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Research Article

A Sharp Bound of the Čebyšev Functional for the Riemann-Stieltjes Integral and Applications

S. S. Dragomir

School of Computer Science and Mathematics, Victoria University, P.O. Box 14428, Melbourne, Victoria 8001, Australia

Correspondence should be addressed to S. S. Dragomir, sever.dragomir@vu.edu.au

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A new sharp bound of the Čebyšev functional for the Riemann-Stieltjes integral is obtained. Applications for quadrature rules including the trapezoid and midpoint rules are given.

1. Introduction

In order the generalise the classical *Čebyšev functional*, namely,

$$T(f,g) := \frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx \cdot \frac{1}{b-a}\int_a^b g(x)dx, \tag{1.1}$$

where f, g, and fg are integrable on $[a,b]$, which has been extensively studied in the literature (see, e.g., the book [1]), the author has introduced in [2] the following functional for Riemann-Stieltjes integrals:

$$T(f,g;u) := \frac{1}{u(b)-u(a)}\int_a^b f(t)g(t)du(t) - \frac{1}{u(b)-u(a)}\int_a^b f(t)du(t) \cdot \frac{1}{u(b)-u(a)}\int_a^b g(t)du(t), \tag{1.2}$$

provided that the involved integrals exist and $u(b) \neq u(a)$.

It has been shown in [2] that

$$|T(f,g;u)| \leq \frac{1}{2}(M-m) \cdot \frac{1}{|u(b)-u(a)|}\left\| g - \frac{1}{u(b)-u(a)}\int_a^b g(s)du(s)\right\|_\infty \bigvee_a^b(u), \tag{1.3}$$

provided that f and g are continuous, $m \le f(t) \le M$ for each $t \in [a,b]$, and u is of bounded variation on $[a,b]$ with the total variation $\vee_a^b(u)$. The constant $1/2$ is sharp in (1.3) in the sense that it cannot be replaced by a smaller quantity.

In the case that u is monotonic nondecreasing,

$$\left|T(f,g;u)\right| \le \frac{1}{2}(M-m)\cdot\frac{1}{\left|u(b)-u(a)\right|}\int_a^b\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)du(s)\right|du(t), \tag{1.4}$$

for which the constant $1/2$ is best possible [2].

Finally, in the case where u is Lipschitzian with the constant L, and in this case we can have f and g Riemann integrable on $[a,b]$, the following result has been obtained as well [2]:

$$\left|T(f,g;u)\right| \le \frac{1}{2}L(M-m)\cdot\frac{1}{\left|u(b)-u(a)\right|}\int_a^b\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)du(s)\right|dt. \tag{1.5}$$

Here $1/2$ is also sharp.

For other results, see [3, 4].

The aim of the present paper is to establish a new sharp bound for the absolute value of the Čebyšev functional (1.2). Applications for the trapezoid and midpoint inequality are pointed out. A general perturbed quadrature rule and error estimates are obtained as well.

2. The results

The following result concerning a sharp bound for the absolute value of the Čebyšev functional $T(f,g;h)$ can be stated.

Theorem 2.1. *Let $f : [a,b] \to \mathbb{R}$ be a function of bounded variation and let $g,h : [a,b] \to \mathbb{R}$ be bounded functions with $h(a) \ne h(b)$ such that the Stieltjes integrals $\int_a^b f(t)g(t)dh(t)$ and $\int_a^b g(t)dh(t)$ exist. Then*

$$\left|T(f,g;h)\right| \le \frac{1}{\left|h(b)-h(a)\right|}\bigvee_a^b(f)\sup_{x\in[a,b]}\left|\int_a^x g(t)dh(t)-\frac{h(x)-h(a)}{h(b)-h(a)}\int_a^b g(s)dh(s)\right|. \tag{2.1}$$

The constant $C = 1$ in the right-hand side of (2.1) cannot be replaced by a smaller quantity.

Proof. We use the following result for the Riemann-Stieltjes integral obtained in [1, page 337].

Let $u,v,w : [a,b] \to \mathbb{R}$ such that u is of bounded variation on $[a,b]$ and v,w are bounded functions with the property that the Riemann-Stieltjes integrals $\int_a^b v(t)dw(t)$ and $\int_a^b u(t)v(t)dw(t)$ exist. Then

$$\left|\int_a^b u(t)v(t)dw(t)\right| \le \left[\left|u(b)\right| + \bigvee_a^b(u)\right]\sup_{x\in[a,b]}\left|\int_a^x v(t)dw(t)\right|. \tag{2.2}$$

We also use the representation (see also [2])

$$T(f,g;h)=\frac{1}{h(b)-h(a)}\int_a^b [f(t)-\gamma]\left[g(t)-\frac{1}{h(b)-h(a)}\int_a^b g(s)dh(s)\right]dh(t), \tag{2.3}$$

which holds for any $\gamma \in \mathbb{R}$.

Now, if we choose $\gamma = f(b)$, $u(t) = f(t) - f(b)$,

$$v(t)=g(t)-\frac{1}{h(b)-h(a)}\int_a^b g(s)dh(s), \tag{2.4}$$

and $w(t) = h(t)$, $t \in [a,b]$, then we get

$$\left|[h(b)-h(a)]T(f,g;h)\right| \le \bigvee_a^b(f)\sup_{x\in[a,b]}\left|\int_a^x g(t)dh(t)-\frac{h(x)-h(a)}{h(b)-h(a)}\int_a^b g(s)dh(s)\right| \tag{2.5}$$

and inequality (2.1) is proved.

For the sharpness of the inequality, assume that $h(t) = t$ and $g(t) = \operatorname{sgn}(t-(a+b)/2)$, $t \in [a,b]$. Then (2.1) becomes

$$\left|\int_a^b f(t)\operatorname{sgn}\left(t-\frac{a+b}{2}\right)dt\right| \le \bigvee_a^b(f)\sup_{x\in[a,b]}\left|\int_a^x \operatorname{sgn}\left(t-\frac{a+b}{2}\right)dt\right|, \tag{2.6}$$

provided that f is of bounded variation on $[a,b]$.

Notice that, if we consider $\lambda(x)$ defined by

$$\lambda(x) := \int_a^x \operatorname{sgn}\left(t-\frac{a+b}{2}\right)dt = \begin{cases} a-x, & \text{if } x\in\left[a,\dfrac{a+b}{2}\right], \\ x-b, & \text{if } x\in\left(\dfrac{a+b}{2},b\right], \end{cases} \tag{2.7}$$

then

$$\sup_{x\in[a,b]}|\lambda(x)| = \frac{b-a}{2}. \tag{2.8}$$

Therefore, (2.6) becomes

$$\left|\int_a^b f(t)\operatorname{sgn}\left(t-\frac{a+b}{2}\right)dt\right| \le \frac{b-a}{2}\cdot\bigvee_a^b(f). \tag{2.9}$$

Now, if in (2.9) we choose $f(t) = \operatorname{sgn}(t-(a+b)/2)$, then $\vee_a^b(f) = 2$,

$$\int_a^b f(t)\operatorname{sgn}\left(t-\frac{a+b}{2}\right)dt = b-a, \tag{2.10}$$

and in both sides of (2.9) we get the same quantity $(b-a)$. $\square$

Remark 2.2. We observe that

$$\begin{aligned}
&\int_a^x g(t)dh(t) - \frac{h(x)-h(a)}{h(b)-h(a)}\int_a^b g(s)dh(s) \\
&\quad = \int_a^x g(t)dh(t) - \frac{h(x)-h(a)}{h(b)-h(a)}\left[\int_a^x g(s)dh(s) + \int_x^b g(s)dh(s)\right] \\
&\quad = \frac{h(b)-h(x)}{h(b)-h(a)}\cdot\int_a^x g(s)dh(s) - \frac{h(x)-h(a)}{h(b)-h(a)}\cdot\int_x^b g(s)dh(s) \\
&\quad = \frac{[h(b)-h(x)][h(x)-h(a)]}{h(b)-h(a)}\Delta(g,h;x,a,b),
\end{aligned} \tag{2.11}$$

where $\Delta(g,h;x,a,b)$ is defined by

$$\Delta(g,h;x,a,b) = \frac{1}{h(x)-h(a)}\int_a^x g(s)dh(s) - \frac{1}{h(b)-h(x)}\int_x^b g(s)dh(s), \tag{2.12}$$

provided that $h(x) \neq h(a),\ h(b)$ for $x \in (a,b)$.

With this notation, inequality (2.1) becomes

$$\begin{aligned}
|T(f,g;h)| &\leq \frac{1}{|h(b)-h(a)|}\bigvee_a^b(f)\sup_{x\in[a,b]}\left\{\left|\frac{[h(b)-h(x)][h(x)-h(a)]}{h(b)-h(a)}\right|\cdot|\Delta(g,h;x,a,b)|\right\} \\
&\leq \frac{1}{|h(b)-h(a)|}\bigvee_a^b(f)\sup_{x\in[a,b]}\left|\frac{[h(b)-h(x)][h(x)-h(a)]}{h(b)-h(a)}\right|\sup_{x\in[a,b]}|\Delta(g,h;x,a,b)|.
\end{aligned} \tag{2.13}$$

Now, if we assume that $h(a) < h(x) < h(b)$ for any $x \in (a,b)$, then on utilising the elementary inequality $\alpha\beta \leq (1/4)(\alpha+\beta)^2$, $\alpha,\beta \in [0,\infty)$, we have

$$[h(b)-h(x)][h(x)-h(a)] \leq \frac{1}{4}[h(b)-h(a)]^2, \tag{2.14}$$

and from (2.9), we deduce the following simpler inequality:

$$|T(f,g;h)| \leq \frac{1}{4}\cdot\bigvee_a^b(f)\sup_{x\in[a,b]}|\Delta(g,h;x,a,b)|. \tag{2.15}$$

The constant $1/4$ is best possible in (2.15).

A sufficient condition for h such that $h(a) < h(x) < h(b)$ for any $x \in (a,b)$ is that h is strictly increasing on $[a,b]$. The sharpness of the constant will follow from a particular case considered in Corollary 2.5 below.

Corollary 2.3. *Let $f, g, w : [a,b] \to \mathbb{R}$ be such that f is of bounded variation and the Riemann integrals $\int_a^b f(t)w(t)dt$, $\int_a^b g(t)w(t)dt$, $\int_a^b f(t)g(t)w(t)dt$, and $\int_a^b w(t)dt$ exist and $\int_a^b w(t)dt \neq 0$. Then, one has the inequality*

$$\begin{aligned}&\left|\frac{1}{\int_a^b w(t)dt}\int_a^b f(t)g(t)w(t)dt - \frac{1}{\int_a^b w(t)dt}\int_a^b f(t)w(t)dt \cdot \frac{1}{\int_a^b w(t)dt}\int_a^b g(t)w(t)dt\right| \\ &\quad \leq \frac{1}{\left|\int_a^b w(t)dt\right|}\bigvee_a^b(f)\sup_{x\in[a,b]}\left|\int_a^x g(t)w(t)dt - \frac{\int_a^x w(t)dt}{\int_a^b w(t)dt}\int_a^b g(t)w(t)dt\right|.\end{aligned} \tag{2.16}$$

The inequality is sharp.

The proof follows by Theorem 2.1 on choosing $h(x) = \int_a^x w(s)ds$.

Remark 2.4. In particular, if $w(s) > 0$ for $s \in [a,b]$, then $h(x) = \int_a^x w(s)ds$ is strictly decreasing on $[a,b]$ and by (2.15) we deduce the inequality

$$\begin{aligned}&\left|\frac{1}{\int_a^b w(t)dt}\int_a^b f(t)g(t)w(t)dt - \frac{1}{\int_a^b w(t)dt}\int_a^b f(t)w(t)dt \cdot \frac{1}{\int_a^b w(t)dt}\int_a^b g(t)w(t)dt\right| \\ &\quad \leq \frac{1}{\int_a^b w(s)ds}\bigvee_a^b(f)\sup_{x\in[a,b]}\left|\int_a^x g(s)w(s)ds - \frac{\int_a^x w(s)ds}{\int_a^b w(s)ds}\int_a^b g(s)w(s)ds\right| \\ &\quad \leq \frac{1}{4}\cdot\bigvee_a^b(f)\sup_{x\in[a,b]}\left|\frac{1}{\int_a^x w(s)ds}\int_a^x g(s)w(s)ds - \frac{1}{\int_x^b w(s)ds}\int_a^b g(s)w(s)ds\right|.\end{aligned} \tag{2.17}$$

The constant $1/4$ is best possible.

Corollary 2.5. *Let $f, g : [a,b] \to \mathbb{R}$ be such that f is of bounded variation and the Riemann integrals $\int_a^b g(t)dt$ and $\int_a^b f(t)g(t)dt$ exist. Then*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)g(t)dt - \frac{1}{b-a}\int_a^b f(t)dt \cdot \frac{1}{b-a}\int_a^b g(t)dt\right| \\ &\quad \leq \frac{1}{b-a}\bigvee_a^b(f)\sup_{x\in[a,b]}\left|\int_a^x g(t)dt - \frac{x-a}{b-a}\int_a^b g(t)dt\right| \\ &\quad \leq \frac{1}{4}\cdot\bigvee_a^b(f)\sup_{x\in(a,b)}\left|\frac{1}{x-a}\int_a^x g(s)ds - \frac{1}{b-x}\int_x^b g(s)ds\right|.\end{aligned} \tag{2.18}$$

The constant 1/4 is best possible in (2.18).

Proof. For the sharpness of the constant, consider $g(t) = \operatorname{sgn}(t - (a+b)/2)$, $t \in [a,b]$. If we denote

$$\mu(x) := \frac{1}{x-a}\int_a^x \operatorname{sgn}\left(t - \frac{a+b}{2}\right)dt - \frac{1}{b-x}\int_x^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right)dt, \quad x \in (a,b), \tag{2.19}$$

then

$$\begin{aligned}\mu(x) &= \frac{1}{x-a}\int_a^x \operatorname{sgn}\left(t - \frac{a+b}{2}\right)dt - \frac{1}{b-x}\left(\int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right)dt - \int_a^x \operatorname{sgn}\left(t - \frac{a+b}{2}\right)dt\right)\\ &= \frac{b-a}{(x-a)(b-x)}\int_a^x \operatorname{sgn}\left(t - \frac{a+b}{2}\right)dt = \frac{b-a}{(x-a)(b-x)}\cdot \lambda(x),\end{aligned} \tag{2.20}$$

where λ has been defined in the proof of Theorem 2.1.

Therefore,

$$\sup_{x\in[a,b]} \left|\mu(x)\right| = (b-a)\sup_{x\in[a,b]} \delta(x), \tag{2.21}$$

where

$$\delta(x) = \begin{cases} \dfrac{1}{b-x}, & \text{if } x \in \left[a, \dfrac{a+b}{2}\right), \\ \dfrac{1}{x-a}, & \text{if } x \in \left(\dfrac{a+b}{2}, b\right]. \end{cases} \tag{2.22}$$

Since $\sup_{x\in[a,b]}\delta(x) = 2$, inequality (2.18) becomes, for g given above,

$$\left|\int_a^b f(t)\operatorname{sgn}\left(t - \frac{a+b}{2}\right)dt\right| \le \frac{1}{2}\bigvee_a^b(f), \tag{2.23}$$

for any function f of bounded variation on $[a,b]$.

If in this inequality we choose $f(t) = \operatorname{sgn}(t - (a+b)/2)$, then we obtain in both sides of (2.23) the same quantity $(b-a)$. □

3. Applications for the trapezoid rule

The following result concerning the error estimate for the trapezoid rule can be stated as follows.

Proposition 3.1. *Assume that $f : [a,b] \to \mathbb{R}$ is absolutely continuous and has the derivative $f' : [a,b] \to \mathbb{R}$ of bounded variation on $[a,b]$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{f(a)+f(b)}{2}\right| \le \frac{1}{8}(b-a)\bigvee_a^b(f'). \tag{3.1}$$

The constant 1/8 is best possible.

Proof. We use the identity (see, e.g., [5])

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt=\frac{1}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)f'(t)dt. \tag{3.2}$$

If we apply inequality (2.18), then we can write that

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f'(t)\left(t-\frac{a+b}{2}\right)dt-\frac{1}{b-a}\int_a^b f'(t)dt\cdot\frac{1}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)dt\right|\\&\quad\le\frac{1}{b-a}\bigvee_a^b(f')\sup_{x\in[a,b]}\left|\int_a^x\left(t-\frac{a+b}{2}\right)dt-\frac{x-a}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)dt\right|.\end{aligned} \tag{3.3}$$

Since

$$\begin{aligned}&\int_a^b\left(t-\frac{a+b}{2}\right)dt=0,\qquad\int_a^x\left(t-\frac{a+b}{2}\right)dt=\frac{1}{2}\left[\left(x-\frac{a+b}{2}\right)^2-\left(\frac{b-a}{2}\right)^2\right],\\&\sup_{x\in[a,b]}\left|\int_a^x\left(t-\frac{a+b}{2}\right)dt\right|=\frac{1}{2}\sup_{x\in[a,b]}\left|\left(x-\frac{a+b}{2}\right)^2-\left(\frac{b-a}{2}\right)^2\right|=\frac{(b-a)^2}{8},\end{aligned} \tag{3.4}$$

hence, by (3.2) and (3.3), we deduce (3.1).

For the sharpness of the constant we choose $f(t)=|t-(a+b)/2|$. For this function, we have

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(t)dt&=\frac{1}{b-a}\int_a^b\left|t-\frac{a+b}{2}\right|dt=\frac{b-a}{4},\\\frac{f(a)+f(b)}{2}&=\frac{b-a}{2},\\f'(t)&=\begin{cases}-1, & \text{if } x\in\left[a,\dfrac{a+b}{2}\right),\\ 1, & \text{if } x\in\left(\dfrac{a+b}{2},b\right],\end{cases}\end{aligned} \tag{3.5}$$

and $\vee_a^b(f')=2$.

If we replace the above quantities in (3.1), we get the same result $(b-a)/4$ in both sides. □

The following result can be stated as well.

Proposition 3.2. *If* $f:[a,b]\to\mathbb{R}$ *is absolutely continuous on* $[a,b]$, *then*

$$\begin{aligned}\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{f(a)+f(b)}{2}\right|&\le\sup_{x\in[a,b]}\left|f(x)-f(a)-(x-a)\cdot\frac{f(b)-f(a)}{b-a}\right|\\&\le\frac{1}{4}(b-a)\cdot\sup_{x\in(a,b)}\left|\frac{f(x)-f(a)}{x-a}-\frac{f(b)-f(x)}{b-x}\right|.\end{aligned} \tag{3.6}$$

Proof. Applying inequality (2.18), we can also write that

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f'(t)\left(t-\frac{a+b}{2}\right)dt-\frac{1}{b-a}\int_a^b f'(t)dt\cdot\frac{1}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)dt\right|\\&\quad\le\frac{1}{b-a}\bigvee_a^b\left(\cdot-\frac{a+b}{2}\right)\cdot\sup_{x\in[a,b]}\left|\int_a^x f'(t)dt-\frac{x-a}{b-a}\int_a^b f'(t)dt\right|\\&\quad\le\frac{1}{4}\bigvee_a^b\left(\cdot-\frac{a+b}{2}\right)\cdot\sup_{x\in[a,b]}\left|\frac{\int_a^x f'(t)dt}{x-a}-\frac{\int_x^b f'(t)dt}{b-x}\right|,\end{aligned}\tag{3.7}$$

which, together with the identity (3.2), produces the desired inequality (3.6). □

For other results on the trapezoid rule, see [5].

4. Applications for the midpoint rule

The following result concerning the error estimates for the midpoint rule can be stated.

Proposition 4.1. *Assume that $f:[a,b]\to\mathbb{R}$ is absolutely continuous and has the derivative $f':[a,b]\to\mathbb{R}$ of bounded variation on $[a,b]$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt-f\left(\frac{a+b}{2}\right)\right|\le\frac{1}{8}(b-a)\bigvee_a^b(f').\tag{4.1}$$

The constant 1/8 is best possible.

Proof. We use the identity (see, e.g., [6])

$$f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt=\frac{1}{b-a}\int_a^b p(t)f'(t)dt,\tag{4.2}$$

where $p:[a,b]\to\mathbb{R}$ is given by

$$p(t)=\begin{cases}t-a, & \text{if } t\in\left[a,\dfrac{a+b}{2}\right],\\ t-b, & \text{if } t\in\left(\dfrac{a+b}{2},b\right].\end{cases}\tag{4.3}$$

If we apply inequality (2.18), we can write that

$$\left|\frac{1}{b-a}\int_a^b f'(t)p(t)dt-\frac{1}{b-a}\int_a^b f'(t)dt\cdot\frac{1}{b-a}\int_a^b p(t)dt\right|\le\frac{1}{b-a}\bigvee_a^b(f')\sup_{x\in[a,b]}\left|\int_a^x p(t)dt-\frac{x-a}{b-a}\int_a^b p(t)dt\right|.\tag{4.4}$$

We notice that

$$\int_a^b p(t)dt = 0,$$

$$\begin{aligned}
\delta(x) &:= \int_a^x p(t)dt \\
&= \begin{cases} \displaystyle\int_a^x (t-a)dt, & \text{if } t \in \left[a, \dfrac{a+b}{2}\right], \\ \displaystyle\int_a^{(a+b)/2} (t-a)dt + \int_{(a+b)/2}^x (t-b)dt, & \text{if } t \in \left(\dfrac{a+b}{2}, b\right], \end{cases} \\
&= \begin{cases} \dfrac{1}{2}(x-a)^2, & \text{if } t \in \left[a, \dfrac{a+b}{2}\right], \\ \dfrac{1}{2}(b-x)^2, & \text{if } t \in \left(\dfrac{a+b}{2}, b\right], \end{cases}
\end{aligned} \tag{4.5}$$

for $x \in [a,b]$.

Since

$$\sup_{x\in[a,b]} |\delta(x)| = \frac{1}{8}(b-a)^2, \tag{4.6}$$

then by (4.2) and (4.4), we deduce (4.1).

For the sharpness of the constant $1/8$, observe that for the absolutely continuous function $f(t) = |t-(a+b)/2|$, we get in both sides of (4.1) the same quantity $(b-a)/4$. □

The following result can be stated as well.

Proposition 4.2. *If $f : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$, then*

$$\begin{aligned}
\left| \frac{1}{b-a} \int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) \right| &\le \sup_{x\in[a,b]} \left| f(x) - f(a) - (x-a) \cdot \frac{f(b)-f(a)}{b-a} \right| \\
&\le \frac{1}{4}(b-a) \cdot \sup_{x\in(a,b)} \left| \frac{f(x)-f(a)}{x-a} - \frac{f(b)-f(x)}{b-x} \right|.
\end{aligned} \tag{4.7}$$

Proof. Applying inequality (2.18), we can write

$$\begin{aligned}
&\left| \frac{1}{b-a} \int_a^b p(t)f'(t)dt - \frac{1}{b-a} \int_a^b p(t)dt \cdot \frac{1}{b-a} \int_a^b f'(t)dt \right| \\
&\quad \le \frac{1}{b-a} \bigvee_a^b (p) \sup_{x\in[a,b]} \left| \int_a^x f'(t)dt - \frac{x-a}{b-a} \int_a^b f'(t)dt \right| \le \frac{1}{4} \bigvee_a^b (p) \sup_{x\in[a,b]} \left| \frac{\int_a^x f'(t)dt}{x-a} - \frac{\int_x^b f'(t)dt}{b-x} \right|,
\end{aligned} \tag{4.8}$$

and since $\vee_a^b(p) = b-a$, we deduce from (4.8) the desired inequality (4.7). □

For other results on the midpoint rule and their applications, see [6–8].

5. Applications for general quadrature rules

Let $h : [a,b] \to \mathbb{R}$ be a Riemann integrable function. Suppose that h is n-time differentiable and that there exists the division $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ and the weights $\alpha_0, \dots, \alpha_n$ such that

$$\int_a^b h(t)dt = \sum_{i=0}^{n} \alpha_i h(x_i) + \int_a^b K_n(t) h^{(n)}(t)dt, \tag{5.1}$$

where $K_n : [a,b] \to \mathbb{R}$ is the *Peano kernel* associated with the quadrature rule $A(h) := \sum_{i=0}^{n} \alpha_i h(x_i)$.

Utilising inequality (2.18), we can produce a "perturbed quadrature rule" by approximating the error terms $\int_a^b K_n(t) h^{(n)}(t)dt$ as follows.

Proposition 5.1. *With the above assumptions and if $h^{(n)}$ is of bounded variation, then*

$$\int_a^b h(t)dt = \sum_{i=0}^{n} \alpha_i h(x_i) + \frac{h^{(n-1)}(b) - h^{(n-1)}(a)}{b-a} \cdot \int_a^b K_n(t)dt + E_n(h) \tag{5.2}$$

and the error term $E_n(h)$ satisfies the bound

$$\begin{aligned} \left|E_n(h)\right| &\le \bigvee_a^b \left(h^{(n)}\right) \sup_{x\in[a,b]} \left| \int_a^x K_n(t)dt - \frac{x-a}{b-a} \int_a^b K_n(t)dt \right| \\ &\le \frac{1}{4} \cdot (b-a) \bigvee_a^b \left(h^{(n)}\right) \sup_{x\in(a,b)} \left| \frac{\int_a^x K_n(t)dt}{x-a} - \frac{\int_x^b K_n(t)dt}{b-x} \right|. \end{aligned} \tag{5.3}$$

The proof is obvious by (2.9) on choosing $f = h^{(n)}$ and $g = K_n$.

The second natural possibility is incorporated in the following proposition.

Proposition 5.2. *With the above assumption and if K_n is of bounded variation on $[a,b]$, then the representation* (5.2) *holds and the error term $E_n(h)$ satisfies the bounds*

$$\begin{aligned} \left|E_n(h)\right| &\le \bigvee_a^b (K_n) \sup_{x\in[a,b]} \left| h^{(n-1)}(x) - h^{(n-1)}(a) - (x-a) \cdot \frac{h^{(n-1)}(b) - h^{(n-1)}(a)}{b-a} \right| \\ &\le \frac{1}{4} \cdot \bigvee_a^b (K_n) \sup_{x\in(a,b)} \left| \frac{h^{(n-1)}(x) - h^{(n-1)}(a)}{x-a} - \frac{h^{(n-1)}(b) - h^{(n-1)}(x)}{b-x} \right|. \end{aligned} \tag{5.4}$$

The proof follows by inequality (2.18) on choosing $f = K_n$ and $g = h^{(n)}$.

Remark 5.3. As noted in the previous section, in practical applications and for a large number of quadrature rules, the Peano kernel K_n is available and the involved quantities in the error estimates (5.3) and (5.4) can be completely specified. In some cases, the new perturbed rules provide a better approximation than the original one. The details are left to the interested reader.

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