation theory, especially in converse and imbedding theorems. Recently, inequalities of the same type have been obtained for various system of functions by Ky [71]. In [71], Ky has shown how to obtain inequalities of Nikolisky type from Bernstein type inequalities. He has also considered arbitrary function systems in symmetric spaces. Borwein and Erdélyi [17] have considered Remez, Nikolisky and Markov type inequalities for polynomials where the zeros are restricted to some specified intervals.

Lawrence Harris [66] has shown, how classical inequalities for the derivative of polynomials can be proved for real and complex Hilbert spaces using simple functional analytic arguments. He also shows a relation between an inequality of norms for symmetric multilinear mappings and van der Corput and Schaake inequality. His general method which relies on a lemma of Hörmander yields extension of inequalities for the derivatives of polynomials. In a subsequent paper Harris [67] proved a Bernstein–Markov theorem for normed spaces.

Jia [70] has obtained a Bernstein type inequality associated with wavelet decomposition and has used it to study the related nonlinear approximation problem on the basis of shift invariant spaces of functions. He has also given examples of piecewise polynomial spaces to illustrate the general theory.

Dense Markov systems and related unbounded Bernstein inequalities were considered by Borwein and Erdélyi [20]. An extension of such theorems for rational systems is done in Borwein *et al.* [18].

Xioming [110] has obtained estimates of Bernstein basis functions and Meyer–Konig and Zeller basis functions and used it to obtain new rates of convergence of Durameyer operators and Meyer–Konig and Zeller operators.

Application of Bernstein inequalities to analytic geometry, differential equations and analytic functions have been carried out in [21,39]. The main goal was to develop new techniques where Bernstein inequality can be used to projections of analytic sets and apply this method to study bifurcations and periodic orbits.

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