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A Remark Over the Converse of Hölder Inequality*

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Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $\mathcal{L}$ be the set of measurable nonnegative real functions defined on Ω. Let $F: \mathcal{L} \to [0, \infty]$ be a positive homogenous functional. Suppose that there are two sets $A, B \in \mathcal{A}$ such that $0 < F(\chi_A) < 1 < F(\chi_B) < \infty$ and let ϕ and ψ be continuous bijective functions of $[0, \infty)$ onto $[0, \infty)$. We prove that if there is no positive real number d such that $\{F(\chi_C): C \in \mathcal{A}, F(\chi_C) > 0\} \subset \{d^k: k \in Z\}$ and

$$F(xy) \leq \phi^{-1}(F(\phi \circ x))\psi^{-1}(F(\psi \circ y))$$

for all $x, y \in \{\alpha\chi_C \in \mathcal{L}: F(\chi_C) < \infty, \alpha \in R\}$, then ϕ and ψ must be conjugate power functions.

In addition, we prove that if there exists a real number $d > 0$ such that $\{F(\chi_C): C \in \mathcal{A}, F(\chi_C) > 0\} \subset \{d^k: k \in Z\}$ then there are nonpower continuous bijective functions ϕ and ψ which the above inequality. Also we give an example which shows that the condition that ϕ and ψ are continuous functions is essential.

Keywords: Measure space; Positive homogenous functional; Hölder inequality; Conjugate functions

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1. INTRODUCTION

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. We denote by $\mathcal{L} = \mathcal{L}(\Omega, \mathcal{A}, \mu)$ the set of measurable nonnegative real functions Ω. Let $F: \mathcal{L} \to [0, \infty]$ be a

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positive homogenous functional, i.e., $F(\lambda x)=\lambda F(x)$, for all $\lambda \geq 0, x \in \mathcal{L}$. Suppose that $\mathcal{A}$ has two sets A, B such that $0<F(\chi_A)<1<F(\chi_B)<\infty$ and let ϕ and ψ be bijective functions of $[0,\infty)$ onto $[0,\infty)$ such that $\phi(0)=0$ and $\psi(0)=0$. In [1] Matkowski proved that if $F(x):=\int_\Omega x\,\mathrm{d}\mu$ for $x\in\mathcal{L}$ and

$$F(xy) \leq \phi^{-1}(F(\phi\circ x))\psi^{-1}(F(\psi\circ y)) \tag{1}$$

for all μ-integrable nonnegative step functions x, y, then ϕ and ψ must be conjugate power functions.

Suppose now that ϕ, ψ are continuous bijective functions of $[0,\infty)$ onto $[0,\infty)$. We will prove that if the following property holds :

(A) *There is no real number $d>0$ such that*

$$\{F(\chi_C): C\in\mathcal{A}, F(\chi_C)>0\} \subset \{d^k : k\in\mathbb{Z}\}, \tag{2}$$

and (1) *is satisfied for all function $x=\alpha\chi_C$, $\alpha>0$, $C\in\mathcal{A}$, $F(\chi_C)<\infty$, then ϕ and ψ are conjugate power functions.*

We want to emphasize that in this paper, we work with a class of functions smaller than those considered in [1] and we do not use the additive property of the integral. Also we will show that the hypotheses:

(i) the property (A) is true, and
(ii) ϕ and ψ are continuous functions

are essential.

Remark It is easy to see that the Hölder inequality holds, with the same proof (cf. [2, p. 95]), when we have a monotone positive homogenous functional $F:\mathcal{L}\to[0,\infty]$. More precisely, if F is a positive homogenous functional which satisfies $F(x)\leq F(y)$ for all $x,y\in\mathcal{L}$, $x\leq y$, and p, q are positive real numbers with $p^{-1}+q^{-1}=1$ then

$$F(xy) \leq (F(x^p))^{1/p}(F(y^q))^{1/q} \quad \text{for all } x, y\in\mathcal{L}. \tag{3}$$

Note that when $\phi(t)=t^p$ and $\psi(t)=t^q$, (1) gives (3).

2. THE MAIN RESULT

We begin with an auxiliary lemma.

LEMMA 1 *Let S be a subset of the positive real numbers such that:*

(i) *there exist $a, b \in S$ with $a < 1 < b$,*
(ii) *$st \in S$, for all $s, t \in S$.*

Then $S \subset \{d^k\colon k \in \mathbb{Z}\}$ for some d, $0 < d \in \mathbb{R}$ or S is dense in $(0, \infty)$.

Proof Consider the following sets $U := \{t \in S\colon t > 1\}$ and $V := \{t \in S\colon t < 1\}$. Clearly, both of them are nonempty sets. Suppose that the set U has a minimum, say d, and let $v \in V$. By definition of d, we have $vd \leq 1$. Note that there exists a $k \in \mathbb{Z}$ such that $vd^k = 1$, otherwise we obtain $vd^k < 1$ for all $k \in \mathbb{N}$, which is a contradiction since $d^k \to \infty$, for $k \to \infty$. Thus we have $V \subset \{d^{-k}\colon k \in \mathbb{N}\}$. Let $d_1 := \max V$. By definition of d_1 we have $d_1 d \geq 1$. On the other hand $d_1 d \leq 1$. It follows that $d_1 = d^{-1}$.

Let $u \in U$ and let $k \in \mathbb{N}$ be such that $d^k \leq u < d^{k+1}$. Since $d^{-k}u \in S$ and $1 \leq d^{-k}u < d$ we get $u = d^k$. Therefore

$$S \subset \{d^k\colon\ k \in \mathbb{Z}\}.$$

If we assume that there exists a maximum element in V, we obtain analogously that $S \subset \{d^k\colon k \in \mathbb{Z}\}$ for some positive real number d.

Suppose that neither there is a minimum element in U nor a maximum element in V. Let $a_1 := \sup V$ and $b_1 := \inf U$. We will prove that $a_1 = b_1 = 1$. First we observe that $a_1 b_1 = 1$. Otherwise if $a_1 b_1 > 1$ we can find two numbers $u \in U$, $v \in V$ such that $1 < uv < a_1 b_1 \leq b_1$, which is a contradiction. Analogously, the case $a_1 b_1 < 1$ gives us a contradiction. Next we see that $b_1 = 1$. Suppose that $1 < b_1$. Once again we can find two numbers $u \in U$, $v \in V$ such that $a_1 < uv < a_1 b_1 = 1$, which is a contradiction. So, $a_1 = b_1 = 1$.

Finally we will show that S is dense in $(0, \infty)$. Let $w \in (0, \infty)$ and define

$$q := \sup\{t \in S\colon t < w\} \quad \text{and} \quad p := \inf\{t \in S\colon t > w\}.$$

Clearly $q \leq p$. Suppose that $q < p$. As $a_1 = b_1 = 1$, there exist $u \in U$ and $v \in V$ such that $u < (pq^{-1})^{1/2}$ and $v > (pq^{-1})^{-1/2}$. By definition of q and p

it follows that there are $r, s \in S$ such that $r < w < s$, $ru > q$ and $sv < p$. Since $s \geq p$, $r \leq q$ we have $sr^{-1} \geq pq^{-1} > uv^{-1}$. Therefore we obtain $p > sv > ru > q$, which is a contradiction. So $q = p = w$. Thus S is dense in $(0, \infty)$. This completes the proof.

The following lemma is interesting in itself.

LEMMA 2 *Let $F: \mathcal{L} \to [0, \infty]$ be a positive homogenous functional and ϕ, ψ be continuous bijective functions of $[0, \infty)$ onto $[0, \infty)$. Suppose that the property (A) holds. If (1) is true for all function $x = \beta\chi_C$, $C \in \mathcal{A}$, $F(\chi_C) < \infty$, $\beta = \phi^{-1}(r)$ or $\beta = \psi^{-1}(r)$, then there exists a constant c such that*

$$\phi^{-1}(t)\psi^{-1}(t) = ct, \quad \text{for all } t \geq 0.$$

Proof Let $C \in \mathcal{A}$ be such that $F(\chi_C) < \infty$. Let $x = \phi^{-1}(r)\chi_C$ and $y = \psi^{-1}(r)\chi_C$. Since $\phi^{-1}(r\chi_C) = \phi^{-1}(r)\chi_C$, $\psi^{-1}(r\chi_C) = \psi^{-1}(r)\chi_C$, and F is a positive homogenous functional, it follows from (1) that

$$\phi^{-1}(r)\psi^{-1}(r)F(\chi_C) \leq \phi^{-1}(rF(\chi_C))\psi^{-1}(rF(\chi_C)). \tag{4}$$

Define the function $f: [0, \infty) \to [0, \infty)$ by $f(t) := \phi^{-1}(t)\psi^{-1}(t)$.

By (4) we have

$$F(\chi_C)f(r) \leq f(rF(\chi_C)). \tag{5}$$

Consider the following set:

$$S := \{t \in (0, \infty): \ tf(r) \leq f(rt) \text{ for all } r > 0\}. \tag{6}$$

If $t, s \in S$ then $tsf(1) \leq tf(s) \leq f(ts)$. Hence, $ts \in S$. Further, it follows from (5) that $F(\chi_C) \in S$, for all $C \in \mathcal{A}$ with $F(\chi_C) < \infty$. Since we have assumed there are sets $A, B \in \mathcal{A}$ which satisfy $0 < F(\chi_A) < 1 < F(\chi_B) < \infty$, we observe that the set S satisfies the hypotheses (a) and (b) of Lemma 1. In consequence under our hypothesis S is dense in $(0, \infty)$.

Since ϕ and ψ are continuous functions we find that f is also a continuous function. So, S is an algebraic group. In fact, let $t \in S$ and (a_n) be a sequence in S such that (a_n) converges to $1/t$. We have $f(a_n r) \geq a_n f(r)$. Since the left member in the last inequality converges to

$f(r/t)$ and the right member converges to $(1/t)f(r)$, we obtain $f(r/t) \geq (1/t)f(r)$. So $1/t \in S$.

If $t \in S$, then

$$f(r) \leq \frac{1}{t} f(rt) \leq f(\frac{r}{t} t) = f(r).$$

Therefore, $f(rt) = tf(r)$ for all $t \in S$ and since f is continuous, we have $f(rt) = tf(r)$ for all $t \geq 0$. This concludes the proof with $c = \phi^{-1}(1)\psi^{-1}(1)$.

THEOREM 3 *Suppose that the hypothesis of Lemma 2 except that the inequality* (1) *holds for all functions of the form* $x = \alpha\chi_C$, $\alpha > 0$, $C \in \mathcal{A}$, $F(\chi_C) < \infty$. *Then* ϕ *and* ψ *are conjugate power functions.*

Proof By Lemma 2 we have for some $c > 0$

$$\phi^{-1}(t)\psi^{-1}(t) = ct, \quad \text{for all } t \geq 0. \tag{7}$$

For $\alpha > 0$, $\beta > 0$ and $C \in \mathcal{A}$ with $F(\chi_C) < \infty$, let $x = \phi^{-1}(\alpha)\chi_C$ and $y = \psi^{-1}(\beta)\chi_C$. From (1) we get

$$\phi^{-1}(\alpha)\psi^{-1}(\beta)F(\chi_C) \leq \phi^{-1}(\alpha F(\chi_C))\psi^{-1}(\beta F(\chi_C)). \tag{8}$$

Consider the following set:

$$S := \{t \in (0,\infty):\ t\phi^{-1}(\alpha)\psi^{-1}(\beta) \leq \phi^{-1}(\alpha t)\psi^{-1}(\beta t),\\ \text{for all } \alpha > 0,\ \beta > 0\}.$$

As in the proof of Lemma 2 it is easy to see that $S = (0,\infty)$ and

$$t\phi^{-1}(\alpha)\psi^{-1}(\beta) = \phi^{-1}(\alpha t)\psi^{-1}(\beta t), \quad \text{for all } \alpha > 0,\ \beta > 0,\ t > 0. \tag{9}$$

From (7) and (9) we get

$$\phi^{-1}(t\alpha)\phi^{-1}(1) = \phi^{-1}(t)\phi^{-1}(\alpha).$$

In this case, it is well known that $\phi(t) = \lambda t^p$ for some $\lambda > 0$ and $p > 0$. Thus (7) implies that $\psi(t) = \eta t^q$, for some $\eta > 0$, $q > 0$ with $1/p + 1/q = 1$.

COROLLARY 4 *Let* $\phi; \psi$ *be continuous bijective functions of* $[0,\infty)$ *onto* $[0,\infty)$. *Suppose that there is a real number* $a > 1$ *such that* *for all* $t \in (0, a]$,

$$t\alpha\beta \leq \phi^{-1}(t\phi(\alpha))\psi^{-1}(t\psi(\beta)) \tag{10}$$

for arbitrary nonnegative α, β. *Then* ϕ *and* ψ *are conjugate power functions.*

Proof Take $0 < s < 1$ such that $\log_a s$ is not a rational number. We observe that there does not exists $d > 0$ satisfying $\{s, a\} \subset \{d^k : k \in \mathbb{Z}\}$. Otherwise, we have $s = d^k$, $a = d^j$, for some $d > 0; j, k \in \mathbb{Z}$. So, $\log_a s = k/j$, which is a contradiction.

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space where $\mathcal{A} = \{\emptyset, A, \Omega\}$ with $A \subset \Omega$, $\mu(A) = s$, $\mu(\Omega) = a$. Clearly, (10) implies (1) with $F(x) = \int_\Omega x \, \mathrm{d}\mu$ for $x \in \mathcal{L}$. By Theorem 3 we find that ϕ and ψ are conjugate power functions.

Remark Hardy *et al.* [3, p. 82, Theorem 101] proved the following theorem:

Let f *and* g *be functions of* $[0, \infty)$ *onto* $[0, \infty)$. *Suppose that* f *is a continuous and strictly increasing function which vanishes at* $x = 0$ *and has a second derivative continuous for* $x > 0$, *and that* g *is its inverse function. Suppose further that* ϕ *and* ψ *are defined by*

$$\phi(t) = tf(t), \qquad \psi(t) = tg(t)$$

or

$$\phi(t) = \int_0^t f(s)\, \mathrm{d}s, \qquad \psi(t) = \int_0^t g(s)\, \mathrm{d}s;$$

and that for every $n \in \mathbb{N}$ *and for all positive real numbers* $r_1, \ldots, r_n$ *such that* $r_1 + \cdots + r_n = 1$,

$$r_1\alpha_1\beta_1 + \cdots + r_n\alpha_n\beta_n \\ \leq \phi^{-1}(r_1\phi(\alpha_1) + \cdots + r_n\phi(\alpha_n))\psi^{-1}(r_1\psi(\beta_1) + \cdots + r_n\psi(\beta_n)) \tag{11}$$

for all nonnegative real number $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$. *Then* f *is a power function.*

As a special case of Corollary 4, we obtain the following result, which is a variation of this theorem:

Let f, g *be continuous and strictly increasing functions of* $[0, \infty)$ *onto* $[0, \infty)$ *which vanishes at* $x = 0$. *Let* ϕ *and* ψ *be defined as above. Suppose* *that there exists a real number* $a > 1$ *such that* (11) *holds for* $n = 2$ *and for all positive real number* r_1, r_2 *with* $r_1 + r_2 = a$. *Then* f *is a power function.*

3. TWO EXAMPLES

In this section we will give examples which show that the conditions:

(i) the property (A) is true, and
(ii) ϕ and ψ are continuous functions

are essential in Theorem 3.

Example 1 Let $(\Omega, \mathcal{A}, \mu)$, $\Omega \neq \emptyset$ be a measure space. Let $F: \mathcal{L} \to [0, \infty]$ be a monotone positive homogenous functional which satisfies (2) for some $d > 1$. Let f be a strictly increasing function on $[1, d^{1/2}]$ such that $f(1) = 1$, $f(d^{1/2}) = d$. Define a one to one continuous function of $[0, \infty)$ onto $[0, \infty)$ by $\phi(t) = d^k f(d^{-k/2} t)$, if $d^{k/2} \leq t \leq d^{(k+1)/2}$, $k \in \mathbb{Z}$ and $\phi(0) = 0$. Let $\psi = \phi$. We will prove that (1) holds. For this, let $C, D \in \mathcal{A}$ be such that $F(\chi_C) < \infty$, $F(\chi_D) < \infty$ and let $\alpha > 0$, $\beta > 0$. Consider the functions $x := \alpha\chi_C$ and $y := \beta\chi_D$. If $F(\chi_C) = 0$ or $F(\chi_D) = 0$, both members in (1) are equal to zero. Clearly if $F(\chi_{C \cap D}) = 0$, (1) holds. Suppose that $F(\chi_C) > 0$, $F(\chi_D) > 0$ and $F(\chi_{C \cap D}) > 0$. Then there are integers k, t such that

$$F(\chi_C) = d^k, \qquad F(\chi_D) = d^t.$$

Let $m, n \in \mathbb{Z}$ be such that

$$d^{m/2} \leq \alpha < d^{(m+1)/2} \quad \text{and} \quad d^{n/2} \leq \beta < d^{(n+1)/2}.$$

Since $d^{(m+k)/2} \leq \alpha d^{k/2} < d^{(m+k+1)/2}$ and $d^{(n+t)/2} \leq \beta d^{t/2} < d^{(n+t+1)/2}$, we have

$$F(xy) = \alpha\beta F(\chi_{C \cap D}) \tag{12}$$

and

$$\begin{aligned} \phi^{-1}(F(\phi \circ x))\psi^{-1}(F(\psi \circ y)) &= \phi^{-1}(\phi(\alpha)d^k)\psi^{-1}(\psi(\beta)d^t) \\ &= \alpha d^{k/2}\beta d^{t/2} = \alpha\beta F(\chi_C)^{1/2} F(\chi_D)^{1/2}. \end{aligned} \tag{13}$$

As F is monotone, from (12) and (13) it follows that

$$F(\chi_{C\cap D})^2 \leq F(\chi_C)F(\chi_D). \tag{14}$$

Next, we give an example which shows that we cannot omit the hypothesis of continuity of the functions ϕ and ψ.

Example 2 Once again we consider a measure space and a functional F as in the beginning of this section. Suppose further that

$$\{F(\chi_C): C \in \mathcal{A}, F(\chi_C) > 0\} \subset \mathbb{Q}_+.$$

Let ϕ and ψ be a couple of bijective functions of $[0,\infty)$ onto $[0,\infty)$ defined by

$$\phi(x) = \begin{cases} x^2 & \text{if } x^2 \in \mathbb{Q}_+, \\ x^2/2 & \text{if } x^2 \notin \mathbb{Q}_+, \end{cases}$$

and $\psi(x) = x^2$ for $x \geq 0$.

Note that (1) is satisfied. Let $x := \alpha\chi_C$ and $y := \beta\ \chi_D$ be as in Example 1. Clearly (12) is satisfied. On the other hand, we have

$$\psi^{-1}(F(\psi(y)) = \psi^{-1}(\psi(\beta)F(\chi_D)) = \psi^{-1}(\beta^2 F(\chi_D)) = \beta(F(\chi_D))^{1/2}.$$

If $\alpha^2 \in \mathbb{Q}_+$ then

$$\phi^{-1}(F(\phi(x)) = \phi^{-1}(\phi(\alpha)F(\chi_C)) = \phi^{-1}(\alpha^2 F(\chi_C)) = \alpha(F(\chi_C))^{1/2}.$$

If $\alpha^2 \notin \mathbb{Q}_+$ then

$$\phi^{-1}(F(\phi(x)) = \phi^{-1}(\phi(\alpha)F(\chi_C)) = \phi^{-1}\left(\frac{\alpha^2}{2}F(\chi_C)\right) = \alpha(F(\chi_C))^{1/2}.$$

In either case, we get

$$\phi^{-1}(F(\phi \circ x))\psi^{-1}(F(\psi \circ y)) = \alpha\beta F(\chi_C)^{1/2} F(\chi_D)^{1/2}.$$

Thus (1) follows from the inequality (14).

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