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Research Article

On Stability of a Functional Equation Connected with the Reynolds Operator

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Let $(X,\circ)$ be an Abelain semigroup, $g : X \to X$, and let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. We prove superstability of the functional equation $f(x \circ g(y)) = f(x)f(y)$ in the class of functions $f : X \to \mathbb{K}$. We also show some stability results of the equation in the class of functions $f : X \to \mathbb{K}^n$.

Throughout this paper n is a positive integer, $(X,\circ)$ is a commutative semigroup, $\mathbb{K}$ is either the field of reals $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$, and $g : X \to X$ is an arbitrary function. We study stability of the functional equation

$$f(x \circ g(y)) = f(x)f(y) \quad \text{for } x, y \in X, \tag{1}$$

in the class of functions $f : X \to \mathbb{K}^n$, where $(a_1,a_2,\dots,a_n)(b_1,b_2,\dots,b_n) = (a_1b_2,a_2b_2,\dots,a_nb_n)$ for $(a_1,a_2,\dots,a_n),(b_1,b_2,\dots,b_n) \in \mathbb{K}^n$. (For details concerning the problem of stability of functional equations we refer to, e.g., [1].)

Particular cases of (1) are the well-known multiplicative Cauchy equation $f(xy) = f(x)f(y)$, exponential equation $f(x+y) = f(x)f(y)$ (see, e.g., [2]) and the equation

$$f(xf(y)) = f(x)f(y). \tag{2}$$

The origin of (2) is in the averaging theory applied to turbulent fluid motion. This equation is connected with some linear operators, that is, the Reynolds operator (see [3] and [4]), the averaging operator, the multiplicatively symmetric operator (see [2]).

Ger and Šemrl in [5] (cf. [6], [7]) considered the problem of stability for the exponential equation in the class of functions mapping X into a semisimple complex commutative

Banach algebra $\mathcal{A}$. They have shown that if a mapping $f : X \to \mathcal{A}$ satisfies

$$\|f(x \circ y) - f(x)f(y)\| \le \epsilon \tag{3}$$

with some $\epsilon > 0$, then there exist a commutative C^*-algebra $\mathcal{B}$ and a continuous monomorphism Λ of $\mathcal{A}$ into $\mathcal{B}$ such that $\mathcal{B}$ is represented as a direct sum $\mathcal{B} = I \oplus J$ where I and J are closed ideals and $P\Lambda f$ is exponential, and $Q\Lambda f$ is norm-bounded where P and Q are projections corresponding to the direct sum decomposition $\mathcal{B} = I \oplus J$. We present a very short and simple proof that a similar result is valid for function $F : X \to \mathbb{K}^n$ satisfying (with any norm in $\mathbb{K}^n$) the following more general condition:

$$\|F(x \circ g(y)) - F(x)F(y)\| \le \epsilon \quad \text{for } x, y \in X. \tag{4}$$

Let us start with the following theorem, showing superstability of (1).

Theorem 1. *Let $f : X \to \mathbb{K}$ be a function satisfying*

$$|f(x \circ g(y)) - f(x)f(y)| \le \epsilon \quad \text{for } x, y \in X. \tag{5}$$

Then either f is bounded or (1) holds.

Proof. Suppose that f is unbounded. Take a sequence $(x_n : n \in \mathbb{N})$ of elements of X with $|f(x_n)| \to \infty$. Replace in (5) x by $x \circ g(x_n)$. Then for $x, y \in X$, we have

$$|f(x \circ g(x_n) \circ g(y)) - f(x \circ g(x_n))f(y)| \le \epsilon. \tag{6}$$

Next (5) implies

$$f(x) = \lim_{n\to\infty} \frac{f(x \circ g(x_n))}{f(x_n)} \quad \text{for } x \in X. \tag{7}$$

Thus from (6) and (7), for every $x, y \in X$, we obtain

$$\begin{aligned} f(x \circ g(y)) &= \lim_{n\to\infty} \frac{f(x \circ g(y) \circ g(x_n))}{f(x_n)} \\ &= \lim_{n\to\infty} \frac{f(x \circ g(x_n) \circ g(y)) - f(x \circ g(x_n))f(y)}{f(x_n)} + \lim_{n\to\infty} \frac{f(x \circ g(x_n))}{f(x_n)} f(y) \\ &= f(x)f(y). \end{aligned} \tag{8}$$

□

Remark 2. If $f : X \to \mathbb{K}$ is a bounded function satisfying (5), then

$$|f(x)| \le \frac{1 + \sqrt{1 + 4\epsilon}}{2} \quad \text{for } x \in X. \tag{9}$$

In fact, suppose that $f : X \to \mathbb{K}$ satisfies (5) and

$$M := \sup\{|f(x)| : x \in X\} > \frac{1 + \sqrt{1 + 4\epsilon}}{2}. \tag{10}$$

There exists a sequence $(x_n : n \in \mathbb{N})$ of elements of X such that $\lim_{n\to\infty} |f(x_n)| = M$. Then for sufficiently large $n \in \mathbb{N}$, we have

$$|f(x_n \circ g(x_n)) - f(x_n)^2| \geq ||f(x_n)|^2 - |f(x_n \circ g(x_n))|| \geq |f(x_n)|^2 - M. \tag{11}$$

Moreover

$$\lim_{n\to\infty} (|f(x_n)|^2 - M) = M^2 - M > \epsilon. \tag{12}$$

Thus $|f(x_n \circ g(x_n)) - f(x_n)^2| > \epsilon$ for some $n \in \mathbb{N}$, which contradicts (5).

Theorem 3. *Let $F : X \to \mathbb{K}^n$, $F = (f_1, f_2, \ldots, f_n)$ be a function satisfying (4). Then there exist ideals $I, J \subset \mathbb{K}^n$ such that $\mathbb{K}^n = I \oplus J$, PF is bounded, and QF satisfies (1) where $P : \mathbb{K}^n \to I$ and $Q : \mathbb{K}^n \to J$ are natural projections.*

Proof. Since every two norms in $\mathbb{K}^n$ are equivalent, (4) implies that there is $\eta > 0$ such that

$$\sum_{i=1}^{n} |f_i(x \circ g(y)) - f_i(x) f_i(y)| \leq \eta \|F(x \circ g(y)) - F(x)F(y)\| \leq \eta\epsilon \quad \text{for } x, y \in X. \tag{13}$$

Let $M := \{i \in \{1,\ldots,n\} : f_i$ is an unbounded solution of (1)$\}$ and $L := \{i \in \{1,\ldots,n\} : f_i$ is bounded$\}$. By Theorem 1, $L \cup M = \{1,\ldots,n\}$. Now it is enough to write $I = \{(a_1,\ldots, a_n) \in \mathbb{K}^n : a_i = 0 \text{ for } i \in M\}$ and $J = \{(a_1,\ldots,a_n) \in \mathbb{K}^n : a_i = 0 \text{ for } i \in L\}$. □

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