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Research Article

Sufficient Conditions for Univalence of an Integral Operator Defined by Al-Oboudi Differential Operator

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We investigate the univalence of an integral operator defined by Al-Oboudi differential operator.

1. Introduction

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$, and $\mathcal{S} := \{f \in \mathcal{A} : f \text{ is univalent in } \mathbb{U}\}$.

For $f \in \mathcal{A}$, Al-Oboudi [1] introduced the following operator:

$$D^0 f(z) = f(z), \tag{1.2}$$

$$D^1 f(z) = (1-\delta) f(z) + \delta z f'(z) = D_\delta f(z), \quad \delta \geq 0, \tag{1.3}$$

$$D^n f(z) = D_\delta\big(D^{n-1} f(z)\big), \quad \big(n \in \mathbb{N} := \{1,2,3,\ldots\}\big). \tag{1.4}$$

If f is given by (1.1), then from (1.3) and (1.4) we see that

$$D^n f(z) = z + \sum_{k=2}^{\infty} \big[1 + (k-1)\delta\big]^n a_k z^k, \quad \big(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}\big), \tag{1.5}$$

with $D^n f(0) = 0$.

When $\delta = 1$, we get Sălăgean's differential operator [2].

By using the Al-Oboudi differential operator, we introduce the following integral operator.

Definition 1.1. Let $n, m \in \mathbb{N}_0$ and $\alpha_i \in \mathbb{C}$, $1 \le i \le m$. We define the integral operator $I(f_1, \ldots, f_m) : \mathcal{A}^m \to \mathcal{A}$,

$$I(f_1, \ldots, f_m)(z) := \int_0^z \left(\frac{D^n f_1(t)}{t}\right)^{\alpha_1} \cdots \left(\frac{D^n f_m(t)}{t}\right)^{\alpha_m} dt \quad (z \in \mathbb{U}), \tag{1.6}$$

where $f_i \in \mathcal{A}$ and D^n is the Al-Oboudi differential operator.

Remark 1.2. (i) For $n = 0$, $m = 1$, $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = \cdots = \alpha_m = 0$, and $D^0 f_1(z) := D^0 f(z) = f(z) \in \mathcal{A}$, we have Alexander integral operator

$$I(f)(z) := \int_0^z \frac{f(t)}{t} dt \tag{1.7}$$

which was introduced in [3].

(ii) For $n = 0$, $m = 1$, $\alpha_1 = \alpha \in [0,1]$, $\alpha_2 = \alpha_3 = \cdots = \alpha_m = 0$, and $D^0 f_1(z) := D^0 f(z) = f(z) \in \mathcal{S}$, we have the integral operator

$$I_\alpha(f)(z) := \int_0^z \left(\frac{f(t)}{t}\right)^\alpha dt \tag{1.8}$$

that was studied in [4].

(iii) For $n = 0$, $m \in \mathbb{N}_0$, $\alpha_i \in \mathbb{C}$, $D^0 f_i(z) = f_i(z) \in \mathcal{S}$, $1 \le i \le m$, we have the integral operator

$$I(f_1, \ldots, f_m)(z) := \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdots \left(\frac{f_m(t)}{t}\right)^{\alpha_m} dt \tag{1.9}$$

which was studied in [5].

(iv) For $n = 0$, $m = 1$, $\alpha_1 = \gamma$, $\alpha_2 = \alpha_3 = \cdots = \alpha_m = 0$ and $D^0 f_1(z) := D^0 f(z) = f(z)$, we have the integral operator

$$I_\gamma(f)(z) := \int_0^z \left(\frac{f(t)}{t}\right)^\gamma dt \tag{1.10}$$

which was studied in [6, 7].

2. Main results

The following lemmas will be required in our investigation.

Lemma 2.1 (see [8]). *If the function f is regular in the unit disk $\mathbb{U}$, $f(z) = z + a_2 z^2 + \cdots$, and*

$$(1 - |z|^2)\left|\frac{z f''(z)}{f'(z)}\right| \le 1, \tag{2.1}$$

for all $z \in \mathbb{U}$, then the function f is univalent in $\mathbb{U}$.

Lemma 2.2 (Schwarz Lemma) (see [9, page 166]). *Let the analytic function $f(z)$ be regular in $\mathbb{U}$ and let $f(0) = 0$. If, in $\mathbb{U}$, $|f(z)| \le 1$, then*

$$\left|f(z)\right| \le |z|, \quad (z \in \mathbb{U}), \tag{2.2}$$

and $|f'(0)| \le 1$.

The equality holds if and only if $f(z) \equiv Kz$ and $|K| = 1$.

Theorem 2.3. *Let $n, m \in \mathbb{N}_0$, $\alpha_i \in \mathbb{C}$, and $f_i \in \mathcal{A}$, $1 \le i \le m$. If*

$$\left|\frac{z(D^n f_i(z))'}{D^n f_i(z)} - 1\right| \le 1, \qquad \left|\alpha_1\right| + \cdots + \left|\alpha_m\right| \le 1, \tag{2.3}$$

then $I(f_1, \ldots, f_m)(z)$ defined in Definition 1.1 is univalent in $\mathbb{U}$.

Proof. Since $f_i \in \mathcal{A}$, $1 \le i \le m$, by (1.5), we have

$$\frac{D^n f_i(z)}{z} = 1 + \sum_{k=2}^{\infty}\left[1 + (k-1)\delta\right]^n a_{k,i} z^{k-1} \quad \left(n \in \mathbb{N}_0\right), \qquad \frac{D^n f_i(z)}{z} \ne 0, \tag{2.4}$$

for all $z \in \mathbb{U}$.

On the other hand, we obtain

$$I'(f_1, \ldots, f_m)(z) = \left(\frac{D^n f_1(z)}{z}\right)^{\alpha_1} \cdots \left(\frac{D^n f_m(z)}{z}\right)^{\alpha_m}, \tag{2.5}$$

for $z \in \mathbb{U}$. This equality implies that

$$\ln I'(f_1, \ldots, f_m)(z) = \alpha_1 \ln \frac{D^n f_1(z)}{z} + \cdots + \alpha_m \ln \frac{D^n f_m(z)}{z} \tag{2.6}$$

or equivalently

$$\ln I'(f_1, \ldots, f_m)(z) = \alpha_1\left[\ln D^n f_1(z) - \ln z\right] + \cdots + \alpha_m\left[\ln D^n f_m(z) - \ln z\right]. \tag{2.7}$$

By differentiating the above equality, we get

$$\frac{I''(f_1, \ldots, f_m)(z)}{I'(f_1, \ldots, f_m)(z)} = \sum_{i=1}^{m} \alpha_i \left[\frac{(D^n f_i(z))'}{D^n f_i(z)} - \frac{1}{z}\right]. \tag{2.8}$$

After some calculus, we obtain

$$\left|\frac{zI''(f_1, \ldots, f_m)(z)}{I'(f_1, \ldots, f_m)(z)}\right| \le \sum_{i=1}^{m} \left|\alpha_i\right| \left|\frac{z(D^n f_i(z))'}{D^n f_i(z)} - 1\right|. \tag{2.9}$$

By hypothesis, since $|z(D^n f_i(z))'/D^n f_i(z)-1| \le 1$, $1 \le i \le m$ $(z \in \mathbb{U})$, and since $|\alpha_1|+\cdots+|\alpha_m| \le 1$ we have

$$\left|\frac{zI''(f_1,\ldots,f_m)(z)}{I'(f_1,\ldots,f_m)(z)}\right| \le \sum_{i=1}^{m}|\alpha_i| \le 1. \tag{2.10}$$

So, we obtain

$$(1-|z|^2)\left|\frac{zI''(f_1,\ldots,f_m)(z)}{I'(f_1,\ldots,f_m)(z)}\right| \le 1-|z|^2 \le 1. \tag{2.11}$$

Thus $I(f_1,\ldots,f_m)(z) \in \mathcal{S}$. □

Remark 2.4. For $n=0$, $D^0 f_i(z) = f_i(z) \in \mathcal{S}$, $1 \le i \le m$, we have [5, Theorem 1].

Corollary 2.5. *Let $n,m \in \mathbb{N}_0$, $\alpha_i > 0$, and $f_i \in \mathcal{A}$, $1 \le i \le m$. If*

$$\left|\frac{z(D^n f_i(z))'}{D^n f_i(z)} - 1\right| \le 1, \quad (z \in \mathbb{U}), \tag{2.12}$$

and $\alpha_1+\cdots+\alpha_m \le 1$, then $I(f_1,\ldots,f_m)(z) \in \mathcal{S}$.

Theorem 2.6. *Let $n,m \in \mathbb{N}_0$, $\alpha_i \in \mathbb{C}$, and $f_i \in \mathcal{A}$, $1 \le i \le m$. If*

(i) $|D^n f_i(z)| \le 1$,

(ii) $|z^2(D^n f_i(z))'/(D^n f_i(z))^2 - 1| \le 1$ $(z \in \mathbb{U})$, *and*

(iii) $|\alpha_1|+\cdots+|\alpha_m| \le 1/3$,

then $I(f_1,\ldots,f_m)(z)$ defined in Definition 1.1 is univalent in $\mathbb{U}$.

Proof. By (2.9), we get

$$(1-|z|^2)\left|\frac{zI''(f_1,\ldots,f_m)(z)}{I'(f_1,\ldots,f_m)(z)}\right| \le (1-|z|^2)\sum_{i=1}^{m}|\alpha_i|\left|\frac{z(D^n f_i(z))'}{D^n f_i(z)} - 1\right|. \tag{2.13}$$

This inequality implies that

$$\begin{aligned}(1-|z|^2)\left|\frac{zI''(f_1,\ldots,f_m)(z)}{I'(f_1,\ldots,f_m)(z)}\right| &\le (1-|z|^2)\sum_{i=1}^{m}\left[|\alpha_i|\left|\frac{z(D^n f_i(z))'}{D^n f_i(z)}\right| + |\alpha_i|\right] \\ &= (1-|z|^2)\sum_{i=1}^{m}\left[|\alpha_i|\left|\frac{z^2(D^n f_i(z))'}{(D^n f_i(z))^2}\right|\frac{|D^n f_i(z)|}{|z|} + |\alpha_i|\right].\end{aligned} \tag{2.14}$$

By Schwarz lemma (Lemma 2.2), we have

$$(1-|z|^2)\left|\frac{zI''(f_1,\ldots,f_m)(z)}{I'(f_1,\ldots,f_m)(z)}\right| \le (1-|z|^2)\sum_{i=1}^{m}\left[|\alpha_i|\left|\frac{z^2(D^n f_i(z))'}{(D^n f_i(z))^2}\right| + |\alpha_i|\right], \tag{2.15}$$

or

$$
\begin{aligned}
(1-|z|^2)\left|\frac{zI''(f_1,\ldots,f_m)(z)}{I'(f_1,\ldots,f_m)(z)}\right| &\le (1-|z|^2)\sum_{i=1}^{m}\left[|\alpha_i|\left|\frac{z^2(D^n f_i(z))'}{(D^n f_i(z))^2}-1\right|+2|\alpha_i|\right] \\
&\le (1-|z|^2)\sum_{i=1}^{m}\left[|\alpha_i|+2|\alpha_i|\right] \\
&= 3(1-|z|^2)\sum_{i=1}^{m}|\alpha_i| \\
&\le 1-|z|^2 \\
&\le 1,
\end{aligned}
\tag{2.16}
$$

for all $z \in \mathbb{U}$.

So, by Lemma 2.1, $I(f_1,\ldots,f_m)(z) \in \mathcal{S}$. □

Remark 2.7. For $n = 0$, $m = 1$, $\alpha_1 = \alpha \in \mathbb{C}$, $|\alpha| \le 1/3$, $\alpha_2 = \alpha_3 = \cdots = \alpha_m = 0$, we have [7, Theorem 1].

Corollary 2.8. *Let $n, m \in \mathbb{N}_0$, $\alpha_i > 0$, and $f_i \in \mathcal{A}$, $1 \le i \le m$. If*

(i) $|D^n f_i(z)| \le 1$,

(ii) $|z^2(D^n f_i(z))'/(D^n f_i(z))^2 - 1| \le 1$ $(z \in \mathbb{U})$, *and*

(iii) $\alpha_1 + \cdots + \alpha_m \le 1/3$,

then $I(f_1,\ldots,f_m)(z) \in \mathcal{S}$.

In [10], similar results are given by using the Ruscheweyh differential operator.

References

[1] F. M. Al-Oboudi, “On univalent functions defined by a generalized Sălăgean operator,” *International Journal of Mathematics and Mathematical Sciences*, vol. 2004, no. 27, pp. 1429–1436, 2004.

[2] G. Ş. Sălăgean, “Subclasses of univalent functions,” in *Complex Analysis—Fifth Romanian-Finnish Seminar, Part 1 (Bucharest, 1981)*, vol. 1013 of *Lecture Notes in Mathematics*, pp. 362–372, Springer, Berlin, Germany, 1983.

[3] J. W. Alexander, “Functions which map the interior of the unit circle upon simple regions,” *Annals of Mathematics*, vol. 17, no. 1, pp. 12–22, 1915.

[4] S. S. Miller, P. T. Mocanu, and M. O. Reade, “Starlike integral operators,” *Pacific Journal of Mathematics*, vol. 79, no. 1, pp. 157–168, 1978.

[5] D. Breaz and N. Breaz, “Two integral operators,” *Studia Universitatis Babeş-Bolyai, Mathematica*, vol. 47, no. 3, pp. 13–19, 2002.

[6] V. Pescar, “On some integral operations which preserve the univalence,” *The Punjab University. Journal of Mathematics*, vol. 30, pp. 1–10, 1997.

[7] V. Pescar and S. Owa, “Sufficient conditions for univalence of certain integral operators,” *Indian Journal of Mathematics*, vol. 42, no. 3, pp. 347–351, 2000.

[8] J. Becker, “Löwnersche Differentialgleichung und quasikonform fortsetzbare schlichte Funktionen,” *Journal für die Reine und Angewandte Mathematik*, vol. 255, pp. 23–43, 1972.

[9] Z. Nehari, *Conformal Mapping*, Dover, New York, NY, USA, 1975.

[10] G. I. Oros, G. Oros, and D. Breaz, “Sufficient conditions for univalence of an integral operator,” *Journal of Inequalities and Applications*, vol. 2008, Article ID 127645, 7 pages, 2008.