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Explicit Exponential Decay Bounds in Quasilinear Parabolic Problems

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This paper deals with classical solutions $u(x, t)$ of some initial boundary value problems involving the quasilinear parabolic equation

$$g(k(t)|\nabla u|^2)\Delta u + f(u) = u_t, \quad \mathbf{x} \in \Omega, \ t > 0,$$

where f, g, k are given functions. In the case of one space variable, i.e. when $\Omega := (-L, L)$, we establish a maximum principle for the auxiliary function

$$\Phi(x,t) := \mathrm{e}^{2\alpha t}\left\{\frac{1}{k(t)}\int_0^{k(t)u_x^2} g(\xi)\,\mathrm{d}\xi + \alpha u^2 + 2\int_0^u f(s)\,\mathrm{d}s\right\},$$

where α is an arbitrary nonnegative parameter. In some cases this maximum principle may be used to derive explicit exponential decay bounds for $|u|$ and $|u_x|$. Some extensions in N space dimensions are indicated. This work may be considered as a continuation of previous works by Payne and Philippin (*Mathematical Models and Methods in Applied Sciences*, **5** (1995), 95–110; Decay bounds in quasilinear parabolic problems, In: *Nonlinear Problems in Applied Mathematics*, Ed. by T.S. Angell, L. Pamela, Cook, R.E., SIAM, 1997).

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1 INTRODUCTION

Using a maximum principle approach Payne and Philippin [8] derived pointwise decay bounds for solutions of some initial boundary value problems involving the parabolic differential equation $\Delta u + f(u) = u_t$, $\mathbf{x} \in \Omega$, $t > 0$, where Ω is a bounded convex domain in $\mathbb{R}^N$. This paper deals with classical solutions $u(\mathbf{x}, t)$ of some initial boundary value problems involving the quasilinear parabolic equation

$$g(k(t)|\nabla u|^2)\Delta u + f(u) = u_t, \quad \mathbf{x} \in \Omega, \ t > 0, \tag{1}$$

where f, g, k are given functions. In Section 2 we consider the case of one space variable $x \in (-L, L)$. Under certain hypotheses we establish a maximum principle for the auxiliary function

$$\Phi(x, t) := \mathrm{e}^{2\alpha t}\left\{\frac{1}{k(t)}\int_0^{k(t)u_x^2} g(\xi)\,\mathrm{d}\xi + \alpha u^2 + 2\int_0^u f(s)\,\mathrm{d}s\right\}, \tag{2}$$

where α is an arbitrary nonnegative parameter.

When f is zero and k is an exponential function we compute in Section 3.1 a critical value α_0 that depends on the boundary conditions and on g in such a way that for $0 \leq \alpha < \alpha_0$, Φ takes its maximum value initially. This fact leads to explicit exponential decay bounds for $|u|$ and $|u_x|$.

When f is not zero and $k = 1$, we show under certain assumptions that if the initial data $u(x, 0)$ is nonnegative and small enough in some sense that will be made precise in Section 3.2, the solution $u(x, t)$ cannot blow up in finite time. Depending on f we then determine $\alpha_1 < \alpha_0$ such that for $0 \leq \alpha < \alpha_1$, Φ takes its maximum value initially. This leads again to explicit exponential decay bounds for $u (\geq 0)$ and $|u_x|$.

In Section 4, the results of Sections 2 and 3.1 are extended in $\mathbb{R}^N$ in the case of the parabolic equation $g(k(t)|\nabla u|^2)\Delta u = u_t$, $\mathbf{x} \in \Omega$, $t > 0$. We refer to [9] for a similar investigation involving the parabolic equation $\nabla(g(|\nabla u|^2)\nabla u) = u_t$, $\mathbf{x} \in \Omega$, $t > 0$.

For maximum principle results related to parabolic partial differential equations we refer to [10–14].

2 MAXIMUM PRINCIPLE FOR $\Phi(x,t)$

In this section we establish the following result.

THEOREM 1 *Let $u(x,t)$ be the solution of the initial boundary value problem*

$$g(ku_x^2)u_{xx} + f(u) = u_t, \quad x \in (-L, L),\ t \in (0, T), \tag{3}$$

$$u(\pm L, t) = 0, \quad t \in [0, T], \tag*{$(4)_1$}$$

$$u(x, 0) = h(x), \quad x \in [-L, L], \tag{5}$$

with $h(\pm L) = 0$, $h \not\equiv 0$. Let $\Phi(x,t)$ be defined on $u(x,t)$ by

$$\Phi(x,t) := e^{2\alpha t}\left\{\frac{1}{k(t)}G(ku_x^2) + \alpha u^2 + 2F(u)\right\}, \tag{6}$$

where α is an arbitrary nonnegative parameter and with

$$F(s) := \int_0^s f(\xi)\,d\xi, \tag{7}$$

$$G(\sigma) := \int_0^\sigma g(\xi)\,d\xi. \tag{8}$$

Assume that the given functions $g \in C^2$, $k \in C^1$ are strictly positive and satisfy the inequality

$$(2\alpha k - k')[\sigma g(\sigma) - G(\sigma)] \geq 0, \quad t \in (0, T),\ \sigma \geq 0, \tag{9}$$

and that the given function $f \in C^1$ satisfies the inequality

$$sf(s) - 2F(s) \geq 0, \quad s \in \mathbb{R}. \tag{10}$$

We then conclude that Φ takes its maximum value either at an interior critical point $(\bar{x}, \bar{t})$ of u, or initially. In other words we have

$$\Phi(x,t) \leq \max\begin{cases} \Phi(\bar{x}, \bar{t}\,), & \text{with } u_x(\bar{x}, \bar{t}\,) = 0 \quad \text{(i)}, \\ \max_{[-L,L]} \Phi(x, 0) & \text{(ii)}. \end{cases} \tag{11}$$

For the proof of Theorem 1 we compute

$$\begin{aligned}\Phi_x &= 2e^{2\alpha t}\{u_x u_{xx} g(ku_x^2) + \alpha\, uu_x + f(u)u_x\}\\ &= 2e^{2\alpha t} u_x(u_t + \alpha u),\end{aligned} \tag{12}$$

$$\begin{aligned}\Phi_{xx} &= 2e^{2\alpha t}\{2k\, u_x^2 u_{xx}^2 g' + g u_{xx}^2 + g u_x u_{xxx} + \alpha u_x^2 + \alpha uu_{xx}\\ &\qquad + f' u_x^2 + f u_{xx}\}\\ &= 2e^{2\alpha t}\{g u_{xx}^2 + u_x u_{xt} + \alpha u_x^2 + \alpha uu_{xx} + f u_{xx}\},\end{aligned} \tag{13}$$

$$\begin{aligned}\Phi_t = e^{2\alpha t}\Bigg\{&\frac{k'}{k^2}\left[gku_x^2 - G(ku_x^2)\right] + 2g u_x u_{xt} + 2\alpha uu_t + 2fu_t\\ &+ 2\alpha\left[\frac{1}{k}G(ku_x^2) + \alpha u^2 + 2F(u)\right]\Bigg\}.\end{aligned} \tag{14}$$

Combining (13) and (14) we obtain after some reduction

$$\begin{aligned}g\Phi_{xx} - \Phi_t = e^{2\alpha t}\Bigg\{&\frac{1}{k^2}(2\alpha k - k')[gku_x^2 - G(ku_x^2)] + 2g^2 u_{xx}^2\\ &- 2f^2 - 2\alpha uf - 2\alpha^2 u^2 - 4\alpha F(u)\Bigg\}.\end{aligned} \tag{15}$$

From (12) we compute

$$g u_{xx} = \tfrac{1}{2} u_x^{-1}\Phi_x e^{-2\alpha t} - (f + \alpha u). \tag{16}$$

Inserting (16) into (15) we obtain the parabolic differential inequality

$$\begin{aligned}&g\Phi_{xx} + u_x^{-2}c(x,t)\Phi_x - \Phi_t\\ &\quad = e^{2\alpha t}\left\{\frac{1}{k^2}(2\alpha k - k')[gku_x^2 - G(ku_x^2)] + 2\alpha[uf - 2F(u)]\right\} \geq 0,\end{aligned} \tag{17}$$

where the last inequality in (17) follows from (9) and (10). In (17), $c(x, t)$ is regular throughout $(-L, L) \times (0, T)$. It then follows from Nirenberg's maximum principle [6,10] that Φ takes its maximum value (i) at a critical point $(\bar{x}, \bar{t})$ of u, or (ii) initially, or (iii) at a boundary point $(\hat{x}, \hat{t})$ with $\hat{x} = \pm L$, $\hat{t} \in (0, T]$. The conclusion of Theorem 1 will follow if we can show that (iii) implies (i) or (ii). If fact (12) and the boundary conditions $(4)_1$, imply $\Phi_x(\pm L, t) = 0$. It then follows from Friedman's maximum principle [3,10] that Φ can take its maximum value at a boundary point $(\hat{x}, \hat{t})$ with $\hat{x} = \pm L$ only if Φ is identically constant in $(-L, L) \times (0, \hat{t})$, in which case the two possibilities (i) and (ii) hold in (11). This achieves the proof of Theorem 1.

It is worthy to note that Φ can be constant only for some particular choice of the data in problem (3), $(4)_1$, (5). In fact $\Phi = \text{const.}$ implies equality in (17), i.e. also in (9) and in (10). This implies then that

$$f(u) = \lambda u, \quad \lambda = \text{const.}, \tag{18}$$

and that

$$\text{either } k(t) = e^{2\alpha t} \quad \text{or} \quad g = \text{const.} \tag{19}$$

Moreover $\Phi \equiv \text{const.}$ implies $\Phi_x = u_x(u_t + \alpha u) \equiv 0$, from which we conclude

$$\text{either } u_x \equiv 0 \quad \text{or} \quad u_t + \alpha u \equiv 0. \tag{20}$$

The first equation in (20), together with (3) and (18), leads to

$$u(x, t) = e^{\lambda t}, \tag{21}$$

which is impossible in view of $(4)_1$. The second equation in (20) leads to

$$u(x, t) = h(x) e^{-\alpha t}, \tag{22}$$

which solves (3) only if we have

$$g(h'^2) h'' + (\lambda + \alpha) h = 0, \quad x \in (-L, L). \tag{23}$$

To conclude this section we note that Theorem 1 remains true even if we replace $(4)_1$ by any one of the following pairs of boundary conditions:

$$u(-L, t) = u_x(L, t) = 0, \tag*{$(4)_2$}$$

$$u_x(-L, t) = u(L, t) = 0, \tag*{$(4)_3$}$$

$$u_x(-L, t) = u_x(L, t) = 0. \tag*{$(4)_4$}$$

Moreover if both inequalities (9) and (10) in Theorem 1 are reversed we then conclude that Φ takes its minimum value either at a critical point of u, or initially.

3 ELIMINATION OF THE FIRST POSSIBILITY (i) IN (11)

We note that the realization of (ii) in (11) leads to lower exponential decay bounds for both $|u|$ and $|u_x|$. In this section we impose restrictions on the parameter $\alpha \geq 0$ so that the first possibility (i) in (11) cannot occur. We shall investigate two particular cases.

3.1 First Case: $f(u) := 0$, $k(t) := e^{2\mu t}$, $\mu \leq \alpha$

We consider the parabolic problem (3), $(4)_k$, $k = 1, 2, 3$ or 4, and (5) with the particular choices $f(u) := 0$, $k(t) := \mathrm{e}^{2\mu t}$, $\mu = \text{const.} \leq \alpha$. We assume (9) so that the conclusion (11) of Theorem 1 holds. Note that if $\mu := \alpha$, assumption (9) is satisfied for any arbitrary function $g > 0$. If $\mu < \alpha$, (9) is satisfied if and only if $g' \geq 0$.

Suppose that we have the first possibility (i) in (11), i.e.

$$\Phi(x,t) \leq \alpha u^2(\bar{x},\bar{t})\mathrm{e}^{2\alpha\bar{t}} = \alpha u_{\mathrm{M}}^2 \mathrm{e}^{2\alpha\bar{t}}, \tag{27}$$

with

$$u_{\mathrm{M}}^2 := \max_{(-L,L)} u^2(x,\bar{t}). \tag{28}$$

We can rewrite (27) evaluated at $t = \bar{t}$ as

$$\mathrm{e}^{-2\mu\bar{t}} G(\mathrm{e}^{2\mu\bar{t}} u_x^2) \leq \alpha(u_{\mathrm{M}}^2 - u^2(x,\bar{t})). \tag{29}$$

Making use of the mean value theorem we may bound the left hand side of (29) as follows:

$$u_x^2(x,\bar{t}) g_{\min} \leq \mathrm{e}^{-2\mu\bar{t}} G(\mathrm{e}^{2\mu\bar{t}} u_x^2), \tag{30}$$

where $g_{\min} > 0$ is the minimum value of g. From (29) and (30) we obtain the inequality

$$u_x^2(x, \bar{t})g_{\min} \leq \alpha(u_{\mathrm{M}}^2 - u^2(x, \bar{t})), \tag{31}$$

that may be rewritten as

$$\frac{|u_x(x, \bar{t})|}{\sqrt{u_{\mathrm{M}}^2 - u^2(x, \bar{t})}} \leq \sqrt{\frac{\alpha}{g_{\min}}}. \tag{32}$$

Integrating (32) from the critical point $\bar{x}$ to the nearest zero $\bar{\bar{x}} \in [-L, L]$ of $u(x, \bar{t})$, we obtain

$$\alpha \geq \alpha_0 := \frac{\pi^2 g_{\min}}{4|\bar{x} - \bar{\bar{x}}|^2}. \tag{33}$$

Since $|\bar{x} - \bar{\bar{x}}|$ is unknown, we need an upper bound for this quantity in (33). Obviously we may use

$$|\bar{x} - \bar{\bar{x}}| \leq \Lambda := \begin{cases} L & \text{if we have } (4)_1, \\ 2L & \text{if we have } (4)_k, k = 2, 3 \text{ or } 4, \end{cases} \tag{34}$$

if we assume the existence of $\bar{\bar{x}} \in [-L, L]$ when we have the boundary conditions $(4)_4$. This will be the case e.g. if the initial data $h(x)$ have zero mean value, i.e. if we have

$$\int_{-L}^{L} h(x)\,\mathrm{d}x = 0. \tag{35}$$

In fact with the auxiliary function $\rho(\sigma)$ defined by

$$\rho(\sigma) := \frac{1}{2}\sigma^{-1/2} \int_0^{\sigma} g(\xi)\xi^{-1/2}\,\mathrm{d}\xi, \tag{36}$$

we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_{-L}^{L} u(x, t)\,\mathrm{d}x &= \int_{-L}^{L} u_t(x, t)\,\mathrm{d}x = \int_{-L}^{L} g(\mathrm{e}^{2\mu t}u_x^2)u_{xx}\,\mathrm{d}x \\ &= \int_{-L}^{L} (\rho(\mathrm{e}^{2\mu t}u_x^2)u_x)_x\,\mathrm{d}x = \rho(\mathrm{e}^{2\mu t}u_x^2)u_x\Big|_{-L}^{L} = 0. \end{aligned} \tag{37}$$

It then follows from (35) and (37) that

$$\int_{-L}^{L} u(x,t)\,\mathrm{d}x = \int_{-L}^{L} h(x)\,\mathrm{d}x = 0, \qquad \forall t \in [0,T], \tag{38}$$

i.e. the zero mean value property of $u(x,t)$ is inherited from the zero mean value property of the initial data if we have the boundary conditions $(4)_4$. But this implies the existence of $\bar{\bar{x}} \in [-L, L]$ such that $u(\bar{\bar{x}}, t) = 0$.

We conclude from the above investigation that, for $0 \le \alpha \le \alpha_0$, the first possibility (i) in (11) cannot hold, so that $|u|$ and $|u_x|$ must decay exponentially. This shows in fact that $u(x,t)$ cannot blow up and will exist for all $t > 0$. These results are summarized next.

THEOREM 2 *Let $u(x,t)$ be the solution of the parabolic problem* (3), $(4)_k$ $k = 1, 2, 3,$ *or* 4, *and* (5). *If we have* $(4)_4$ *we require that the initial data $h(x)$ satisfy the further condition* (35). *Assume either*

$$\mu = \alpha < \alpha_0, \tag{39}$$

or

$$\mu < \alpha < \alpha_0 \quad \text{and} \quad g' \ge 0, \tag{40}$$

with

$$\alpha_0 := \frac{\pi^2 g_{\min}}{4\Lambda^2}, \tag{41}$$

where Λ is defined in (34). *Then we may take $T = \infty$ in* (3) *and* (4). *Moreover the function Φ defined as*

$$\Phi(x,t) := \mathrm{e}^{2\alpha t}\{\mathrm{e}^{-2\mu t} G(\mathrm{e}^{2\mu t} u_x^2) + \alpha u^2\}, \tag{42}$$

takes its maximum value initially. The resulting inequality (11) *with $\alpha \to \alpha_0$ takes the form*

$$\mathrm{e}^{-2\mu t} G(\mathrm{e}^{2\mu t} u_x^2) + \alpha_0 u^2 \le H^2 \mathrm{e}^{-2\alpha_0 t}, \qquad \forall \mu \le \alpha_0, \tag{43}$$

with

$$H^2 := \max_{[-L,L]} \{G(h'^2) + \alpha_0 h^2\}. \tag{44}$$

We note that the quantities α_0 and H^2 are explicitely computable in terms of the initial and boundary data.

A weaker but more practical version of (43) is

$$g_{\min} u_x^2(x,t) + \alpha_0 u^2 \leq H^2 \mathrm{e}^{-2\alpha_0 t}. \tag{45}$$

Integrating (45) over $(-L, L)$ we obtain

$$g_{\min} \int_{-L}^{L} u_x^2(x,t)\,\mathrm{d}x + \alpha_0 \int_{-L}^{L} u^2\,\mathrm{d}x \leq 2LH^2 \mathrm{e}^{-2\alpha_0 t}. \tag{46}$$

Moreover depending of the boundary conditions (4), $u(x,t)$ is admissible for the variational characterization of the first or second eigenvalue of a vibrating string of length $2L$ with fixed or free ends. We have actually

$$g_{\min} \int_{-L}^{L} u_x^2(x,t)\,\mathrm{d}x \geq \alpha_0 \int_{-L}^{L} u^2\,\mathrm{d}x, \tag{47}$$

valid in all cases considered in Theorem 2. From (46) and (47) we obtain the following decay bound for $\int_{-L}^{L} u^2\,\mathrm{d}x$:

$$\int_{-L}^{L} u^2(x,t)\,\mathrm{d}x \leq L\alpha_0^{-1} H^2 \mathrm{e}^{-2\alpha_0 t}. \tag{48}$$

We shall now derive a pointwise lower bound for $|u(x,t)|$ that is proportional to the distance $|x - \bar{\bar{x}}|$ from x to the nearest zero $\bar{\bar{x}}$ of $u(x,t)$. To this end we rewrite (45) as

$$\frac{|u_x|}{\sqrt{(H^2/\alpha_0)\mathrm{e}^{-2\alpha_0 t} - u^2}} \leq \sqrt{\frac{\alpha_0}{g_{\min}}}. \tag{49}$$

Integrating (49) for fixed t from x to $\bar{\bar{x}}$ we obtain

$$\arcsin\left(\frac{\sqrt{\alpha_0}|u|}{H\mathrm{e}^{-\alpha_0 t}}\right) = \int_0^{|u|} \frac{\mathrm{d}\xi}{\sqrt{(H^2/\alpha_0)\mathrm{e}^{-2\alpha_0 t} - \xi^2}} \leq \sqrt{\frac{\alpha_0}{g_{\min}}}|x - \bar{\bar{x}}|, \tag{50}$$

or

$$|u(x,t)| \leq \frac{H}{\sqrt{g_{\min}}} |x - \bar{\bar{x}}| e^{-\alpha_0 t}, \quad x \in (-L, L), \ t > 0. \tag{51}$$

Of course we can substitute $\bar{\bar{x}}$ by $+L$ or $-L$ if we have the boundary conditions $(4)_k$, $k = 1, 2, 3$.

3.2 Second Case: $f(u) \not\equiv 0$, $k(t) = 1$

In this section we consider the parabolic problem (3), $(4)_k$, $k = 1, 2, 3$ or 4, and (5) with the particular choice $k(t) = 1, f(u) \not\equiv 0$. It is well known that the solution $u(x, t)$ of this problem may not exist for all time. In fact $u(x, t)$ may blow up at some time t^* which may be finite or infinite [1,3]. However if blow-up does occur at $t = t^*$, then $u(x, t)$ will exist on the time interval $(0, t^*)$.

We want to establish conditions involving the data sufficient to prevent blow-up of $u(x, t)$ and even sufficient to guarantee its exponential decay. To this end we first establish the following comparison result.

LEMMA 1 *Let $u(x, t)$ be the solution of the parabolic problem* (3), $(4)_k$ $k = 1, 2, 3,$ *or* 4, *and* (5) *with* $h(x) \geq 0$ *and* $k(t) = 1$. *Assume moreover the following conditions on f and g*:

$$sf'(s) \geq f(s) > 0, \quad s > 0, \ f(0) = 0, \tag{52}$$

$$\mu := \frac{f(u_M)}{u_M} \leq \alpha_0 := \frac{\pi^2 g_{\min}}{4\Lambda^2}, \tag{53}$$

$$g'(\sigma) \geq 0, \quad \sigma \geq 0, \tag{54}$$

where u_M^2 has been defined in (28). *We then have the following bounds for $u(x, t)$*:

$$0 \leq u(x,t) \leq U \exp\left\{ \left(\frac{f(u_M)}{u_M} - \alpha_0 \right) t \right\}, \quad t \in (0, T), \tag{55}$$

with

$$U := \max_{(-L,L)} \sqrt{h^2 + \frac{1}{\alpha_0} G(h'^2)}. \tag{56}$$

We note that condition (52) implies condition (10) and the fact that the ratio $f(s)/s$ is a nondecreasing function of s.

The lower bound in (55) follows from Nirenberg's and Friedman's maximum principles [3,6,10]. To establish the upper bound in (55) we introduce an auxiliary function $v(x,t)$ defined as

$$u(x,t) = v(x,t)e^{\mu t}, \tag{57}$$

with $\mu := f(u_M)/u_M$. Inserting (57) into (3) we obtain

$$e^{\mu t}\left[g(e^{2\mu t}v_x^2)v_{xx} - v_t\right] = u\left(\mu - \frac{f(u)}{u}\right) \geq 0, \tag{58}$$

where the above inequality results from the definition of μ together with the monotonicity of $f(s)/s$. The auxiliary function $v(x,t)$ then satisfies

$$g(e^{2\mu t}v_x^2)v_{xx} - v_t \geq 0, \quad x \in (-L,L),\ t \in (0,T), \tag{59}$$

$$v(x,0) = h(x), \quad x \in (-L,L). \tag{60}$$

Moreover $v(x,t)$ satisfies the same boundary conditions (4) as $u(x,t)$. Let $w(x,t)$ satisfy

$$g(e^{2\mu t}w_x^2)w_{xx} - w_t = 0, \quad x \in (-L,L),\ t \in (0,T), \tag{61}$$

$$w(x,0) = h(x), \quad x \in (-L,L), \tag{62}$$

with the same boundary conditions as u and v. From (59) and (61) we have

$$g(e^{2\mu t}v_x^2)v_{xx} - g(e^{2\mu t}w_x^2)w_{xx} - (v-w)_t \geq 0. \tag{63}$$

Using the mean value theorem we may rewrite the first two terms in (63) as follows:

$$
\begin{aligned}
&g(e^{2\mu t}v_x^2)v_{xx} - g(e^{2\mu t}w_x^2)w_{xx} \\
&\quad = g(e^{2\mu t}v_x^2)(v-w)_{xx} + w_{xx}[g(e^{2\mu t}v_x^2) - g(e^{2\mu t}w_x^2)] \\
&\quad = g(e^{2\mu t}v_x^2)(v-w)_{xx} + w_{xx}g'(\xi)e^{2\mu t}(v-w)_x(v+w)_x,
\end{aligned}
\tag{64}
$$

for some intermediate value ξ. We conclude from (63) and (64) that the function $\omega := v - w$ satisfies a parabolic inequality of the following form:

$$
g(e^{2\mu t}v_x^2)\omega_{xx} + C(x,t)\omega_x - \omega_t \geq 0, \quad x \in (-L, L), \ t \in (0, T), \tag{65}
$$

where $C(x,t)$ is regular throughout $(-L, L) \times (0, T)$. Since $\omega(x, 0) = 0$ and since ω satisfies zero Dirichlet or Neumann boundary conditions, it follows that

$$
\omega := v - w \leq 0, \quad x \in (-L, L), \ t \in (0, T). \tag{66}
$$

From (57) and (66) we obtain

$$
0 \leq u(x,t) \leq e^{\mu t}w(x,t). \tag{67}
$$

Finally since we assume (53) and (54) we may use (43) to bound $w(x,t)$. Dropping the first term in (43) we obtain

$$
\alpha_0 w^2 \leq H^2 e^{-2\alpha_0 t}, \tag{68}
$$

where H^2 is defined in (44). The desired inequality (55) follows now from (67) and (68).

Lemma 1 is the main tool in the derivation of the following result.

THEOREM 3 *Let* $u(x,t)$ *be the solution of problem* (3), $(4)_k$, $k = 1, 2, 3$ *or* 4, *and* (5) *with* $h(x) \geq 0$, *and* $k(t) = 1$. *Assume* (52)–(54). *Moreover assume that the data in problem* (3)–(5) *are small enough in the following sense*:

$$
\frac{f(U)}{U} < \alpha_0 := \frac{\pi^2 g_{\min}}{4\Lambda^2}, \tag{69}
$$

where U is defined in (56). *Then $u(x,t)$ exists for all time $t>0$* (*i.e. we may take $T=\infty$ in problem* (3)–(5)). *Moreover we have*

$$\max_{(-L,L)} \frac{f(u(x,t))}{u(x,t)} < \alpha_0, \quad \forall t > 0. \tag{70}$$

For the proof of Theorem 3 we assume that (70) is not valid and show that this invalidity is self-contradictory. From the definition of U we have

$$U \geq \max_{(-L,L)} h(x). \tag{71}$$

Since $f(s)/s$ is nondecreasing, (71) and (69) imply

$$\frac{f(h)}{h} \leq \frac{f(U)}{U} < \alpha_0. \tag{72}$$

If (70) is violated, there exists in view of (72) a first time $t=\tau$ for which we have

$$\max_{(-L,L)} \frac{f(u)}{u} = \alpha_0. \tag{73}$$

With $f(u_{\mathrm{M}})/u_{\mathrm{M}} \leq \max_{(-L,L)} (f(u)/u)$, we obtain

$$\frac{f(u_{\mathrm{M}})}{u_{\mathrm{M}}} \leq \alpha_0. \tag{74}$$

From (55) and (74) we obtain

$$u(x,t) \leq U, \quad x \in (-L,L), \ 0 \leq t \leq \tau, \tag{75}$$

and we conclude that

$$\max_{(-L,L)} \frac{f(u(x,\tau))}{u(x,\tau)} \leq \frac{f(U)}{U} < \alpha_0, \tag{76}$$

so that (70) cannot actually be violated in a finite time τ. This establishes (70) with $\tau=\infty$.

We are now prepared to establish the following result:

THEOREM 4 *Let $u(x,t)$ be the solution of the parabolic problem* (3), $(4)_k$, $k=1,2$ *or* 3, *and* (5) *with* $h(x)\geq 0$, $k(t)=1$. *Assume* (52)–(54). *Moreover assume that the data in problem* (3)–(5) *are small enough in the sense that there exists a constant* $\alpha_1>0$ *such that the inequality*

$$\frac{f(U)}{U}<\alpha_0-\alpha_1 \tag{77}$$

is satisfied, where U is defined in (56). *Then we conclude that the first possibility* (i) *in* (11) *cannot hold* $\forall\alpha$, $0\leq\alpha\leq\alpha_1$. *We are then led to the following decay bound for $u^2$and u_x^2:*

$$G(u_x^2)+\alpha_1 u^2+2F(u)\leq\mathcal{H}^2\mathrm{e}^{-2\alpha_1 t},\quad x\in(-L,L),\ t>0 \tag{78}$$

(*valid for all time* $t>0$) *with*

$$\mathcal{H}^2:=\max_{(-L,L)}\{G(h'^2)+\alpha_1 h^2+2F(h)\}. \tag{79}$$

Before proving Theorem 4 we show that the realization of (i) in (11) with $\alpha:=\alpha_1$ implies the inequality

$$\frac{f(u_\mathrm{M})}{u_\mathrm{M}}\geq\alpha_0-\alpha_1. \tag{80}$$

In fact the realization of (i) in (11) with $\alpha:=\alpha_1$ implies the inequality

$$\{G(u_x^2(x,t))+\alpha_1 u^2+2F(u)\}\mathrm{e}^{2\alpha_1 t}\leq[\alpha_1 u_\mathrm{M}^2+2F(u_\mathrm{M})]\mathrm{e}^{2\alpha_1\bar{t}}, \tag{81}$$

where u_M^2 is defined in (28). Evaluated at $t=\bar{t}$, we obtain

$$G(u_x^2(x,\bar{t}))\leq\alpha_1[u_\mathrm{M}^2-u^2(x,\bar{t})]+2[F(u_\mathrm{M})-F(u(x,\bar{t}))]. \tag{82}$$

Using the generalized mean value theorem and the monotonicity of $f(s)/s$ we may rewrite the last term in (82) as follows:

$$\begin{aligned}F(u_\mathrm{M})-F(u(x,\bar{t}))&=\frac{F(u_\mathrm{M})-F(u(x,\bar{t}))}{u_\mathrm{M}^2-u^2(x,\bar{t})}[u_\mathrm{M}^2-u^2(x,\bar{t})]\\&=\frac{f(\xi)}{2\xi}[u_\mathrm{M}^2-u^2(x,\bar{t})]\leq\frac{1}{2}\frac{f(u_\mathrm{M})}{u_\mathrm{M}}[u_\mathrm{M}^2-u^2(x,\bar{t})],\end{aligned} \tag{83}$$

where ξ is some intermediate value of u. Moreover the left hand side of (82) may be bounded as follows:

$$g_{\min} u_x^2(x,\bar{t}) \le G(u_x^2(x,\bar{t})), \tag{84}$$

with $g_{\min} = g(0)$. From (82)–(84) we obtain the inequality

$$g_{\min} u_x^2(x,\bar{t}) \le \left(\alpha_1 + \frac{f(u_M)}{u_M}\right)[u_M^2 - u^2(x,\bar{t})], \tag{85}$$

that may be rewritten as

$$\frac{|u_x(x,\bar{t})|}{\sqrt{u_M^2 - u^2(x,\bar{t})}} \le \sqrt{g_{\min}^{-1}\left(\alpha_1 + \frac{f(u_M)}{u_M}\right)}. \tag{86}$$

Integrating (86) from the critical point $\bar{x}$ to the nearest end $\bar{\bar{x}} = \pm L$ of the interval $(-L, L)$ with $u(\bar{\bar{x}}, t) = 0$, we obtain (80).

For the proof of Theorem 4 we note that the assumption (77) implies (69), so that conclusion (70) of Theorem 3 holds. In particular we have

$$\frac{f(u_M)}{u_M} \le \alpha_0, \quad \forall t > 0, \tag{87}$$

and (55) leads to

$$u_M \le U, \tag{88}$$

from which we obtain using the monotonicity of $f(s)/s$ and assumption (77)

$$\frac{f(u_M)}{u_M} \le \frac{f(U)}{U} < \alpha_0 - \alpha_1, \tag{89}$$

in contradiction to (80), so that (i) in (11) cannot hold. The inequality (78) follows now from (ii) in (11). This achieves the proof of Theorem 4.

As an example, let $u(x, t)$ be the solution of the following parabolic differential equation:

$$u_{xx}\sqrt{1+u_x^2}+u^{1+\varepsilon}=u_t, \quad x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right),\ t>0, \tag{90}$$

with the boundary conditions

$$u\left(-\frac{\pi}{2},t\right)=u\left(\frac{\pi}{2},t\right)=0, \tag{91}$$

and with the initial condition

$$u(x,0)=a\cos x, \quad a=\text{const.}>0. \tag{92}$$

With $\varepsilon:=\text{const.}\geq 0$, the function $f(s):=s^{1+\varepsilon}$ satisfies (52). With $g(\sigma):=(1+\sigma)^{1/2}$, condition (54) is satisfied. Since $g(s)$ is increasing we have $g_{\min}=1$. From (56) with $\alpha_0=1$ and $h(x)=a\cos x$ we compute

$$U=\max_{(-\pi/2,\pi/2)}\left\{a^2\cos^2 x+\int_0^{a^2\sin^2 x}\sqrt{1+\xi}\,\mathrm{d}\xi\right\}^{1/2}=\left\{\tfrac{2}{3}[(1+a^2)^{3/2}-1]\right\}^{1/2}. \tag{93}$$

From Theorem 3 we conclude that $u(x, t)$ exists for all time $t>0$ if (69) is satisfied, i.e. if we have $0<a<\sqrt{(5/2)^{2/3}-1}\cong 0.917$. From Theorem 4 we have the decay estimate (78) with $\alpha_1:=1-\{2/3[(1+a^2)^{3/2}-1]\}^{\varepsilon/2}>0$.

4 EXTENSION TO THE N-DIMENSIONAL CASE

The results of Sections 2 and 3.1 may be extended in case of N space variables $\mathbf{x}:=(x_1,\ldots,x_N)$, $N\geq 2$. In this section we establish the following maximum principle analogous to Theorem 1.

THEOREM 5 *Let Ω be a bounded convex domain in $\mathbb{R}^N$ with a $C^{2+\varepsilon}$ boundary $\partial\Omega$. Let $u(\mathbf{x}, t)$ be the solution of the initial boundary value problem*

$$g(k(t)|\nabla u|^2)\Delta u=u_t, \quad \mathbf{x}\in\Omega,\ t\in(0,T), \tag{94}$$

$$u(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \partial\Omega, \ t \in (0, T), \tag{95}$$

$$u(\mathbf{x}, 0) = h(x), \quad \mathbf{x} \in \Omega, \tag{96}$$

where g and k are given positive functions, $g \in C^2$, $k \in C^1$. *Let* $\Phi(\mathbf{x}, t)$ *be defined on* $u(\mathbf{x}, t)$ *by*

$$\Phi(\mathbf{x}, t) := \left\{\frac{1}{k(t)} G(k(t)|\nabla u|^2) + \alpha u^2\right\} \mathrm{e}^{2\alpha\beta t}, \tag{97}$$

with

$$G(\sigma) := \int_0^\sigma g(\xi)\, \mathrm{d}\xi. \tag{98}$$

In (97), α *is an arbitrary nonnegative parameter, and* β *is a constant to be chosen in* $(0, 1)$ *as indicated below. We distinguish two cases.*

If $g'(\sigma) \geq 0$, *we assume*

$$2\alpha k - k' \geq 0, \tag{99}$$

and we assume that two constants $\lambda > 0$ *and* $\beta \in (0, 1)$ *can be determined such that*

$$g(\sigma) - A(\lambda, N, \beta)\sigma g'(\sigma) \geq 0, \quad \sigma \geq 0, \tag{100}$$

with

$$A(\lambda, N, \beta) := \max\left\{\lambda N, \frac{\lambda N + \lambda^{-1} - 2}{1 - \beta}\right\}. \tag{101}$$

If $g'(\sigma) \leq 0$, *we assume*

$$k'(t) \geq 0, \tag{102}$$

and we assume that $\beta \in (0, 1)$ *can be determined such that*

$$\sigma g(\sigma) - \beta G(\sigma) \geq 0, \quad \sigma \geq 0, \tag{103}$$

$$g(\sigma) + B(N, \beta)\sigma g'(\sigma) \geq 0, \quad \sigma \geq 0, \tag{104}$$

with

$$B(N,\beta) := \max\left\{N-1, \frac{1}{1-\beta}\right\}. \tag{105}$$

We then conclude that $\Phi(\mathbf{x},t)$ takes its maximum value either at an interior critical point $(\bar{\mathbf{x}},\bar{t})$ of u, or initially. In other words we have

$$\Phi(\mathbf{x},t) \le \max\begin{cases} \Phi(\bar{\mathbf{x}},\bar{t}) & \text{with } \nabla u(\bar{\mathbf{x}},\bar{t}) = 0 \quad \text{(i)}, \\ \max_\Omega \Phi(\mathbf{x},0) & \text{(ii)}. \end{cases} \tag{106}$$

We note the presence of a factor β in the decay exponent of $\Phi(\mathbf{x},t)$. This factor makes Theorem 5 less sharp than Theorem 1 corresponding to the one-dimensional case.

The existence of a classical solution of (94)–(96) will not be investigated in this paper. We refer to [1,5] for such existence results.

For the proof of Theorem 5 we proceed in two steps. We first construct a parabolic inequality of the following type:

$$\mathcal{L}\Phi := g(k(t)|\nabla u|^2)\Delta\Phi + |\nabla u|^{-2}\mathbf{c}(\mathbf{x},t)\cdot\nabla\Phi - \Phi_{,t} \ge 0, \tag{107}$$

where the vector field $\mathbf{c}(\mathbf{x},t)$ is regular throughout $\Omega\times(0,T)$. Using the following notations: $u_{,i} := \partial u/\partial x_i$, $i=1,\ldots,N$, $u_{,ik} := \partial^2 u/\partial x_i\partial x_k$, $i,k = 1,\ldots,N$, $u_{,t} = \partial u/\partial t$, $u_{,i}v_{,i} = \sum_{i=1}^N u_{,i}v_{,i} = \nabla u\cdot\nabla v$, etc., we compute

$$\begin{aligned}\Phi_{,t} = \Bigg\{ & \frac{k'}{k^2}[gk|\nabla u|^2 - G(k(t)|\nabla u|^2)] + 2\alpha u u_t + 2g u_{,kt}u_{,k} \\ & + 2\alpha\beta\left[\frac{1}{k}G(k|\nabla u|^2) + \alpha u^2\right]\Bigg\}e^{2\alpha\beta t},\end{aligned} \tag{108}$$

$$\Phi_{,k} = 2\{g u_{,ik}u_{,i} + \alpha u u_{,k}\}e^{2\alpha\beta t}, \tag{109}$$

$$\begin{aligned}\Delta\Phi = 2\{& 2g'k u_{,ik}u_{,k}u_{,i\ell}u_{,\ell} + g u_{,i}(\Delta u)_{,i} \\ & + g u_{,ik}u_{,ik} + \alpha|\nabla u|^2 + \alpha u\Delta u\}e^{2\alpha\beta t}.\end{aligned} \tag{110}$$

Moreover differentiating (94) we obtain

$$g u_{,i}(\Delta u)_{,i} = -2kg' u_{,ik}u_{,i}u_{,k}\Delta u + u_{,tk}u_{,k}. \tag{111}$$

Combining (108), (110), and (111), we obtain after some reduction

$$g\Delta\Phi - \Phi_{,t} = \Big\{ 4gg'k[u_{,ik}u_{,k}u_{,i\ell}u_{,\ell} - u_{,ik}u_{,i}u_{,k}\Delta u] + 2g^2 u_{,ik}u_{,ik} + \frac{2\alpha}{k}[gk|\nabla u|^2 - \beta G(k|\nabla u|^2)] - \frac{k'}{k^2}[gk|\nabla u|^2 - G(k|\nabla u|^2)] - 2\alpha^2\beta u^2 \Big\} e^{2\alpha\beta t}. \tag{112}$$

In contrast to the one-dimensional case the quantity $u_{,ik}u_{,k}u_{,i\ell}u_{,\ell} - u_{,ik}u_{,i}u_{,k}\Delta u$ is not identically zero. Depending on the sign of g', it seems convenient to substitute an upper bound or a lower bound for $u_{,ik}u_{,i}u_{,k}\Delta u$.

If $g' \geq 0$, we use the arithmetic–geometric mean inequality in the following form:

$$\begin{aligned} 2u_{,ik}u_{,i}u_{,k}\Delta u &\leq \lambda|\nabla u|^2(\Delta u)^2 + \lambda^{-1}|\nabla u|^{-2}(u_{,ik}u_{,i}u_{,k})^2 \\ &\leq \lambda N|\nabla u|^2 u_{,ik}u_{,ik} + \lambda^{-1}|\nabla u|^{-2}(u_{,ik}u_{,i}u_{,k})^2, \end{aligned} \tag{113}$$

where λ is an arbitrary positive constant. Combining (112) and (113) we obtain

$$g\Delta\Phi - \Phi_{,t} \geq \Big\{ 4gg'ku_{,ik}u_{,k}u_{,i\ell}u_{,\ell} + 2g[g - N\lambda g'k|\nabla u|^2]u_{,ik}u_{,ik} - 2\lambda^{-1}gg'k|\nabla u|^{-2}(u_{,ik}u_{,i}u_{,k})^2 + \frac{2\alpha}{k}[gk|\nabla u|^2 - \beta G(k|\nabla u|^2)] - \frac{k'}{k^2}[gk|\nabla u|^2 - G(k|\nabla u|^2)] - 2\alpha^2\beta u^2 \Big\} e^{2\alpha\beta t}. \tag{114}$$

Since $g - N\lambda g'k|\nabla u|^2 \geq 0$ by assumption (100) we may use the Cauchy–Schwarz inequality

$$|\nabla u|^2 u_{,ik}u_{,ik} \geq u_{,ik}u_{,k}u_{,i\ell}u_{,\ell}. \tag{115}$$

We then obtain

$$g\Delta\Phi - \Phi_{,t} \geq \Big\{ 2g|\nabla u|^{-2}[g + (2 - \lambda N)g'k|\nabla u|^2]u_{,ik}u_{,k}u_{,i\ell}u_{,\ell} - 2\lambda^{-1}gg'k|\nabla u|^{-2}(u_{,ik}u_{,i}u_{,k})^2 + \frac{2\alpha}{k}[gk|\nabla u|^2 - \beta G(k|\nabla u|^2)] - \frac{k'}{k^2}[gk|\nabla u|^2 - G(k|\nabla u|^2)] - 2\alpha^2\beta u^2 \Big\} e^{2\alpha\beta t}. \tag{116}$$

We now make use of (109) to represent $u_{,ik}u_{,i}$ as follows:

$$gu_{,ik}u_{,i} = -\alpha uu_{,k} + \cdots, \qquad k = 1, \ldots, N, \tag{117}$$

where dots stand for a term containing $\Phi_{,k}$. From (117) we compute

$$g^2 u_{,ik}u_{,k}u_{,i\ell}u_{,\ell} = \alpha^2|\nabla u|^2 u^2 + \cdots, \tag{118}$$

$$g^2(u_{,ik}u_{,i}u_{,k})^2 = \alpha^2|\nabla u|^4 u^2 + \cdots \tag{119}$$

Inserting (118) and (119) into (116) we obtain after some reduction

$$\begin{aligned} &g\Delta\Phi - \Phi_{,t} + \cdots \\ &\geq \Big\{ 2g^{-1}\alpha^2u^2[g + (2 - N\lambda - \lambda^{-1})g'k|\nabla u|^2] + \frac{2\alpha}{k}[gk|\nabla u|^2 - \beta G(k|\nabla u|^2)] \\ &\quad - \frac{k'}{k^2}[gk|\nabla u|^2 - G(k|\nabla u|^2)] - 2\alpha^2\beta u^2 \Big\} e^{2\alpha\beta t}. \end{aligned} \tag{120}$$

Using (100) we obtain

$$g^{-1}\alpha^2u^2[g + (2 - N\lambda - \lambda^{-1})g'k|\nabla u|^2] \geq \beta\alpha^2u^2. \tag{121}$$

Combining (120) and (121) we are led to the desired inequality

$$g\Delta\Phi - \Phi_{,t} + \cdots \geq k^{-2}(2\alpha k - k')[gk|\nabla u|^2 - G(k|\nabla u|^2)]e^{2\alpha\beta t} \geq 0. \tag{122}$$

If $g' \leq 0$, we use the inequality

$$\begin{aligned} 2\Delta u u_{,ik}u_{,i}u_{,k} \geq &-(N-1)|\nabla u|^2 u_{,ik}u_{,ik} + |\nabla u|^{-2}(u_{,ik}u_{,i}u_{,k})^2 \\ &+ (N-1)u_{,ik}u_{,k}u_{,i\ell}u_{,\ell}, \end{aligned} \tag{123}$$

derived in [7]. Combining (112) and (123) we obtain

$$\begin{aligned} g\Delta\Phi - \Phi_{,t} \geq &\Big\{ 2(3-N)gg'ku_{,ik}u_{,k}u_{,i\ell}u_{,\ell} + 2g[g + (N-1)g'k|\nabla u|^2]u_{,ik}u_{,ik} \\ &- 2gg'k|\nabla u|^{-2}(u_{,ik}u_{,i}u_{,k})^2 + \frac{2\alpha}{k}[gk|\nabla u|^2 - \beta G(k|\nabla u|^2)] \\ &- \frac{k'}{k^2}[gk|\nabla u|^2 - G(k|\nabla u|^2)] - 2\alpha^2\beta u^2 \Big\} e^{2\alpha\beta t} \\ \geq &\{ 2(3-N)gg'ku_{,ik}u_{,k}u_{,i\ell}u_{,\ell} + 2g[g + (N-1)g'k|\nabla u|^2]u_{,ik}u_{,ik} \\ &- 2gg'k|\nabla u|^{-2}(u_{,ik}u_{,i}u_{,k})^2 - 2\alpha^2\beta u^2\} e^{2\alpha\beta t}, \end{aligned} \tag{124}$$

where the last inequality in (124) follows from assumptions (102) and (103). Now since $g+(N-1)g'k|\nabla u|^2 \geq 0$ by assumption (104) we may use (115). Moreover inserting (118) and (119) we obtain after some reduction

$$g\Delta\Phi - \Phi_{,t} + \cdots \geq \{2\alpha^2 u^2 g^{-1}[g+g'k|\nabla u|^2] - 2\alpha^2\beta u^2\}e^{2\alpha\beta t} \geq 0, \tag{125}$$

where the last inequality follows from (104). The inequality (125) is again of the desired type.

It follows from Nirenberg's maximum principle [6,10] that Φ takes its maximum value (i) at an interior critical point $(\bar{\mathbf{x}}, \bar{t})$ of u, or (ii) initially, or (iii) at a boundary point $(\tilde{\mathbf{x}}, \tilde{t})$ with $\tilde{\mathbf{x}} \in \partial\Omega$. The second step of the proof of Theorem 5 consists in showing that the later possibility (iii) cannot hold. To this end we compute the outward normal derivative of Φ on $\partial\Omega$. Using (94) rewritten in normal coordinates we obtain

$$\frac{\partial\Phi}{\partial n} = 2e^{2\alpha\beta t}u_n u_{nn} g = -2(N-1)e^{2\alpha\beta t} gK|\nabla u|^2 \leq 0 \quad \text{on } \partial\Omega, \tag{126}$$

where $K(\geq 0)$ is the average curvature of $\partial\Omega$. Let $(\tilde{\mathbf{x}}, \tilde{t})$ be a point at which Φ takes its maximum value with $\tilde{\mathbf{x}} \in \partial\Omega$. Friedman's boundary lemma [3,10] implies that $\Phi \equiv \text{const.}$ in $\Omega \times [0, \tilde{t}]$, so that we must actually have $\partial\Phi/\partial n = 0$ on $\partial\Omega$. Since we have $|\nabla u|^2 > 0$ on $\partial\Omega$, we conclude then that the average curvature K vanishes identically on $\partial\Omega$, which is clearly impossible. This achieves the proof of Theorem 5.

Now we want to select $\alpha \geq 0$ in such a way that the first possibility (i) in (106) cannot occur. To this end we proceed as in Section 3.1. In the particular case of $k(t) \equiv 1$, this leads to the following result.

THEOREM 6 *Let Ω be a bounded convex domain in $\mathbb{R}^N$ whose boundary is $C^{2+\varepsilon}$. Let d be the radius of the greatest ball contained in Ω. Let $u(\mathbf{x}, t)$ be the solution of the parabolic problem* (94)–(96) *with $k(t) \equiv 1$. Assume that the hypotheses of Theorem* 5 *are satisfied. We then conclude that if*

$$0 \leq \alpha < \alpha_0 := \frac{\pi^2 g_{\min}}{4d^2}, \tag{127}$$

the first possibility (i) *in* (106) *cannot occur. With $\alpha \longrightarrow \alpha_0$ we are then led to the following decay bound for Φ:*

$$G(|\nabla u|^2) + \alpha_0 u^2 \leq H^2 e^{-2\alpha_0\beta t}, \tag{128}$$

with

$$H^2 := \max_{\Omega}\{G(|\nabla h|^2) + \alpha_0 h^2\}. \tag{129}$$

We note that in the context of Theorem 6, the quantity

$$\psi := |\nabla u|^2 \tag{130}$$

satisfies the parabolic inequality

$$g\Delta\psi - \psi_{,t} + \psi^{-1}\nabla\psi \cdot \tilde{\mathbf{c}} \geq 0, \tag{131}$$

where the vector field $\tilde{\mathbf{c}}$ is regular throughout $\Omega \times (0, \infty)$. Moreover we have

$$\frac{\partial\psi}{\partial n} = -2(N-1)Ku_n^2 \leq 0 \quad \text{on } \partial\Omega. \tag{132}$$

It then follows from (131) and (132) that ψ takes its maximum value initially. This shows that if $g' \leq 0$, we have

$$g_{\min} = g(\psi_{\max}), \tag{133}$$

with $\psi_{\max} = \max_{\Omega}|\nabla h|^2$.

As a first example consider $g(\sigma) := (1+\sigma)^{1/2}$. Since $g'(\sigma) = \frac{1}{2}(1+\sigma)^{-1/2} \geq 0$, we have to determine the (greatest) $\beta \in (0,1)$ such that (100) is satisfied, i.e. such that $A(\lambda, N, \beta) \leq 2$, where A is defined in (101). This condition is satisfied only for $N \leq 4$. We are then led to $\beta = 2 - \sqrt{N} > 0$ if $N = 2$ or $N = 3$.

As a second example, consider $g(\sigma) := (1+\sigma)^{-\varepsilon}$, $0 \leq \varepsilon \leq E := \min\{\frac{1}{2}, 1/(N-1)\}$. Since $g' = -\varepsilon(1+\sigma)^{-1-\varepsilon} \leq 0$, we have to determine the (greatest) $\beta \in (0,1)$ such that (103) and (104) are both satisfied. This will be the case for $\beta = 1 - \varepsilon$.

We refer to [9] for similar results involving solutions of the parabolic differential equation

$$(g(|\nabla u|^2)u_{,i})_{,i} = u_{,t}. \tag{134}$$

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