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Research Article

The Nonzero Solutions and Multiple Solutions for a Class of Bilinear Variational Inequalities

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Some existence theorems of nonzero solutions and multiple solutions for a class of bilinear variational inequalities are studied in reflexive Banach spaces by fixed point index approach. The results presented in this paper improve and extend some known results in the literature.

1. Introduction and preliminaries

The fundamental theory of the variational inequalities, since it was founded in the 1960's, has made powerful progress and has played an important role in nonlinear analysis. It has been applied intensively to mechanics, partial differential equation problems with boundary conditions, control theory, game theory, optimization methods, nonlinear programming, and so forth (see [1]).

In 1987, Noor [2] studied Signorini problem in the framework of the following variational inequality:

$$a(u,u-v)+b(u,u)-b(u,v)\leq\langle g(u),u-v\rangle,\quad\forall v\in K,\tag{1.1}$$

and proved the existence of solutions of Signorini problem in Hilbert spaces.

In 1991, Zhang and Xiang [3] studied the existence of solutions of bilinear variational inequality (1.1) in reflexive Banach space X. As an application, they discussed the existence of solutions for Signorini problem.

Throughout this paper, we assume that X is a reflexive space, X^* is the dual space of X, $\langle\cdot,\cdot\rangle$ is the pair between X^* and X, K is a nonempty closed convex subset of X with

$\theta \in K$, and for $r > 0$, $K^r = \{x \in K;\ \|x\| < r\}$. Suppose that $a : X \times X \to \mathbb{R} = (-\infty, +\infty)$ is a coercive and bilinear continuous mapping, that is, there exist constants $\alpha, \beta > 0$ such that the following condition holds:

$$a(u,u) \geq \alpha \|u\|^2, \quad |a(u,v)| \leq \beta \|u\| \, \|v\|, \quad \forall u, v \in X \tag{A}$$

and $b : X \times X \to \mathbb{R}$ satisfies the following condition including (i), (ii), (iii), and (iv):

$$\begin{aligned} &\text{(i) } b \text{ is linear with respect to the first argument;} \\ &\text{(ii) } b \text{ is convex lower semicontinuous with respect to the second argument;} \\ &\text{(iii) there exists } \gamma \in (0,\alpha) \text{ such that } b(u,v) \leq \gamma \|u\| \, \|v\| \text{ for all } u, v \in X; \\ &\text{(iv) for all } u, v, w \in K,\ b(u,v) - b(u,w) \leq b(u, v - w). \end{aligned} \tag{B}$$

Obviously, the (iii) and (iv) of condition (B) imply that $b(u,\theta) = 0$.

Theorem 1.1 (see [3]). *Let $a : X \times X \to \mathbb{R}$ be a coercive and bilinear continuous mapping satisfying condition (A) and let $b : X \times X \to \mathbb{R}^+ = [0, +\infty)$ satisfy condition (B). If $g : K \to X^*$ is a semicontinuous mapping and antimonotone (i.e., $\langle g(u) - g(v), u - v \rangle \leq 0$, $\forall u, v \in K$), then there exists a unique solution of variational inequality (1.1) in K.*

On the other hand, the existence of nonzero solutions for variational inequalities is an important topic of variational inequality theory. Recently, several authors discussed the existence of nonzero solutions for variational inequalities in Hilbert or Banach spaces (see [4] and the references therein).

In this paper, we will study the existence of nonzero solutions and multiple solutions for the following class of bilinear variational inequalities in reflexive Banach spaces by fixed point index approach, which has been applied intensively to famous Signorini problem in mechanics (see [2, 3]).

Given a mapping $g : K \to X^*$ and a point $f \in X^*$, we consider the following problem (in short, VI (1.2)): find $u \in K \setminus \{\theta\}$ such that

$$a(u, u - v) + b(u,u) - b(u,v) \leq \langle g(u), u - v \rangle + \langle f, u - v \rangle, \quad \forall v \in K. \tag{1.2}$$

For $x \in K \setminus \{\theta\}$, if $g(\theta) = 0$, $b(u,K) \subset \mathbb{R}^+$ for any $u \in K$ and $\langle f, x \rangle < 0$, then θ is a solution of VI (1.2). Hence, a natural problem can be raised: do the nonzero solutions of VI (1.2) exist? Do any other solutions of VI (1.2) exist in K?

By Theorem 1.1, for each $p \in X^*$, the variational inequality

$$a(u, u - v) + b(u,u) - b(u,v) \leq \langle p, u - v \rangle + \langle f, u - v \rangle, \quad \forall v \in K \tag{1.3}$$

has a unique solution u in K. Thus, we may define mappings $K_a : X^* \to K$ and $K_a g : K \to K$, respectively, as follows:

$$K_a(p) = u, \quad (K_a g)(u) = K_a(g(u)). \tag{1.4}$$

Clearly, the nonzero fixed point u of $K_a g$ is a nonzero solution of VI (1.2).

Lemma 1.2. *The mapping $K_a : X^* \to K$ has the following property:*

$$\|K_a(p) - K_a(q)\| \le \frac{1}{\alpha - \gamma}\|p - q\|, \quad \forall p,q \in X^*. \tag{1.5}$$

Consequently, K_a is $1/(\alpha-\gamma)$ set-contractive (see [5]).

Proof. Let $u_1 = K_a(p)$, $u_2 = K_a(q)$. Then for each $v \in K$,

$$a(u_1,u_1-v) + b(u_1,u_1) - b(u_1,v) \le \langle u_1,u_1-v\rangle + \langle f,u_1-v\rangle, \tag{1.6}$$

$$a(u_2,u_2-v) + b(u_2,u_2) - b(u_2,v) \le \langle u_2,u_2-v\rangle + \langle f,u_2-v\rangle. \tag{1.7}$$

Setting $v = u_2$ in (1.6) and $v = u_1$ in (1.7), respectively, we have

$$\begin{aligned} a(u_1,u_1-u_2) + b(u_1,u_1) - b(u_1,u_2) &\le \langle g(u_1),u_1-u_2\rangle + \langle f,u_1-u_2\rangle, \\ a(u_2,u_2-u_1) + b(u_2,u_2) - b(u_2,u_1) &\le \langle g(u_2),u_2-u_1\rangle + \langle f,u_2-u_1\rangle. \end{aligned} \tag{1.8}$$

Adding (1.8), it follows from condition (iv) of mapping b that

$$a(u_1-u_2,u_1-u_2) \le \langle p-q,u_1-u_2\rangle + b(u_1-u_2,u_2-u_1). \tag{1.9}$$

Thus,

$$\alpha\|u_1-u_2\|^2 \le \|p-q\|\|u_1-u_2\| + \gamma\|u_1-u_2\|^2. \tag{1.10}$$

This implies that (1.5) is true. This completes the proof. □

To state our theorems, we recall the following notes and the well-known conclusions. Let X be a normed linear space and let D be a subset of X. A continuous bounded mapping $T : D \to X$ is said to be k-set-contractive on D if there exists a constant number $k > 0$ such that $\alpha(T(A)) \le k\alpha(A)$ holds for each bounded subset A of D, where $\alpha(\cdot)$ is Kuratowski measure of noncompactness. T is said to be strictly set-contractive if $k < 1$. T is said to be condensing on D if $\alpha(T(A)) < \alpha(A)$ holds for each bounded subset A of D with $\alpha(A) \ne 0$.

Let X be a Banach space, let K be a nonempty closed convex subset of X, and let U be an open bounded subset of X with $U \cap K \ne \emptyset$. The closure and boundary of U relative to K are denoted by $\overline{U}_K$ and ∂U_K, respectively. Assume that $T : \overline{U}_K \to K$ is a strictly set-contractive mapping and $x \ne T(x)$ for each $x \in \partial U_K$. It is well known that the fixed point index $i_K(T,U)$ is well defined and $i_K(T,U)$ has the following properties (see [5, 6]).

(i) If $i_K(T,U) \ne 0$, then T has a fixed point in U_K;

(ii) For mapping $\hat{x}_0$ with constant value, if $x_0 \in U_K$, then $i_K(\hat{x}_0,U) = 1$;

(iii) Let U_1, U_2 be two open and bounded subsets of X with $U_1 \cap U_2 = \emptyset$, if $x \ne T(x)$ for $x \in \partial U_{1K} \cup \partial U_{2K}$, then $i_K(T,U_1 \cup U_2) = i_K(T,U_1) + i_K(T,U_2)$;

(iv) Let $H : [0,1] \times \overline{U}_K \to K$ be a continuous bounded mapping and for each $t \in [0,1]$, $H(t,\cdot)$ be a k-set-contractive mapping. Suppose that $H(t,x)$ is uniformly continuous with respect to t for all $x \in U$ and for all $(t,x) \in [0,1] \times \partial U_K$, $x \ne H(t,x)$. Then $i_K(H(1,\cdot),U) = i_K(H(0,\cdot),U)$.

Since K_a is a $1/(\alpha-\gamma)$-set contraction, if g is a k-set contraction, where $k<\alpha-\gamma$, then $K_a g$ is a strictly set contraction. If $K_a g$ has not fixed point in ∂U_K, then the fixed point index $i_K(K_a g, U)$ of $K_a g$ in U is well defined.

From Lemma 1.2 and the property (ii) of fixed point index, it is easy to see that $i_K(K_a g, U)=1$ for the constant mapping $g(u)\equiv p\in X^*$ and $K_a(p)\in U$.

2. The nonzero solutions for VI (1.2)

In this section, we discuss the nonzero solutions of VI (1.2). For convex subset K of X, the recession cone of K is defined by $\mathrm{rc}(K)=\{w\in X;\ w+u\in K,\ \forall u\in K\}$.

Theorem 2.1. *Let $f\in X^*$ be a linear continuous functional with $\langle f,z\rangle<0$ for all $z\in K$ and let $g:K\to X^*$ be a bounded continuous k-set-contractive mapping with $k<\alpha-\gamma$ satisfying the following conditions:*

(G_1) *there exists $h\in X^*$ such that $\|(g(u)/\|u\|)-h\|<\alpha$ for any $u\in K$ with $\|u\|$ small enough;*

(G_2) *there exist $u_0\in \mathrm{rc}(K)$ and $l>0$ such that $\langle g(u),u_0\rangle>(\beta+\gamma)\|u\|\|u_0\|$ for all $u\in K$ with $\|u\|>l$, where α,β are two constants which satisfy condition (A).*

Then, variational inequality (1.2) has nonzero solutions in K.

Proof. Define a mapping $K_a g:K\to K$ by $K_a g(u)=K_a(g(u))$ for all $u\in K$. It follows from Lemma 1.2 that $K_a g$ is continuous bounded strictly set-contractive. We will show that there exists $R_0>r_0>0$ such that $i_K(K_a g, K^{r_0})=1$ and $i_K(K_a g, K^{R_0})=0$.

First, for $r>0$, let $H:[0,1]\times \overline{K}^r\to K$ and $H(t,u)=K_a(tg(u))$. Obviously, $H(t,\cdot)$ is strictly set-contractive for fixed $t\in[0,1]$ and $H(t,u)$ is uniformly continuous with respect to t for all $u\in K^r$. We will claim that there exists $r_0>0$ small enough such that $u\neq H(t,u)$ for all $t\in[0,1]$ and $u\in\partial K^{r_0}$. Otherwise, for any natural number n, there exist $t_n\in[0,1]$, $u_n\in K$ satisfying $\|u_n\|=1/n$ such that $u_n=H(t_n,u_n)$, that is,

$$a(u_n,u_n-v)+b(u_n,u_n)-b(u_n,v)\le t_n\langle g(u_n),u_n-v\rangle+\langle f,u_n-v\rangle,\quad \forall v\in K. \tag{2.1}$$

Setting $v=\theta$ in (2.1) and $w_n=u_n/\|u_n\|=nu_n$, we know that

$$a(w_n,w_n)+n^2b(u_n,u_n)\le t_n\langle ng(u_n)-h,w_n\rangle+t_n\langle h,w_n\rangle+n\langle f,w_n\rangle \tag{2.2}$$

and so

$$\alpha\le \|ng(u_n)-h\|+t_n\langle h,w_n\rangle+n\langle f,w_n\rangle. \tag{2.3}$$

Since X is a reflexive Banach space, there exists a weakly convergent subsequence of $\{w_n\}$ in X. Without loss of generality, we may assume that $w_n\to w$ weakly. If $w=\theta$, by condition (G_1), the first term on the right-hand side in (2.3) is less than α for $\|u_n\|$ small enough; the second one tends to 0 and the third one is less than 0. This is a contradiction. If $w\neq\theta$, the second term on the right-hand side in (2.3) is bounded and the third term tends to negative infinity as $n \to \infty$, which is a contradiction. Thus,

$$i_K(K_a g, K^{r_o}) = i_K(H(1,\cdot), K^{r_0}) = i_K(H(0,\cdot), K^{r_0}) = 1. \tag{2.4}$$

Therefore, $K_a g$ has a fixed point $u_1 \in K^{r_0}$, and so u_1 is a solution of VI (1.2).

Next, we claim that there exists $R_0 > r_0$ large enough such that $i_K(K_a g, K^{R_0}) = 0$. Define a mapping $H : [0,1] \times K^r \to K$ as follows:

$$H(t,u) = K_a(g(u) - tNf), \tag{2.5}$$

where $N > 0$. Clearly, for any fixed $t \in [0,1]$, $H(t,\cdot)$ is strictly set-contractive and $H(t,u)$ is uniformly continuous with respect to t for all $u \in K^r$. We now show that there exists $R_0 > r_0$ such that $u \neq H(t,u)$ for all $t \in [0,1]$ and $u \in \partial K^{R_0}$. Otherwise, for any natural number n, there exist $t_n \in [0,1]$ and $u_n \in K^n$ such that $u_n = K_a(g(u_n) - t_n N f)$, that is,

$$a(u_n, u_n - v) + b(u_n, u_n) - b(u_n, v) \le \langle g(u_n), u_n - v\rangle + (1 - t_n N)\langle f, u_n - v\rangle \quad \forall v \in K. \tag{2.6}$$

Putting $v = u_n + u_0$ in (2.6) and $w_n = u_n / \|u_n\|$, we have

$$a(u_n, u_0) + b(u_n, u_n + u_0) - b(u_n, u_n) \ge \langle g(u_n), u_0\rangle + (1 - t_n N)\langle f, u_0\rangle. \tag{2.7}$$

Thus,

$$a(w_n, u_0) \ge \frac{\langle g(u_n), u_0\rangle}{||u_n||} + ||u_n||^{-1}(1 - t_n N)\langle f, u_0\rangle - b(w_n, u_0). \tag{2.8}$$

Since X is a reflexive Banach space, we assume that $w_n \to w$ weakly. Without loss of generality, we may assume that $\|w\| \le 1$. It follows that

$$a(w, u_0) \ge \limsup_{n\to\infty} \frac{\langle g(u_n), u_0\rangle}{||u_n||} - \gamma||u_0|| > \beta||u_0||. \tag{2.9}$$

But, we know that $a(w, u_0) \le \beta\|u_0\|$. This is a contradiction. It follows from property (iv) of fixed point index that

$$i_K(K_a g, K^{R_0}) = i_K(H(0,\cdot), K^{R_0}) = i_K(H(1,\cdot), K^{R_0}). \tag{2.10}$$

If $i_K(K_a g, K^{R_0}) \neq 0$, then there exists $u \in K^{R_0}$ such that $u = K_a(g(u) - Nf)$, that is,

$$a(u, u - v) + b(u,u) - b(u,v) \le \langle g(u), u - v\rangle + (1 - N)\langle f, u - v\rangle. \tag{2.11}$$

Let $v = u + u_0$ in (2.11). We obtain

$$a(u, u_0) + b(u, u + u_0) - b(u,u) \ge \langle g(u), u_0\rangle + (1 - N)\langle f, u_0\rangle, \tag{2.12}$$

and so

$$\beta\|u\|||u_0|| + \gamma\|u\|||u_0|| \ge \langle g(u), u_0\rangle + (1 - N)\langle f, u_0\rangle. \tag{2.13}$$

If $\|u\| \geq l$, then $(1-N)\langle f,u_0\rangle \leq 0$, which is a contradiction. If $\|u\| < l$, since g is bounded, there exists $C>0$ such that $\|g(u)\| \leq C$ for $\|u\| < l$. We take $N>0$ large enough such that

$$(1-N)\langle f,u_0\rangle > (l\beta + l\gamma + C)||u_0||. \tag{2.14}$$

On the other hand, (2.13) implies that

$$(1-N)\langle f,u_0\rangle \leq (l\beta + l\gamma + C)||u_0||, \tag{2.15}$$

which contradicts (2.14). Therefore, $i_K(K_a g, K^{R_0}) = 0$. It follows from property (iii) of fixed point index that $i_K(K_a g, K^{R_0} \setminus K^{r_0}) = -1$. Thus, $K_a g$ has a fixed point $u_2 \in K^{R_0} \setminus K^{r_0}$, which is a nonzero solution of VI (1.2). This completes the proof. □

Theorem 2.2. *Suppose that there exists $u_0 \in rcK \setminus \{\theta\}$ such that $\langle f,u_0\rangle > 0$ and $g: K \to X^*$ is a bounded continuous k-set contractive mapping with $k < \alpha - \gamma$ which satisfies condition (G_2) and the following condition:*

(G_3) $\exists C > 0$ *such that* $|\langle g(u)/\|u\|, u_0\rangle| \leq C$, *for each* $u \in K$ *with* $\|u\|$ *small enough.*

Then VI (1.2) has nonzero solutions in K.

Proof. For $r>0$, let $H:[0,1]\times \overline{K}^r \to K$ and $H(t,u) = K_a(tg(u))$. We claim that there exists $r_0 > 0$ small enough such that $u \neq H(t,u)$ for all $t \in [0,1]$ and $u \in \partial K^{r_0}$. Otherwise, for any natural number n, there exist $t_n \in [0,1]$ and $u_n \in K$ with $\|u_n\| = 1/n$ such that $u_n = H(t_n,u_n)$, that is,

$$a(u_n,u_n - v) + b(u_n,u_n) - b(u_n,v) \leq t_n\langle g(u_n), u_n - v\rangle + \langle f, u_n - v\rangle, \quad \forall v \in K. \tag{2.16}$$

Setting $v = u_n + u_0$ in (2.16), we obtain

$$a(u_n,u_0) + b(u_n,u_0) \geq a(u_n,u_0) + b(u_n,u_n + u_o) - b(u_n,u_n) \geq t_n\langle g(u_n),u_0\rangle + \langle f,u_0\rangle. \tag{2.17}$$

Therefore,

$$\beta + \gamma \geq t_n\left\langle \frac{g(u_n)}{||u_n||}, z_0\right\rangle + ||u_n||^{-1}\langle f,z_0\rangle, \tag{2.18}$$

where $z_0 = u_0/\|u_0\|$. The first term on the right-hand side in (2.18) is bounded by condition (G_3) and the second one tends to $+\infty$. This is a contradiction. Therefore,

$$i_K(K_a g, K^{r_0}) = i_K(H(1,\cdot),K^{r_0}) = i_K(H(0,\cdot),K^{r_0}) = 1. \tag{2.19}$$

For $r>0$, let $H:[0,1]\times\overline{K}^r \to K$ and $H(t,u) = K_a(g(u)+tNf)$. Similar to the second part of the proof of Theorem 2.1, there exists $R_0 > r_0$ such that $u \neq H(t,u)$ for any $t \in [0,1]$ and $u \in \partial K^{R_0}$ with $i_K(K_a g, K^{R_0}) = 0$. Thus, we have

$$i_K(K_a g, K^{R_0} \setminus K^{r_0}) = -1 \tag{2.20}$$

and so there exists $u_2 \in K^{R_0} \setminus R^{r_0}$, which is the nonzero fixed point of $K_a g$. This implies that it is also the nonzero solution of VI (1.2). This completes the proof. □

Remark 2.3. The fixed point index in Theorem 2.1 is based on the strictly set contraction mapping $K_a g$. When $K_a g$ is condensing mapping, the fixed point index $i_K(K_a g, U)$ is well defined. But it is necessary to require K as a star-shaped convex closed set (see [5]). Similarly, we may show the existence of nonzero solutions for VI (1.2) as $K_a g$ is condensing mapping.

3. Multiple solutions of VI (1.2)

In this section, we study the existence of multiple solutions of VI (1.2).

THEOREM 3.1. *Suppose that conditions of Theorem 2.1 are satisfied and $g : K \to X^*$ satisfies the following conditions:*

(G_4) $\limsup_{\|u\|\to\infty} \langle g(u), u_0\rangle/\|u\| = +\infty$.

(G_5) *There exists $h \in X^*$ such that $(\|g(u)/\|u\|) - h\|$ is bounded in $X \setminus K^n$.*

Then, there exist three solutions of VI (1.2), at least two of which are nonzero solutions.

Proof. We can prove that there exists $R_1 > R_0$ such that $i_K(K_a g, K^{R_1}) = 1$ (where R_0 is the same as in Theorem 2.1). In fact, setting $H : [0,1] \times \overline{K}^r \to K$ as follows:

$$H(t,u) = K_a(tg(u)), \tag{3.1}$$

then there exists $R_1 > R_0$ such that $u \neq H(t,u)$ ($\forall t \in [0,1]$ and $u \in \partial K^{R_1}$). Otherwise, there exist $t_n \in [0,1]$ and $u_n \in \partial K^n$ such that $u_n = H(t_n, u_n)$ for each natural number n, that is,

$$a(u_n, u_n - v) + b(u_n, u_n) - b(u_n, v) \le t_n \langle g(u_n), u_n - v\rangle + \langle f, u_n - v\rangle, \quad \forall v \in K. \tag{3.2}$$

Let $v = \theta$ in (3.2) and $w_n = u_n/\|u_n\|$. Then,

$$\alpha < a(w_n, w_n) + \frac{1}{\|u_n\|^2} b(u_n, u_n) \le t_n \left\langle \frac{g(u_n)}{||u_n||}, w_n \right\rangle + \frac{1}{||u_n||} \langle f, w_n\rangle, \tag{3.3}$$

and so

$$\alpha < t_n \left\| \frac{g(u_n)}{||u_n||} - h \right\| + t_n \langle h, w_n\rangle + \frac{1}{||u_n||} \langle f, w_n\rangle. \tag{3.4}$$

Letting $v = u + u_0$ in (3.2), we have

$$a(u_n, u_0) + b(u_n, u_0) \ge t_n \langle g(u_n), u_0\rangle + \langle f, u_0\rangle, \tag{3.5}$$

and so

$$(\beta + \gamma) \ge t_n \left\langle \frac{g(u_n)}{||u_n||}, z_0 \right\rangle + \frac{1}{||u_n||} \langle f, z_0\rangle, \tag{3.6}$$

where $z_0 = u_0/\|u_0\|$.

If there exists $\varepsilon_0 > 0$ such that $t_n \in [\varepsilon_0, 1]$, then the first term of the right-hand side in (3.6) tends to $+\infty$ and the second one tends to 0. This is a contradiction.

If there exist $0 \le t_{n_k} < (1/k)$ for $k = 1,2,\dots$, then

$$\alpha < t_{n_k}\left\|\frac{g(u_{n_k})}{\|u_{n_k}\|} - h\right\| + t_{n_k}\langle h, w_{n_k}\rangle + \frac{1}{\|u_{n_k}\|}\langle f, w_{n_k}\rangle. \tag{3.7}$$

Since X is a reflexive space, we may assume that $\{w_{n_k}\}$ weakly converges to some w without loss of generality. It is easy to see that every term of the right-hand side in (3.7) tends to 0, which is a contradiction. Therefore,

$$i_K(K_a g, K^{R_1}) = i_K(H(1,\cdot), K^{R_1}) = i_K(H(0,\cdot), K^{R_1}) = 1. \tag{3.8}$$

It follows from property (iii) of fixed point index that $i_K(K_a g, K^{R_1}\backslash K^{R_0}) = 1$. Thus, $K_a g$ has a fixed point $u_3 \in K^{R_1}\backslash K^{R_0}$, which is a nonzero solution of VI (1.2). From Theorem 2.1, we have three solutions of VI (1.2), at least u_2 and u_3 are nonzero solutions of VI (1.2). This completes the proof. □

Theorem 3.2. *Let all the conditions of Theorem 2.2 be satisfied and let $g : K \to X^*$ satisfy the conditions (G_4) and (G_5). Then, there exist three solutions of VI (1.2), at least two of which are nonzero solutions.*

The proof of Theorem 3.2 is similar to the proof of Theorem 3.1 and so we omit it.

4. An example about mapping g

In this section, we give a mapping g which satisfies all the conditions in the above theorems.

Let $\Omega \subset \mathbb{R}^n$ be a bounded subset with $\text{mes}(\Omega) \le 1$. Suppose that $X = L_p(\Omega)$, where $2 \le p < +\infty$. Then $X^* = L_q(\Omega)$, $(1/p) + (1/q) = 1$ and $1 < q \le p < +\infty$. We know that $L_p(\Omega) \subset L_q(\Omega)$ and $\|u\|_{L_q} \le c\|u\|_{L_p}$, where $c = \mu(\Omega)^{(1/q)-(1/p)}$ (see [7]).

Suppose that K is a closed convex subset of X with $\theta \in K$. For each $u_0 \in rcK \setminus \{\theta\}$, there exists a continuous linear functional $v_0 \in L_q(\Omega)$ such that $\langle v_0, u_0\rangle = \|u_0\|_p$ and $\|v_0\|_q = 1$ by the Hahn-Banach theorem. Define $g : K \to L_q(\Omega)$ as follows:

$$g(u) = \|u\|_p^2 v_0 - ku, \quad \forall u \in K,\ 0 < k < \alpha - \gamma, \tag{4.1}$$

where $\|u\|_p = \|u\|_{L_P}$ and $\|u\|_q = \|u\|_{L_q}$.

We can prove that the mapping $g_1(u) = \|u\|_p^2 v_0$ is compact and so is 0-set-contractive. In fact, for any bounded subset A of K, there exists $M > 0$ such that $\|u\|_p \le M$ for all $u \in A$. Thus, $g_1(u) = \|u\|^2 v_0 \subset M^2 \operatorname{co}\{v_0, \theta\}$ ($\operatorname{co}\{v_0,\theta\}$ denotes the convex hull of v_0 and θ). This implies that g_1 is a compact mapping. It is easy to see that the mapping $g_2(u) = ku$ is k-set-contractive and so g is k-set-contractive.

Now we show that g satisfies all conditions in the above theorems.

First, when $\|u\|_p < \alpha - kc$, we have

$$\left\|\frac{g(u)}{\|u\|_p}\right\|_q = \left\|\|u\|_p v_0 - k\frac{u}{\|u\|_p}\right\|_q = \|u\|_p\|v_0\|_q + k\frac{\|u\|_q}{\|u\|_p} \le | \ \|u\|_p + kc < \alpha. \tag{4.2}$$

This implies that condition (G_1) holds for $h = \theta$.

Next, taking $\|u\|_p > \beta+\gamma+kc$, we have

$$\frac{\langle g(u),u_0\rangle}{\|u\|_p} = \|u\|_p\|u_0\|_p - k\left\langle \frac{u}{\|u\|_p},u_0\right\rangle \geq (\|u\|_p - kc)\|u_0\|_p > (\beta+\gamma)\|u_0\|_p. \tag{4.3}$$

Hence, the condition (G_2) holds.

Finally, since

$$\left|\frac{\langle g(u),u_0\rangle}{\|u\|_p}\right| \leq \|u\|_p\|u_0\|_p + k\left\langle \frac{u}{\|u\|_p},u_0\right\rangle \leq (\|u\|_p + kc)\|u_0\|_p \tag{4.4}$$

when $\|u\|_p$ small enough, the $\langle g(u),u_0\rangle/\|u\|_p$ is bounded.

Similarly, we can prove that the conditions (G_4) and (G_5) are satisfied.

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