

*Research Article*

## One Method for Proving Inequalities by Computer

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We consider a numerical method for proving a class of analytical inequalities via mini-max rational approximations. All numerical calculations in this paper are given by Maple computer program.

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### 1. Some particular inequalities

In this section we prove two new inequalities given in Theorems 1.2 and 1.10. While proving these theorems we use a method for inequalities of the form  $f(x) \geq 0$ , for the continuous function  $f : [a, b] \rightarrow R$ .

**1.1.** Let us consider some inequalities for the gamma function which is defined by the integral

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt \quad (1.1)$$

which converges for  $Re(z) > 0$ . In [1], the following statement is proved.

**LEMMA 1.1.** For  $x \in [0, 1]$  the following inequalities are true:

$$\begin{aligned} \Gamma(x+1) &< x^2 - \frac{7}{4}x + \frac{9}{5}, \\ (x+2)\Gamma(x+1) &> \frac{9}{5}. \end{aligned} \quad (1.2)$$

The previous statement [1, Lemma 4.1] is proved by the approximative formula for the gamma function  $\Gamma(x + 1)$  by the polynomial of the fifth order:

$$P_5(x) = -0.1010678x^5 + 0.4245549x^4 - 0.6998588x^3 + 0.9512363x^2 - 0.5748646x + 1 \tag{1.3}$$

which has the numerical bound of the absolute error  $\varepsilon = 5 \cdot 10^{-5}$  for values of argument  $x \in [0, 1]$  [2, Formula 6.1.35, page 257].

In the Maple computer program we use numapprox package [3] for obtaining the minimax rational approximation  $R(x) = P_m(x)/Q_n(x)$  of the continuous function  $f(x)$  over segment  $[a, b]$  ( $m$  is the degree of the polynomial  $P_m(x)$  and  $n$  is the degree of the polynomial  $Q_n(x)$ ). Let  $\varepsilon(x) = f(x) - R(x)$  be the error function of an approximation over segment  $[a, b]$ . Numerical computation of  $R(x)$  is given by Maple command

$$R := \text{minimax} (f(x), x = a..b, [m, n], \text{“err”}). \tag{1.4}$$

The result of the previous command is the minimax rational approximation  $R(x)$  and an estimate for the value of the minimax norm of  $\varepsilon(x)$  as the number err (computation is realized without the weight function). With the Maple minimax command a realization of the Remez algorithm is given [4]. If it is not possible to determine minimax approximation in Maple program, there appears a message that it is necessary to increase decimal degrees.

Let us assume that for the function  $f(x)$  the minimax rational approximation  $R(x)$  is determined. Then, in the Maple the same estimate for the minimax norm of the error function  $\varepsilon(x)$  is given by the command

$$\text{err} := \text{infnorm} (\varepsilon(x), x = a..b). \tag{1.5}$$

The result of the previous command is number  $\text{err} := \max_{x \in [a, b]} |f(x) - R(x)|$ . Practically for the bound of the absolute error function  $|\varepsilon(x)|$  we use  $\varepsilon = \text{err}$ . Let us remark that the bound of the absolute error  $\varepsilon$  is a numerical bound in the sense of [5] (approximation errors on page 4), see also [6, 7].

Let us notice, as it is emphasized by [1, Remark 4.2], that for the proof of Lemma 1.1, it is possible to use other polynomial approximations (of lower degree) of the functions  $\Gamma(x + 1/2)$  and  $\Gamma(x + 1)$  for values  $x \in [0, 1]$ . That idea is implemented in the next statement for Kurepa’s function which is defined by the integral

$$K(z) = \int_0^\infty e^{-t} \frac{t^z - 1}{t - 1} dt, \tag{1.6}$$

which converges for  $\text{Re}(z) > 0$  [8]. It is possible to make an analytical continuation of Kurepa’s function  $K(z)$  to the meromorphic function with simple poles at  $z = -1$  and  $z = -n$  ( $n \geq 3$ ). Practically for computation values of Kurepa’s function we use the following formula:

$$K(z) = \frac{\text{Ei}(1) + i\pi}{e} + \frac{(-1)^z \Gamma(1 + z) \Gamma(-z, -1)}{e} \tag{1.7}$$

which is cited in [9]. In the previous formula  $Ei(z)$  and  $\Gamma(z, a)$  are the exponential integral and the incomplete gamma function, respectively. Let us numerically prove the following statement.

**THEOREM 1.2.** *For  $x \in [0, 1]$ , the following inequality is true:*

$$K(x) \leq K'(0)x, \tag{1.8}$$

where  $K'(0) = 1.432205735\dots$  is the best possible constant.

*Proof.* Let us define the function  $f(x) = K'(0)x - K(x)$  for  $x \in [0, 1]$ . Let us prove  $f(x) \geq 0$  for  $x \in [0, 1]$ . Let us consider the continuous function

$$g(x) = \begin{cases} \alpha & : x = 0, \\ \frac{f(x)}{x^2} & : x \in (0, 1], \end{cases} \tag{1.9}$$

for constant  $\alpha = -K''(0)/2$ . Let us notice that the constant

$$\alpha = -\frac{K''(0)}{2} = \lim_{x \rightarrow 0^+} \frac{K'(0) - K'(x)}{2x} = \lim_{x \rightarrow 0^+} \frac{K'(0)x - K(x)}{x^2} = \lim_{x \rightarrow 0^+} \frac{f(x)}{x^2} \tag{1.10}$$

is determined in the sense that  $g(x)$  is a continuous function over segment  $[0, 1]$ . The numerical value of that constant is

$$\alpha = -\frac{1}{2} \int_0^\infty e^{-t} \frac{\log^2 t}{t-1} dt = 0.963321189\dots (> 0). \tag{1.11}$$

Using Maple we determine the minimax rational approximation for the function  $g(x)$  by the polynomial of the first order:

$$P_1(x) = -0.531115454x + 0.921004887 \tag{1.12}$$

which has the bound of the absolute error  $\varepsilon_1 = 0.04232$  for values  $x \in [0, 1]$ . The following is true:

$$g(x) - (P_1(x) - \varepsilon_1) \geq 0, \quad P_1(x) - \varepsilon_1 > 0, \tag{1.13}$$

for values  $x \in [0, 1]$ . Hence, for  $x \in [0, 1]$  it is true that  $g(x) > 0$  and  $f(x) \geq 0$  as well.  $\square$

*Remark 1.3.* Numerical values of constants  $K'(0)$  and  $K''(0)$  are determined by Maple program. The numerical value of  $K'(0)$  was first determined by Slavić in [10].

**COROLLARY 1.4.** *Petojević in [11] used an auxiliary result  $K(x) \leq 9/5x$ , for values  $x \in [0, 1]$ , from [1, Lemma 4.3], for proving new inequalities for Kurepa's function. Based on the previous theorem, all appropriate inequalities from [11] can be improved with a simple change of fraction  $9/5$  with constant  $K'(0)$ .*

**1.2.** Mitrinović and Vasić considered in [12] the lower bound of the arc sin-function, which belongs to Shafer. Namely, the following statement is true.

**THEOREM 1.5.** For  $0 \leq x \leq 1$  the following inequalities are true:

$$\frac{3x}{2 + \sqrt{1-x^2}} \leq \frac{6(\sqrt{1+x} - \sqrt{1-x})}{4 + \sqrt{1+x} + \sqrt{1-x}} \leq \arcsin x. \tag{1.14}$$

Fink proved the following statement in paper [13].

**THEOREM 1.6.** For  $0 \leq x \leq 1$ , the following inequalities are true:

$$\frac{3x}{2 + \sqrt{1-x^2}} \leq \arcsin x \leq \frac{\pi x}{2 + \sqrt{1-x^2}}. \tag{1.15}$$

Malešević proved the following statement in [14].

**THEOREM 1.7.** For  $0 \leq x \leq 1$ , the following inequalities are true:

$$\frac{3x}{2 + \sqrt{1-x^2}} \leq \arcsin x \leq \frac{(\pi/(\pi-2))x}{2/(\pi-2) + \sqrt{1-x^2}} \leq \frac{\pi x}{2 + \sqrt{1-x^2}}. \tag{1.16}$$

*Remark 1.8.* The upper bound of the arc sin-function

$$\phi(x) = \frac{(\pi/(\pi-2))x}{2/(\pi-2) + \sqrt{1-x^2}} \tag{1.17}$$

is determined in paper [14] by  $\lambda$ -method of Mitrinović and Vasić [12].

Zhu proved the following statement in [15].

**THEOREM 1.9.** For  $x \in [0, 1]$ , the following inequalities are true:

$$\begin{aligned} \frac{3x}{2 + \sqrt{1-x^2}} &\leq \frac{6(\sqrt{1+x} - \sqrt{1-x})}{4 + \sqrt{1+x} + \sqrt{1-x}} \leq \arcsin x \\ &\leq \frac{\pi(\sqrt{2}+1/2)(\sqrt{1+x} - \sqrt{1-x})}{4 + \sqrt{1+x} + \sqrt{1-x}} \leq \frac{\pi x}{2 + \sqrt{1-x^2}}. \end{aligned} \tag{1.18}$$

In this paper we give an improved statement of Zhu. Let us numerically prove the following statement.

**THEOREM 1.10.** For  $x \in [0, 1]$ , the following inequalities are true:

$$\begin{aligned} \frac{3x}{2 + \sqrt{1-x^2}} &\leq \frac{6(\sqrt{1+x} - \sqrt{1-x})}{4 + \sqrt{1+x} + \sqrt{1-x}} \leq \arcsin x \\ &\leq \frac{(\pi(2 - \sqrt{2})/(\pi - 2\sqrt{2}))(\sqrt{1+x} - \sqrt{1-x})}{(\sqrt{2}(4 - \pi)/(\pi - 2\sqrt{2})) + \sqrt{1+x} + \sqrt{1-x}} \\ &\leq \frac{\pi(\sqrt{2}+1/2)(\sqrt{1+x} - \sqrt{1-x})}{4 + \sqrt{1+x} + \sqrt{1-x}} \leq \frac{\pi x}{2 + \sqrt{1-x^2}}. \end{aligned} \tag{1.19}$$

*Proof.* Inequality

$$\frac{\pi(\sqrt{2}+1/2)(\sqrt{1+x}-\sqrt{1-x})}{4+\sqrt{1+x}+\sqrt{1-x}} \geq \frac{(\pi(2-\sqrt{2})/(\pi-2\sqrt{2}))(\sqrt{1+x}-\sqrt{1-x})}{\sqrt{2}(4-\pi)/(\pi-2\sqrt{2})+\sqrt{1+x}+\sqrt{1-x}}, \quad (1.20)$$

for  $x \in [0, 1]$ , is directly verifiable by algebraic manipulations. Let us define the following function:

$$f(x) = \frac{(\pi(2-\sqrt{2})/(\pi-2\sqrt{2}))(\sqrt{1+x}-\sqrt{1-x})}{(\sqrt{2}(4-\pi)/(\pi-2\sqrt{2})+\sqrt{1+x}+\sqrt{1-x})} - \arcsin x, \quad (1.21)$$

for  $x \in [0, 1]$ . Let us prove  $f(x) \geq 0$  for  $x \in [0, 1]$ , that is,  $f(\sin t) \geq 0$  for  $t \in [0, \pi/2]$ . Let us define the function

$$g(t) = \begin{cases} \alpha & : t = 0, \\ \frac{f(\sin t)}{t^3(\pi/2-t)} & : t \in \left(0, \frac{\pi}{2}\right), \\ \beta & : t = \frac{\pi}{2}, \end{cases} \quad (1.22)$$

where  $\alpha$  and  $\beta$  are constants determined with limits:

$$\alpha = \lim_{t \rightarrow 0^+} \frac{f(\sin t)}{t^3(\pi/2-t)} = \frac{(4+\sqrt{2})\pi - 12\sqrt{2}}{(24-12\sqrt{2})\pi^2} > 0, \quad (1.23)$$

$$\beta = \lim_{t \rightarrow \pi/2^-} \frac{f(\sin t)}{t^3(\pi/2-t)} = \frac{(16\sqrt{2}-16) + (8-4\sqrt{2})\pi - \sqrt{2}\pi^2}{(2\sqrt{2}-2)\pi^3} > 0.$$

The previously determined function  $g(t)$  is continuous over  $[0, \pi/2]$ . Using Maple we determine the minimax rational approximation for the function  $g(t)$  by the polynomial of the first order:

$$P_1(t) = 0.000410754 t + 0.000543606 \quad (1.24)$$

which has the bound of the absolute error  $\varepsilon_1 = 1.408 \cdot 10^{-5}$  for values  $t \in [0, \pi/2]$ . It is true that

$$g(t) - (P_1(t) - \varepsilon_1) \geq 0, \quad P_1(t) - \varepsilon_1 > 0, \quad (1.25)$$

for values  $t \in [0, \pi/2]$ . Hence, for  $t \in [0, \pi/2]$  it is true that  $g(t) > 0$  and therefore  $f(\sin t) \geq 0$  for  $t \in [0, \pi/2]$ . Finally  $f(x) \geq 0$  for  $x \in [0, 1]$ .  $\square$

*Remark 1.11.* The paper [16] considers the upper bound of the arc sin-function

$$\varphi(x) = \frac{(\pi(2-\sqrt{2})/(\pi-2\sqrt{2}))(\sqrt{1+x}-\sqrt{1-x})}{\sqrt{2}(4-\pi)/(\pi-2\sqrt{2})+\sqrt{1+x}+\sqrt{1-x}} \quad (1.26)$$

via  $\lambda$ -method of Mitrinović and Vasić [12].

## 2. A method for proving inequalities

In this section we expose a numerical method for proving inequalities in following form:

$$f(x) \geq 0 \quad (2.1)$$

for the continuous function  $f : [a, b] \rightarrow R$ . Let us assume that  $x = a$  is the root of the order  $n$  and  $x = b$  is the root of the order  $m$  of the function  $f(x)$  (if  $x = a$  is not the root, then we determine that  $n = 0$ , that is, if  $x = b$  is not the root, then we determine that  $m = 0$ ). The method is based on the first assumption that there exist finite and nonzero limits:

$$\alpha = \lim_{x \rightarrow a^+} \frac{f(x)}{(x-a)^n(b-x)^m}, \quad \beta = \lim_{x \rightarrow b^-} \frac{f(x)}{(x-a)^n(b-x)^m}. \quad (2.2)$$

If for the function  $f(x)$  (over extended domain of  $[a, b]$ ) at the point  $x = a$  there is an approximation of the function by Taylor polynomial of  $n$ th order and at point  $x = b$  there is an approximation of the function by Taylor polynomial of  $m$ th order, then

$$\alpha = \frac{f^{(n)}(a)}{n!(b-a)^m}, \quad \beta = (-1)^m \frac{f^{(m)}(b)}{m!(b-a)^n}. \quad (2.3)$$

Let us define the function

$$g(x) = g_{a,b}^f(x) = \begin{cases} \alpha & : x = a, \\ \frac{f(x)}{(x-a)^n(b-x)^m} & : x \in (a, b), \\ \beta & : x = b \end{cases} \quad (2.4)$$

which is continuous over segment  $[a, b]$ . For proving inequality (2.1) we use the equivalence

$$f(x) \geq 0 \iff g(x) \geq 0, \quad (2.5)$$

which is true for all values  $x \in [a, b]$ . Thus if  $\alpha < 0$  or  $\beta < 0$ , the inequality (2.1) is not true. Hence, we consider only the cases  $\alpha > 0$  and  $\beta > 0$ . Let us notice that if the function  $f(x)$  has only roots at some end-points of the segment  $[a, b]$ , then (2.5) becomes  $f(x) \geq 0$  if and only if  $g(x) > 0$  for  $x \in [a, b]$ . The second assumption of the method is that there is the minimax (polynomial) rational approximation  $R(x) = P_m(x)/Q_n(x)$  of the function  $g(x)$  over  $[a, b]$ , which has the bound of the absolute error  $\varepsilon > 0$  such that

$$R(x) - \varepsilon > 0, \quad (2.6)$$

for  $x \in [a, b]$ . Then  $g(x) > 0$ , for  $x \in [a, b]$ . Finally, on the basis (2.5), we can conclude that  $f(x) \geq 0$ , for  $x \in [a, b]$ .

Let us emphasize that the minimax (polynomial) rational approximation of the function  $g(x)$  over  $[a, b]$  can be computed by Remez algorithm (via Maple minimax function [3]). For applying Remez algorithm to the function  $g(x)$ , it is sufficient that the function

is continuous. If  $g(x)$  is a differentiable function, then the second Remez algorithm is applicable [17]. According to the previous consideration, the problem of proving inequality (2.1), in some cases, becomes a problem of existence of the minimax (polynomial) rational approximation  $R(x)$  for  $g = g_{a,b}^f(x)$  function with the bound of the absolute error  $\varepsilon > 0$  such that (2.6) is true. Let us notice that the problem of verification of inequality (2.6) reduces to boolean combination of the polynomial inequalities.

Let us consider practical usages of the previously described method. For the function of one variable the previous method can be applied to the inequality of the infinite interval using the appropriate substitute variable, which transforms inequality to the new one over the finite interval. Next, if some limits in (2.2) are infinite, then, in some cases, the initial inequality can be transformed, by the means of appropriate substitute variable, to the case when the both limits in (2.2) are finite and nonzero.

The advantage of the described method is that for the function  $f(x)$  we do not have to use some regularities concerning derivatives. Besides, the present method enables us to obtain computer-assisted proofs of appropriate inequalities, which have been published in journals which consider these topics. With this method we have obtained numerical proofs of the appropriate inequalities from the following articles [18–28].

Finally, let us emphasize that the mentioned method can be extended and applied to inequalities for multivariate functions by the means of appropriate multivariate minimax rational approximations.

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