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Research Article

Extension of Oppenheim's Problem to Bessel Functions

Árpád Baricz and Ling Zhu

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Our aim is to extend some trigonometric inequalities to Bessel functions. Moreover, we deduce the hyperbolic analogue of these trigonometric inequalities, and we extend these inequalities to modified Bessel functions.

1. Introduction and main results

In 1957, Ogilvy et al. [1] (or see [2, page 238]) asked the following question: *for each* $a_1 > 0$, *there is a greatest* a_2 *and a least* a_3 *such that for all* $x \in [0,\pi/2]$, *the inequality*

$$a_2 \frac{\sin x}{1 + a_1 \cos x} \le x \le a_3 \frac{\sin x}{1 + a_1 \cos x} \tag{1.1}$$

holds. Determine a_2 *and* a_3 *as functions of* a_1.

In 1958, Oppenheim and Carver [3] (or see [2, page 238]) gave a partial solution to Oppenheim's problem by showing that, for all $a_1 \in (0,1/2]$ and $x \in [0,\pi/2]$, (1.1) holds when $a_2 = a_1 + 1$ and $a_3 = \pi/2$. Recently, Zhu [4, Theorem 7] solved, completely, this problem of Oppenheim, proving that (1.1) holds in the following cases:

(i) if $a_1 \in (0,1/2)$, then $a_2 = a_1 + 1$ and $a_3 = \pi/2$;
(ii) if $a_1 \in [1/2,\pi/2 - 1)$, then $a_2 = 4a_1(1 - a_1^2)$ and $a_3 = \pi/2$;
(iii) if $a_1 \in [\pi/2 - 1, 2/\pi)$, then $a_2 = 4a_1(1 - a_1^2)$ and $a_3 = a_1 + 1$;
(iv) if $a_1 \ge 2/\pi$, then $a_2 = \pi/2$ and $a_3 = a_1 + 1$,

where a_2 and a_3 are the best constants in (i) and (iv), while a_3 is also the best constant in (ii) and (iii).

Recently, Baricz [5, Theorem 2.20] extended the Carver solution to Bessel functions (see also [6] for further results). In this note, our aim is to extend the above-mentioned

Zhu solution to Bessel functions too. For this, let us consider the function $\mathcal{J}_p : \mathbb{R} \to (-\infty, 1]$, defined by

$$\mathcal{J}_p(x) := 2^p\Gamma(p+1)x^{-p}J_p(x) = \sum_{n\geq 0}\frac{(-1/4)^n}{(p+1)_n n!}x^{2n}, \tag{1.2}$$

where

$$J_p(x) := \sum_{n\geq 0}\frac{(-1)^n}{n!\cdot\Gamma(p+n+1)}\left(\frac{x}{2}\right)^{2n+p} \quad \forall\ x\in\mathbb{R} \tag{1.3}$$

is the Bessel function of the first kind [7], and

$$(p+1)_n = (p+1)(p+2)\cdots(p+n) = \Gamma(p+n+1)/\Gamma(p+1) \tag{1.4}$$

is the well-known Pochhammer (or Appell) symbol defined in terms of Euler gamma function. It is worth mentioning here that, in particular, we have

$$\begin{aligned}\mathcal{J}_{1/2}(x) &:= \sqrt{\pi/2}\cdot x^{-1/2}J_{1/2}(x) = \frac{\sin x}{x},\\ \mathcal{J}_{-1/2}(x) &:= \sqrt{\pi/2}\cdot x^{1/2}J_{-1/2}(x) = \cos x.\end{aligned} \tag{1.5}$$

Now, the extension of Zhu solution reads as follows.

THEOREM 1.1. *Let $p \geq -1/2$, $|x| \leq \pi/2$ and a_1, a_2, a_3 such that*

(i) *if $a_1 \in (0, 1/2)$, then $a_2 = a_1 + 1$ and $a_3 = \pi/2$;*
(ii) *if $a_1 \in [1/2, \pi/2 - 1)$, then $a_2 = 4a_1(1 - a_1^2)$ and $a_3 = \pi/2$;*
(iii) *if $a_1 \in [\pi/2 - 1, 2/\pi)$, then $a_2 = 4a_1(1 - a_1^2)$ and $a_3 = a_1 + 1$;*
(iv) *if $a_1 \geq 2/\pi$, then $a_2 = \pi/2$ and $a_3 = a_1 + 1$.*

Then the following inequality holds true:

$$[a_1(2p+1)+a_2]\mathcal{J}_{p+1}(x) \leq 1 + 2a_1(p+1)\mathcal{J}_p(x) \leq [a_1(2p+1)+a_3]\mathcal{J}_{p+1}(x), \tag{1.6}$$

where a_2 and a_3 are the best constants in (i) and (iv), while a_3 is also the best constant in (ii) and (iii).

We note that, in particular, we have

$$\mathcal{J}_{3/2}(x) := 3\sqrt{\pi/2}\cdot x^{-3/2}J_{3/2}(x) = 3\left(\frac{\sin x}{x^3} - \frac{\cos x}{x^2}\right), \tag{1.7}$$

thus, choosing $p = 1/2$ in Theorem 1.1, we obtain the following interesting result.

COROLLARY 1.2. *If a_1, a_2, a_3 are as in Theorem 1.1, then, for all $|x| \leq \pi/2$,*

$$\frac{3(2a_1+a_2)(\sin x/x - \cos x)}{1+3a_1(\sin x/x)} \leq x^2 \leq \frac{3(2a_1+a_3)(\sin x/x - \cos x)}{1+3a_1(\sin x/x)}. \tag{1.8}$$

The hyperbolic analogue of (1.1) is the following result.

Theorem 1.3. *Let $x \geq 0$ and a_1, a_2 such that*
(i) *if $a_1 \geq 1/2$, then $a_2 = a_1 + 1$;*
(ii) *if $a_1 \in (0, 1/2)$, then $a_2 = 4a_1(1 - a_1^2)$.*
Then the following inequality holds true:

$$a_2 \frac{\sinh x}{1 + a_1 \cosh x} \leq x, \tag{1.9}$$

where a_2 is the best constant in (i). Moreover, when $x \leq 0$, the above inequality is reversed.

For $p > -1$, let us consider the function $\mathcal{I}_p : \mathbb{R} \to [1, \infty)$, defined by

$$\mathcal{I}_p(x) := 2^p \Gamma(p+1) x^{-p} I_p(x) = \sum_{n \geq 0} \frac{(1/4)^n}{(p+1)_n n!} x^{2n}, \tag{1.10}$$

where

$$I_p(x) := \sum_{n \geq 0} \frac{1}{n! \cdot \Gamma(p+n+1)} \left(\frac{x}{2}\right)^{2n+p} \quad \forall \ x \in \mathbb{R} \tag{1.11}$$

is the modified Bessel function of the first kind [7]. On the other hand, it is worth mentioning that, in particular, we have

$$\begin{aligned} \mathcal{I}_{1/2}(x) &:= \sqrt{\pi/2} \cdot x^{-1/2} I_{1/2}(x) = \frac{\sinh x}{x}, \\ \mathcal{I}_{-1/2}(x) &:= \sqrt{\pi/2} \cdot x^{1/2} I_{-1/2}(x) = \cosh x, \end{aligned} \tag{1.12}$$

respectively.

The following inequality for $a_1 = 1$ was proved recently by Baricz [6, Theorem 4.9], and provides the extension of Theorem 1.3 to modified Bessel functions.

Theorem 1.4. *Let $p \geq -1/2$, $x \in \mathbb{R}$, and a_1, a_2 be as in Theorem 1.3. Then the following inequality holds true:*

$$[a_1(2p+1) + a_2] \mathcal{I}_{p+1}(x) \leq 1 + 2a_1(p+1) \mathcal{I}_p(x), \tag{1.13}$$

where a_2 is the best constant in (i).

Finally, observe that, in particular, we have

$$\mathcal{I}_{3/2}(x) := 3\sqrt{\pi/2} \cdot x^{-3/2} I_{3/2}(x) = -3\left(\frac{\sinh x}{x^3} - \frac{\cosh x}{x^2}\right), \tag{1.14}$$

thus, choosing $p = 1/2$ in Theorem 1.4, we obtain the following interesting result.

Corollary 1.5. *If a_1, a_2 are as in Theorem 1.4, then, for all $x \in \mathbb{R}$,*

$$\frac{3(2a_1 + a_2)(\cosh x - \sinh x/x)}{1 + 3a_1 (\sinh x/x)} \leq x^2. \tag{1.15}$$

2. Proof of main results

Proof of Theorem 1.1. First, observe that each (1.6) is even, thus we can suppose that $x \in [0,\pi/2]$. On the other hand, when $p = -1/2$ from (1.5), it follows that (1.6) reduces to

$$a_2\mathcal{J}_{1/2}(x) \le 1 + a_1\mathcal{J}_{-1/2}(x) \le a_3\mathcal{J}_{1/2}(x), \tag{2.1}$$

which is equivalent to (1.1), and was proved by Zhu [4, Theorem 7], as we mentioned above. Recall the well-known Sonine integral formula [7, page 373] for Bessel functions:

$$J_{q+p+1}(x) = \frac{x^{p+1}}{2^p\Gamma(p+1)}\int_0^{\pi/2} J_q(x\sin\theta)\sin^{q+1}\theta\cos^{2p+1}\theta\,d\theta, \tag{2.2}$$

where $p,q > -1$ and $x \in \mathbb{R}$. From this, we obtain the following formula

$$\mathcal{J}_{q+p+1}(x) = \frac{2}{B(p+1,q+1)}\int_0^{\pi/2} \mathcal{J}_q(x\sin\theta)\sin^{2q+1}\theta\cos^{2p+1}\theta\,d\theta, \tag{2.3}$$

which will be useful in the sequel. Here, $B(p,q) = \Gamma(p)\Gamma(q)/\Gamma(p+q)$ is the well-known Euler beta function. Changing, in (2.3), p with $p - 1/2$, and taking $q = -1/2$ ($q = 1/2$, resp.) one has, for all $p > -1/2, x \in \mathbb{R}$,

$$\begin{aligned}\mathcal{J}_p(x) &= \frac{2}{B(p+1/2,1/2)}\int_0^{\pi/2} \mathcal{J}_{-1/2}(x\sin\theta)\cos^{2p}\theta\,d\theta,\\ \mathcal{J}_{p+1}(x) &= \frac{2}{B(p+1/2,3/2)}\int_0^{\pi/2} \mathcal{J}_{1/2}(x\sin\theta)\sin^2\theta\cos^{2p}\theta\,d\theta.\end{aligned} \tag{2.4}$$

Now, changing x with $x\sin\theta$ in (2.1), multiplying (2.1) with $\sin^2\theta\cos^{2p}\theta$ and integrating, it follows that the expression (using (2.4))

$$\begin{aligned}\Delta_p(x) &:= \int_0^{\pi/2} \sin^2\theta\cos^{2p}\theta\,d\theta + a_1\int_0^{\pi/2} \mathcal{J}_{-1/2}(x\sin\theta)(1-\cos^2\theta)\cos^{2p}\theta\,d\theta\\ &= \frac{1}{2}B\left(p+\frac{1}{2},\frac{3}{2}\right) + \frac{a_1}{2}B\left(p+\frac{1}{2},\frac{1}{2}\right)\mathcal{J}_p(x) - \frac{a_1}{2}B\left(p+\frac{3}{2},\frac{1}{2}\right)\mathcal{J}_{p+1}(x)\end{aligned} \tag{2.5}$$

satisfies

$$\frac{a_2}{2}B\left(p+\frac{1}{2},\frac{3}{2}\right)\mathcal{J}_{p+1}(x) \le \Delta_p(x) \le \frac{a_3}{2}B\left(p+\frac{1}{2},\frac{3}{2}\right)\mathcal{J}_{p+1}(x). \tag{2.6}$$

After simplifications, we obtain that (1.6) holds. □

Proof of Theorem 1.3. Let us consider the functions $f,g,Q : \mathbb{R}\to\mathbb{R}$ defined by $f(x) := (1 + a_1\cosh x)x$, $g(x) := \sinh x$ and $Q(x) := f(x)/g(x)$. Clearly, we have

$$\begin{aligned}Q(x) &= \frac{f(x)}{g(x)} = \frac{f(x)-f(0)}{g(x)-g(0)},\\ \varphi(x) &:= \frac{f'(x)}{g'(x)} = \frac{1+a_1\cosh x + a_1 x\sinh x}{\cosh x}.\end{aligned} \tag{2.7}$$

Now, in what follows, we try to find the minimum values of Q using the monotone form of l'Hospital's rule discovered by Anderson et al. [8, Lemma 2.2]. Easy computations show that $\varphi'(x) = u(x)/\cosh^2 x$, where $u : [0,\infty) \to \mathbb{R}$ is defined by $u(x) = a_1 x + a_1(\sinh x)(\cosh x) - \sinh x$. Moreover, we have $u'(x) = (\cosh x)(2a_1 \cosh x - 1)$. For convenience, let us consider $\cosh x = t$ and define the function $v : [1,\infty) \to \mathbb{R}$ by $v(t) := t(2a_1 t - 1)$.

There are two cases to consider.

Case 1 ($a_1 \geq 1/2$). Since $t \geq 1 \geq 1/2a_1$, it follows that $u'(x) = v(t) \geq 0$ for all $x \geq 0$, and, consequently, the function u is increasing. This implies that $u(x) \geq u(0) = 0$, that is, the function φ is increasing on $[0,\infty)$. Using the monotone form of l'Hospital's rule [8, Lemma 2.2], we conclude that Q is increasing too on $[0,\infty)$, that is, $Q(x) \geq Q(0) = 1 + a_1$ for all $x \geq 0$. Now, because Q is an even function, clearly, Q is decreasing on $(-\infty, 0]$, that is, $Q(x) \geq Q(0) = 1 + a_1$ for all $x \leq 0$.

Case 2 ($a_1 \in (0,1/2)$). Because Q is even, it is enough again to consider its restriction to $[0,\infty)$. However, at this moment, the function Q is not fully monotone on $[0,\infty)$. Let α be the minimum point of the function Q. We can obtain, by direct calculation,

$$(\sinh^2 x)Q'(x) = \sinh x + a_1(\sinh x)(\cosh x) - x\cosh x - a_1 x. \tag{2.8}$$

Since $Q'(\alpha) = 0$, we have $\sinh\alpha + a_1(\sinh\alpha)(\cosh\alpha) - \alpha\cosh\alpha - a_1\alpha = 0$, that is,

$$\frac{\alpha}{\sinh\alpha} = \frac{1 + a_1\cosh\alpha}{a_1 + \cosh\alpha}. \tag{2.9}$$

Using this relation, we deduce that

$$Q(\alpha) = \frac{(1 + a_1\cosh\alpha)\alpha}{\sinh\alpha} = \frac{(1 + a_1\cosh\alpha)^2}{a_1 + \cosh\alpha}. \tag{2.10}$$

Finally, because the minimum of the function $x \mapsto (1 + a_1 x)^2/(a_1 + x)$ on $[1,\infty)$ is $4a_1(1 - a_1^2)$, we have $Q(\alpha) \geq 4a_1(1 - a_1^2)$, and with this, the proof is complete. $\square$

Proof of Theorem 1.4. In analogy to the proof of Theorem 1.1, we can prove Theorem 1.4 For this, let us recall that, recently, András and Baricz proved [9, Lemma 1] that if $x \in \mathbb{R}$ and $p > q > -1$, then

$$\mathcal{I}_p(x) = \frac{2}{B(q+1, p-q)} \int_0^1 \mathcal{I}_q(tx)t^{2q+1}(1 - t^2)^{p-q-1}\,dt. \tag{2.11}$$

Taking, in the above relation, $t = \sin\theta$, we obtain the hyperbolic analogue of (2.3), that is,

$$\mathcal{I}_p(x) = \frac{2}{B(q+1, p-q)} \int_0^{\pi/2} \mathcal{I}_q(x\sin\theta)\sin^{2q+1}\theta\cos^{2p-2q-1}\theta\,d\theta. \tag{2.12}$$

In particular, taking, in the above relation, $q = -1/2$, changing p with $p+1$ and taking $q = 1/2$, respectively, we get that, for all $p > -1/2$ and $x \in \mathbb{R}$,

$$\begin{aligned}\mathcal{I}_p(x) &= \frac{2}{B(p+1/2,1/2)}\int_0^{\pi/2} \mathcal{I}_{-1/2}(x\sin\theta)\cos^{2p}\theta\, d\theta,\\ \mathcal{I}_{p+1}(x) &= \frac{2}{B(p+1/2,3/2)}\int_0^{\pi/2} \mathcal{I}_{1/2}(x\sin\theta)\sin^2\theta\cos^{2p}\theta\, d\theta.\end{aligned} \tag{2.13}$$

Now, using Theorem 1.3, in view of relations (1.12), we deduce that the inequality $a_2\mathcal{I}_{1/2}(x) \le 1 + a_1\mathcal{I}_{-1/2}(x)$ holds for all x real number. Thus changing, in this inequality, x with $x\sin\theta$ and multiplying both sides with $\sin^2\theta\cos^{2p}\theta$, after integration, we obtain

$$\frac{a_2}{2}B\left(p+\frac{1}{2},\frac{3}{2}\right)\mathcal{I}_{p+1}(x) \le \frac{1}{2}B\left(p+\frac{1}{2},\frac{3}{2}\right)+\frac{a_1}{2}B\left(p+\frac{1}{2},\frac{1}{2}\right)\mathcal{I}_p(x) - \frac{a_1}{2}B\left(p+\frac{3}{2},\frac{1}{2}\right)\mathcal{I}_{p+1}(x), \tag{2.14}$$

where we have used (2.13). Finally, simplifying this inequality, we obtain the required inequality. □

Remark 2.1. New, researches, which are concerned with Oppenheim's problem, are in active progress, readers can refer to [4, 10–13].

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Árpád Baricz: Faculty of Economics, Babeş-Bolyai University, RO-400591 Cluj-Napoca, Romania
Email address: bariczocsi@yahoo.com

Ling Zhu: Department of Mathematics, Zhejiang Gongshang University, Hangzhou 310018, China
Email address: zhuling0571@163.com