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Research Article

Double Subordination-Preserving Properties for Certain Integral Operators

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The purpose of the present paper is to obtain the sandwich-type theorem which contains the subordination- and superordination-preserving properties for certain integral operators defined on the space of normalized analytic functions in the open unit disk.

1. Introduction

Let $\mathcal{H} = \mathcal{H}(\mathbb{U})$ denote the class of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For $a \in \mathbb{C}$, let

$$\mathcal{H}[a,n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}. \tag{1.1}$$

Let f and F be members of $\mathcal{H}$. The function f is said to be subordinate to F, or F is said to be superordinate to f, if there exists a function w analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$, and such that $f(z) = F(w(z))$. In such a case, we write $f \prec F$ or $f(z) \prec F(z)$. If the function F is univalent in $\mathbb{U}$, then $f \prec F$ if and only if $f(0) = F(0)$ and $f(\mathbb{U}) \subset F(\mathbb{U})$ (cf. [1, 2]).

Let $\phi : \mathbb{C}^2 \to \mathbb{C}$ and let h be univalent in $\mathbb{U}$. If p is analytic in $\mathbb{U}$ and satisfies the differential subordination

$$\phi(p(z), zp'(z)) \prec h(z) \quad (z \in \mathbb{U}), \tag{1.2}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant if $p \prec q$ for all p satisfying (1.2). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.2) is said to be the best dominant [1].

Let $\varphi : \mathbb{C}^2 \to \mathbb{C}$ and let h be analytic in $\mathbb{U}$. If p and $\varphi(p(z), zp'(z))$ are univalent in $\mathbb{U}$ and satisfy the differential superordination

$$h(z) \prec \varphi(p(z), zp'(z)) \quad (z \in \mathbb{U}), \tag{1.3}$$

then p is called a solution of the differential superordination. An analytic function q is called a subordinant of the solutions of the differential superordination, or more simply a subordinant if $q \prec p$ for all p satisfying (1.3). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (1.3) is said to be the best subordinant [3].

We denote by $\mathcal{Q}$ the class of functions f that are analytic and injective on $\overline{\mathbb{U}} \backslash E(f)$, where

$$E(f) = \left\{ \zeta \in \partial\mathbb{U} : \lim_{z \to \zeta} f(z) = \infty \right\}, \tag{1.4}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U} \backslash E(f)$ [3].

Let $\mathcal{A}$ denote the subclass of $\mathcal{H}[a,1]$ with the usual normalization $f(0) = f'(0) - 1 = 0$. We also denote by $\mathcal{K}(\alpha)$ $(\alpha < 1)$ the class of convex functions of order α in $\mathbb{U}$. That is,

$$\mathcal{K}(\alpha) := \left\{ f \in \mathcal{A} : \operatorname{Re}\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \ (z \in \mathbb{U}) \right\}. \tag{1.5}$$

The class of starlike functions of order α $(\alpha < 1)$, denoted by $\mathcal{S}^*(\alpha)$, is defined by

$$\mathcal{S}^*(\alpha) := \left\{ f \in \mathcal{A} : \operatorname{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \ (z \in \mathbb{U}) \right\}. \tag{1.6}$$

In particular, the classes $\mathcal{K} \equiv \mathcal{K}(0)$ and $\mathcal{S}^* \equiv \mathcal{S}^*(0)$, respectively, represent the classes of convex functions and starlike functions in $\mathbb{U}$.

For a function $f \in \mathcal{A}$, we introduce the following integral operator $I_{\beta,\gamma}$ defined by

$$I_{\beta,\gamma}(f)(z) := \left(\frac{\beta+\gamma}{z^\gamma} \int_0^z t^{\gamma-1} f^\beta(t)\,dt \right)^{1/\beta} \quad (f \in \mathcal{A};\ \beta \in \mathbb{C}\backslash\{0\};\ \gamma \in \mathbb{C};\ \operatorname{Re}\{\beta+\gamma\} > 0). \tag{1.7}$$

The integral operators defined by (1.7) have been extensively studied by many authors [4–8] with suitable restriction on the parameters β and γ, and for f belonging to some favored classes of analytic functions.

Miller et al. [9] obtained some subordination theorems involving certain integral operators for analytic functions in $\mathbb{U}$. Recently, Bulboacă [5] considered superordination-preserving properties of the integral operator defined by (1.7) as the dual problem of subordination. In the present paper, we investigate the subordination- and superordination-preserving properties of the integral operator $I_{\beta,\gamma}$ defined by (1.7) with the sandwich-type theorem.

2. A set of lemmas

The following lemmas will be required in our present investigation.

Lemma 2.1 [10]. *Let* $\beta,\gamma \in \mathbb{C}$ *with* $\beta \neq 0$ *and let* $h \in \mathcal{H}(\mathbb{U})$ *with* $h(0) = c$. *If* $\operatorname{Re}\{\beta h(z)+\gamma\} > 0$ $(z \in \mathbb{U})$, *then the solution of the differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z)+\gamma} = h(z) \quad (z \in \mathbb{U}) \tag{2.1}$$

with $q(0) = c$ *is analytic in* $\mathbb{U}$ *and satisfies* $\operatorname{Re}\{\beta q(z)+\gamma\} > 0$ $(z \in \mathbb{U})$.

Lemma 2.2 [1]. *Let* $p \in \mathcal{Q}$ *with* $p(0) = a$ *and let* $q(z) = a + a_n z^n + \cdots$ *be analytic in* $\mathbb{U}$ *with* $q(z) \not\equiv a$ *and* $n \geq 1$. *If* q *is not subordinate to* p, *then there exist points* $z_0 = r_0 e^{i\theta} \in \mathbb{U}$ *and* $\zeta_0 \in \partial\mathbb{U}\backslash E(f)$, *for which* $q(\mathbb{U}_{r_0}) \subset p(\mathbb{U})$,

$$q(z_0) = p(\zeta_0), \qquad z_0 q'(z_0) = m\zeta_0 p'(\zeta_0) \quad (m \geq n). \tag{2.2}$$

Our next lemma deals with the notion of subordination chain. A function $L(z,t)$ defined on $\mathbb{U} \times [0,\infty)$ is the subordination chain (or Löwner chain) if $L(\cdot,t)$ is analytic and univalent in $\mathbb{U}$ for all $t \in [0,\infty)$, $L(z,\cdot)$ is continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{U}$, and $L(z,s) \prec L(z,t)$ for $z \in \mathbb{U}$ and $0 \leq s < t$.

Lemma 2.3 [3]. *Let* $q \in \mathcal{H}[a,1]$, *let* $\varphi : \mathbb{C}^2 \to \mathbb{C}$, *and set* $\varphi(q(z), zq'(z)) \equiv h(z)$. *If* $L(z,t) = \varphi(q(z), tzq'(z))$ *is a subordination chain and* $p \in \mathcal{H}[a,1] \cap \mathcal{Q}$, *then*

$$h(z) \prec \varphi(p(z), zp'(z)) \quad (z \in \mathbb{U}) \tag{2.3}$$

implies that

$$q(z) \prec p(z) \quad (z \in \mathbb{U}). \tag{2.4}$$

Furthermore, if $\varphi(q(z), zp'(z)) = h(z)$ *has a univalent solution* $q \in \mathcal{Q}$, *then* q *is the best subordinant.*

We now recall that the Gauss hypergeometric function ${}_2F_1(a,b;c;z)$ is defined by ([11], see also [12, Chapter 14])

$${}_2F_1(a,b;c;z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \quad (z \in \mathbb{U};\ b \in \mathbb{C};\ c \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ \mathbb{Z}_0^- := \{0,-1,-2,\dots\}), \tag{2.5}$$

where $(\lambda)_\nu$ denotes the Pochhammer symbol (or the shifted factorial) defined (for $\lambda,\nu \in \mathbb{C}$ and in terms of the Gamma function) by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C}\backslash\{0\}), \\ \lambda(\lambda+1)\cdots(\lambda+\nu-1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}). \end{cases} \tag{2.6}$$

LEMMA 2.4 [13]. *Let $\beta > 0$, $\beta+\gamma > 0$ and let $I_{\beta,\gamma}$ be the integral operator defined by (1.7). If $\alpha \in [-\gamma/\beta, 1)$, then the order of starlikeness of the class $I_{\beta,\gamma}(\mathcal{S}^*(\alpha))$, that is, the largest number $\delta = \delta(\alpha;\beta,\gamma)$ such that*

$$I_{\beta,\gamma}(\mathcal{S}^*(\alpha)) \subset \mathcal{S}^*(\delta), \tag{2.7}$$

is given by the number $\delta(\alpha;\beta,\gamma) = \inf\{\operatorname{Re} q(z) : z \in \mathbb{U}\}$, where

$$q(z) = \frac{1}{\beta Q(z)} - \frac{\gamma}{\beta}, \qquad Q(z) = \int_0^1 \left(\frac{1-z}{1-tz}\right)^{2\beta(1-\alpha)} t^{\beta+\alpha-1} dt. \tag{2.8}$$

Moreover, if $\alpha \in [\alpha_0, 1)$, where

$$\alpha_0 := \max\left\{\frac{\beta-\gamma-1}{2\beta}, -\frac{\gamma}{\beta}\right\} \tag{2.9}$$

and $f \in \mathcal{S}^*(\alpha)$, then

$$\operatorname{Re}\left\{\frac{z(I_{\beta,\gamma}(f)(z))'}{I_{\beta,\gamma}(f)(z)}\right\} > \delta(\alpha;\beta,\gamma) = \frac{1}{\beta}\left[\frac{\beta+\gamma}{{}_2F_1(1,2\beta(1-\alpha),\beta+\gamma+1;1/2)} - \gamma\right], \tag{2.10}$$

where ${}_2F_1$ represents the Gauss hypergeometric function defined by (2.5).

LEMMA 2.5 [14]. *The function $L(z,t) = a_1(t)z + \cdots$, with $a_1(t) \neq 0$ and $\lim_{t\to\infty} |a_1(t)| = \infty$, is a subordination chain if and only if*

$$\operatorname{Re}\left\{\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right\} > 0 \quad (z \in \mathbb{U};\ 0 \le t < \infty). \tag{2.11}$$

Throughout this paper, we will denote $\mathcal{A}_{\beta,\gamma}$ by

$$\mathcal{A}_{\beta,\gamma} := \left\{f \in \mathcal{A} : \frac{f(z)}{z} \neq 0,\ \frac{I_{\beta,\gamma}(f)(z)}{z} \neq 0\ (z \in \mathbb{U};\ \beta \neq 1)\right\}, \tag{2.12}$$

where $I_{\beta,\gamma}$ is the integral operator defined by (1.7). For various interesting developments involving functions in the class $\mathcal{A}_{\beta,\gamma}$, the reader may be referred, for example, to the recent work of Miller and Mocanu [1].

3. Main results

Subordination theorem involving the integral operator $I_{\beta,\gamma}$ defined by (1.7) is contained in Theorem 3.1 below.

THEOREM 3.1. *Let $f, g \in \mathcal{A}_{\beta,\gamma}$ with $\beta > 0$ and $0 < \beta+\gamma \le 1$. Suppose that*

$$\operatorname{Re}\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\frac{\beta+\gamma}{2} \quad \left(z \in \mathbb{U};\ \phi(z) := \left(\frac{g(z)}{z}\right)^{\beta}\right). \tag{3.1}$$

Then

$$\left(\frac{f(z)}{z}\right)^{\beta} \prec \left(\frac{g(z)}{z}\right)^{\beta} \quad (z \in \mathbb{U}) \tag{3.2}$$

implies that

$$\left(\frac{I_{\beta,\gamma}(f)(z)}{z}\right)^{\beta} \prec \left(\frac{I_{\beta,\gamma}(g)(z)}{z}\right)^{\beta} \quad (z \in \mathbb{U}), \tag{3.3}$$

where the integral operator $I_{\beta,\gamma}$ is defined by (1.7). Moreover, the function $(I_{\beta,\gamma}(g)(z)/z)^{\beta}$ is the best dominant.

Proof. Let us define the functions F and G by

$$F(z) := \left(\frac{I_{\beta,\gamma}(f)(z)}{z}\right)^{\beta}, \qquad G(z) := \left(\frac{I_{\beta,\gamma}(g)(z)}{z}\right)^{\beta}, \tag{3.4}$$

respectively. Without loss of generality, we can assume that G is analytic and univalent on $\overline{\mathbb{U}}$, and $G'(\zeta) \neq 0$ for $|\zeta| = 1$.

We first show that if the function q is defined by

$$q(z) := 1 + \frac{zG''(z)}{G'(z)} \quad (z \in \mathbb{U}), \tag{3.5}$$

then

$$\operatorname{Re}\{q(z)\} > 0 \quad (z \in \mathbb{U}). \tag{3.6}$$

From the definition of (1.7), we obtain

$$(I_{\beta,\gamma}g(z))^{\beta}\left(\beta\frac{z(I_{\beta,\gamma}(g)(z))'}{I_{\beta,\gamma}(g)(z)} + \gamma\right)\frac{1}{\beta+\gamma} = g^{\beta}(z). \tag{3.7}$$

We also have

$$\beta\frac{z(I_{\beta,\gamma}(g)(z))'}{I_{\beta,\gamma}(g)(z)} = \beta + \frac{zG'(z)}{G(z)}. \tag{3.8}$$

It follows from (3.7) and (3.8) that

$$(\beta+\gamma)\phi(z) = (\beta+\gamma)G(z) + zG'(z). \tag{3.9}$$

Now, by differentiating both sides of (3.9), we obtain

$$q(z) + \frac{zq'(z)}{q(z)+\beta+\gamma} = 1 + \frac{z\phi''(z)}{\phi'(z)} \equiv h(z). \tag{3.10}$$

From (3.1), we have

$$\operatorname{Re}\{h(z)+\beta+\gamma\} > \frac{\beta+\gamma}{2} > 0 \quad (z \in \mathbb{U}), \tag{3.11}$$

and by using Lemma 2.1, we conclude that the differential equation (3.10) has a solution $q \in \mathcal{H}(\mathbb{U})$ with $q(0) = h(0) = 1$.

Now, we will use Lemma 2.4 to prove that, under the assumption, the inequality (3.6) holds. Replacing β by $\widetilde{\beta} = 1$ and γ by $\widetilde{\gamma} = \beta+\gamma$ in Lemma 2.4, we have

$$\alpha_0 = \max\left\{\frac{\widetilde{\beta}-\widetilde{\gamma}-1}{2\widetilde{\beta}}, -\frac{\widetilde{\gamma}}{\widetilde{\beta}}\right\} = -\frac{\beta+\gamma}{2}. \tag{3.12}$$

For the differential equation (3.10), by using Lemma 2.4 in the case

$$\alpha = \alpha_0 = -\frac{\beta+\gamma}{2}, \tag{3.13}$$

we obtain that

$$\operatorname{Re}\{q(z)\} > \frac{\beta+\gamma+1}{{}_2F_1(1,\beta+\gamma+2,\beta+\gamma+2;1/2)} - (\beta+\gamma) = \frac{1-(\beta+\gamma)}{2} \geq 0 \quad (z \in \mathbb{U}). \tag{3.14}$$

That is, G defined by (3.4) is convex(univalent) in $\mathbb{U}$.

Next, we prove that the subordination condition (3.2) implies that

$$F(z) \prec G(z) \quad (z \in \mathbb{U}) \tag{3.15}$$

for the functions F and G defined by (3.4). For this purpose, we consider the function $L(z,t)$ given by

$$L(z,t) := G(z) + \frac{1+t}{\beta+\gamma} zG'(z) \quad (z \in \mathbb{U};\ 0 \leq t < \infty). \tag{3.16}$$

We note that

$$\left.\frac{\partial L(z,t)}{\partial z}\right|_{z=0} = G'(0)\left(\frac{\beta+\gamma+1+t}{\beta+\gamma}\right) \neq 0 \quad (0 \leq t < \infty;\ \beta+\gamma > 0). \tag{3.17}$$

This shows that the function

$$L(z,t) = a_1(t)z + \cdots \tag{3.18}$$

satisfies the condition $a_1(t) \neq 0$ for all $t \in [0,\infty)$. Furthermore, we have

$$\operatorname{Re}\left\{\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right\} = \operatorname{Re}\left\{\beta+\gamma+(1+t)\left(1+\frac{zG''(z)}{G'(z)}\right)\right\} > 0, \tag{3.19}$$

since G is convex and $\beta+\gamma>0$. Therefore, by virtue of Lemma 2.5, $L(z,t)$ is a subordination chain. We observe from the definition of a subordination chain that

$$\phi(z)=G(z)+\frac{1}{\beta+\gamma}zG'(z)=L(z,0),\quad L(z,0)\prec L(z,t)\ (z\in\mathbb{U};\ 0\leq t<\infty). \tag{3.20}$$

This implies that

$$L(\zeta,t)\notin L(\mathbb{U},0)=\phi(\mathbb{U}) \tag{3.21}$$

for $\zeta\in\partial\mathbb{U}$ and $t\in[0,\infty)$.

Now, suppose that F is not subordinate to G. Then, by Lemma 2.2, there exist points $z_0\in\mathbb{U}$ and $\zeta_0\in\partial\mathbb{U}$ such that

$$F(z_0)=G(\zeta_0),\qquad z_0F'(z_0)=(1+t)\zeta_0G'(\zeta_0)\quad(0\leq t<\infty). \tag{3.22}$$

Hence, we have

$$L(\zeta_0,t)=G(\zeta_0)+\frac{1+t}{\beta+\gamma}\zeta_0G'(\zeta_0)=F(z_0)+\frac{1}{\beta+\gamma}z_0F'(z_0)=\left(\frac{f(z_0)}{z_0}\right)^{\beta}\in\phi(\mathbb{U}) \tag{3.23}$$

by virtue of the subordination condition (3.2). This contradicts the above observation that $L(\zeta_0,t)\notin\phi(\mathbb{U})$. Therefore, the subordination condition (3.2) must imply the subordination given by (3.15). Considering $F(z)=G(z)$, we see that the function G is the best dominant. Therefore, we complete the proof of Theorem 3.1. □

We next prove a dual problem of Theorem 3.1 in the sense that the subordinations are replaced by superordinations.

THEOREM 3.2. *Let $f,g\in\mathcal{A}_{\beta,\gamma}$ with $\beta>0$ and $0<\beta+\gamma\leq 1$. Suppose that*

$$\operatorname{Re}\left\{1+\frac{z\phi''(z)}{\phi'(z)}\right\}>-\frac{\beta+\gamma}{2}\quad\left(z\in\mathbb{U};\ \phi(z):=\left(\frac{g(z)}{z}\right)^{\beta}\right). \tag{3.24}$$

If $(f(z)/z)^{\beta}$ is univalent in $\mathbb{U}$ and $(I_{\beta,\gamma}(f)(z)/z)^{\beta}\in\mathfrak{Q}$, then

$$\left(\frac{g(z)}{z}\right)^{\beta}\prec\left(\frac{f(z)}{z}\right)^{\beta}\quad(z\in\mathbb{U}) \tag{3.25}$$

implies that

$$\left(\frac{I_{\beta,\gamma}(g)(z)}{z}\right)^{\beta}\prec\left(\frac{I_{\beta,\gamma}(f)(z)}{z}\right)^{\beta}\quad(z\in\mathbb{U}), \tag{3.26}$$

where the integral operator $I_{\beta,\gamma}$ is defined by (1.7). Moreover, the function $(I_{\beta,\gamma}(g)(z)/z)^{\beta}$ is the best subordinant.

Proof. The first part of the proof is similar to that of Theorem 3.1 and so we will use the same notation as in the proof of Theorem 3.1.

Now, let us define the functions F and G, respectively, by (3.4). We first note that from (3.7) and (3.8), we obtain

$$\phi(z) = G(z) + \frac{1}{\beta+\gamma} zG'(z) =: \varphi(G(z), zG'(z)). \tag{3.27}$$

After a simple calculation, (3.27) yields the following relationship:

$$1 + \frac{z\phi''(z)}{\phi'(z)} = q(z) + \frac{zq'(z)}{q(z)+\beta+\gamma}, \tag{3.28}$$

where the function q is defined by (3.5). Then, by using the same method as in the proof of Theorem 3.1, we can prove that $\operatorname{Re}\{q(z)\} > 0$ for all $z \in \mathbb{U}$. That is, G defined by (3.4) is convex(univalent) in $\mathbb{U}$.

Next, we prove that the subordination condition (3.25) implies that

$$F(z) \prec G(z) \quad (z \in \mathbb{U}) \tag{3.29}$$

for the functions F and G defined by (3.4). Now consider the function $L(z,t)$ defined by

$$L(z,t) := G(z) + \frac{t}{\beta+\gamma} zG'(z) \quad (z \in \mathbb{U};\ 0 \le t < \infty). \tag{3.30}$$

Since G is convex and $\beta+\gamma > 0$, we can easily prove that $L(z,t)$ is a subordination chain as in the proof of Theorem 3.1. Therefore, according to Lemma 2.3, we conclude that the superordination condition (3.25) must imply the superordination given by (3.29). Furthermore, since the differential equation (3.27) has the univalent solution G, it is the best subordinant of the given differential superordination. Therefore, we complete the proof of Theorem 3.2. □

If we combine Theorems 3.1 and 3.2, then we obtain the following sandwich-type theorem.

Theorem 3.3. *Let* $f, g_k \in \mathcal{A}_{\beta,\gamma}$ $(k = 1,2)$ *with* $\beta > 0$ *and* $0 < \beta+\gamma \le 1$. *Suppose that*

$$\operatorname{Re}\left\{1 + \frac{z\phi_k''(z)}{\phi_k'(z)}\right\} > -\frac{\beta+\gamma}{2} \quad \left(z \in \mathbb{U};\ \phi_k(z) := \left(\frac{g_k(z)}{z}\right)^{\beta};\ k = 1,2\right). \tag{3.31}$$

If $(f(z)/z)^{\beta}$ *is univalent in* $\mathbb{U}$ *and* $(I_{\beta,\gamma}(f)(z)/z)^{\beta} \in \mathcal{Q}$, *then*

$$\left(\frac{g_1(z)}{z}\right)^{\beta} \prec \left(\frac{f(z)}{z}\right)^{\beta} \prec \left(\frac{g_2(z)}{z}\right)^{\beta} \quad (z \in \mathbb{U}) \tag{3.32}$$

implies that

$$\left(\frac{I_{\beta,\gamma}(g_1)(z)}{z}\right)^{\beta} \prec \left(\frac{I_{\beta,\gamma}(f)(z)}{z}\right)^{\beta} \prec \left(\frac{I_{\beta,\gamma}(g_2)(z)}{z}\right)^{\beta} \quad (z \in \mathbb{U}), \tag{3.33}$$

where $I_{\beta,\gamma}$ *is the integral operator defined by (1.7). Moreover, the functions* $(I_{\beta,\gamma}(g_1)(z)/z)^{\beta}$ *and* $(I_{\beta,\gamma}(g_2)(z)/z)^{\beta}$ *are the best subordinant and the best dominant, respectively.*

Since the assumption of Theorem 3.3, that the functions $(f(z)/z)^\beta$ and $(I_{\beta,\gamma}(f)(z)/z)^\beta$ need to be univalent in $\mathbb{U}$, is not so easy to check, we will replace these conditions by another conditions in the following result.

COROLLARY 3.4. *Let $f, g_k \in \mathcal{A}_{\beta,\gamma}$ $(k = 1,2)$ with $\beta > 0$ and $0 < \beta + \gamma \leq 1$. Suppose that the condition (3.31) is satisfied and*

$$\operatorname{Re}\left\{1 + \frac{z\psi''(z)}{\psi'(z)}\right\} > -\frac{\beta+\gamma}{2} \quad \left(z \in \mathbb{U};\ \psi(z) := \left(\frac{f(z)}{z}\right)^\beta;\ f \in \mathcal{Q}\right). \tag{3.34}$$

Then

$$\left(\frac{g_1(z)}{z}\right)^\beta \prec \left(\frac{f(z)}{z}\right)^\beta \prec \left(\frac{g_2(z)}{z}\right)^\beta \quad (z \in \mathbb{U}) \tag{3.35}$$

implies that

$$\left(\frac{I_{\beta,\gamma}(g_1)(z)}{z}\right)^\beta \prec \left(\frac{I_{\beta,\gamma}(f)(z)}{z}\right)^\beta \prec \left(\frac{I_{\beta,\gamma}(g_2)(z)}{z}\right)^\beta \quad (z \in \mathbb{U}), \tag{3.36}$$

where $I_{\beta,\gamma}$ is the integral operator defined by (1.7). Moreover, the functions $(I_{\beta,\gamma}(g_1)(z)/z)^\beta$ and $(I_{\beta,\gamma}(g_2)(z)/z)^\beta$ are the best subordinant and the best dominant, respectively.

Proof. In order to prove Corollary 3.4, we have to show that the condition (3.34) implies the univalence of $\psi(z)$ and $F(z) := (I_{\beta,\gamma}(f)(z)/z)^\beta$. Since the condition (3.34) means that ψ is a close-to-convex function in $\mathbb{U}$ (see [15]), it follows that ψ is univalent in $\mathbb{U}$. Furthermore, by using the same techniques as in the proof of Theorem 3.1, we can prove the convexity (univalence) of F and so the details may be omitted. Therefore, from Theorem 3.3, we obtain Corollary 3.4. □

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