

# INVERSES OF NEW HILBERT-PACHPATTE-TYPE INEQUALITIES

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We establish a new inequality of Hilbert type for a finite double number of nonnegative sequences of real numbers and some interrelated results, which are inverse and general forms of Pachpatte's and Handley's results. An integral version and some interrelated results are also obtained. These results provide some new estimates on such types of inequalities.

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## 1. Introduction

The various generalizations and sharpenings of Hilbert's double series inequality and its integral version are obtained by Pachpatte, Handley et al., Gao et al. (see [1, 3, 5, 7, 10, 12, 16–18]). An elegant survey on this kind of inequalities was provided by Yang and Rassias (see [13]). Moreover, Pachpatte [9] established a new Hilbert-type inequality and its integral version as follows.

**THEOREM 1.1.** *Let  $\{a_m\}$ ,  $\{b_n\}$  be two nonnegative sequences of real numbers defined for  $m = 1, 2, \dots, k$  and  $n = 1, 2, \dots, r$  with  $a_0 = b_0 = 0$ , and let  $\{p_m\}$ ,  $\{q_n\}$  be two positive sequences of real numbers defined for  $m = 1, 2, \dots, k$ ,  $n = 1, 2, \dots, r$ , where  $k, r$  are natural numbers. Define  $P_m = \sum_{s=1}^m p_s$  and  $Q_n = \sum_{t=1}^n q_t$ . Let  $\phi$  and  $\psi$  be two real-valued nonnegative, convex, and submultiplicative functions defined on  $\mathbb{R}_+ = [0, \infty)$ . Then*

$$\sum_{m=1}^k \sum_{n=1}^r \frac{\phi(a_m)\psi(b_n)}{m+n} \leq M(k,r) \left( \sum_{m=1}^k (k-m+1) \left( p_m \phi \left( \frac{\nabla a_m}{p_m} \right) \right)^2 \right)^{1/2} \times \left( \sum_{n=1}^r (r-n+1) \left( q_n \psi \left( \frac{\nabla b_n}{q_n} \right) \right)^2 \right)^{1/2}, \quad (1.1)$$

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where

$$M(k, r) = \frac{1}{2} \left( \sum_{m=1}^k \left( \frac{\phi(P_m)}{P_m} \right)^2 \right)^{1/2} \left( \sum_{n=1}^r \left( \frac{\phi(Q_n)}{Q_n} \right)^2 \right)^{1/2}, \quad (1.2)$$

and  $\nabla a_m = a_m - a_{m-1}$ ,  $\nabla b_n = b_n - b_{n-1}$ .

**THEOREM 1.2.** Let  $f \in C^1[[0, x], \mathbb{R}^+]$ ,  $g \in C^1[[0, y], \mathbb{R}^+]$  with  $f(0) = g(0) = 0$  and let  $p(\sigma)$ ,  $q(\tau)$  be two positive functions defined for  $\sigma \in [0, x]$  and  $\tau \in [0, y]$ . Let  $P(s) = \int_0^s p(\sigma) d\sigma$  and  $Q(t) = \int_0^t q(\tau) d\tau$  for  $s \in [0, x]$  and  $t \in [0, y]$ , where  $x, y$  are positive real numbers. Let  $\phi$ , and  $\psi$  be as in Theorem 1.1. Then

$$\int_0^x \int_0^y \frac{\phi(f(s))\psi(g(t))}{s+t} ds dt \leq L(x, y) \left( \int_0^x (x-s) \left( p(s) \phi \left( \frac{f'(s)}{p(s)} \right) \right)^2 ds \right)^{1/2} \times \left( \int_0^y (y-t) \left( q(t) \phi \left( \frac{g'(t)}{q(t)} \right) \right)^2 dt \right)^{1/2}, \quad (1.3)$$

where

$$L(x, y) = \frac{1}{2} \left( \int_0^x \left( \frac{\phi(P(s))}{P(s)} \right)^2 ds \right)^{1/2} \left( \int_0^y \left( \frac{\phi(Q(t))}{Q(t)} \right)^2 dt \right)^{1/2}, \quad (1.4)$$

and  $'$  denotes the derivative of a function.

In [4], Handley et al. gave general versions of inequalities (1.1) and (1.3) as follows.

**THEOREM 1.3.** Let  $\{a_{i,m_i}\}$  ( $i = 1, 2, \dots, n$ ) be  $n$  sequences of nonnegative real numbers defined for  $m_i = 1, 2, \dots, k_i$  with  $a_{1,0} = a_{2,0} = \dots = a_{n,0} = 0$ , and let  $\{p_{i,m_i}\}$  be  $n$  sequences of positive real numbers defined for  $m_i = 1, 2, \dots, k_i$ , where  $k_i$  ( $i = 1, 2, \dots, n$ ) are natural numbers. Set  $P_{i,m_i} = \sum_{s=1}^{m_i} p_{i,s}$ . Let  $\phi_i$  ( $i = 1, 2, \dots, n$ ) be  $n$  real-valued nonnegative convex and sub-multiplicative functions defined on  $\mathbb{R}_+$ . Let  $\alpha_i \in (0, 1)$ , and set  $\alpha'_i = 1 - \alpha_i$  ( $i = 1, 2, \dots, n$ ),  $\alpha = \sum_{i=1}^n \alpha_i$ , and  $\alpha' = \sum_{i=1}^n \alpha'_i = n - \alpha$ . Then

$$\sum_{m_1=1}^{k_1} \dots \sum_{m_n=1}^{k_n} \frac{\prod_{i=1}^n \phi_i(a_{i,m_i})}{\left( \sum_{i=1}^n \alpha'_i m_i \right)^{\alpha'}} \leq M(k_1, \dots, k_n) \prod_{i=1}^n \left( \sum_{m_i=1}^{k_i} (k_i - m_i + 1) \left( p_{i,m_i} \phi_i \left( \frac{\nabla a_{i,m_i}}{P_{i,m_i}} \right) \right)^{1/\alpha'_i} \right)^{\alpha_i}, \quad (1.5)$$

where

$$M(k_1, \dots, k_n) = \frac{1}{(\alpha')^{\alpha'}} \prod_{i=1}^n \left( \sum_{m_i=1}^{k_i} \left( \frac{\phi_i(P_{i,m_i})}{P_{i,m_i}} \right)^{1/\alpha'_i} \right)^{\alpha'_i}, \quad (1.6)$$

and  $\nabla a_{i,m_i} = a_{i,m_i} - a_{i,m_i-1}$  ( $i = 1, 2, \dots, n$ ).

**THEOREM 1.4.** Let  $f_i \in C^1[[0, k_i], \mathbb{R}_+]$ ,  $i = 1, 2, \dots, n$ , with  $f_i(0) = 0$  ( $i = 1, 2, \dots, n$ ), let  $p_i(\sigma_i)$  be  $n$  positive functions defined for  $\sigma_i \in [0, x_i]$  ( $i = 1, \dots, n$ ). Set  $P_i(s_i) = \int_0^{s_i} p_i(\sigma_i) d\sigma_i$  for  $s_i \in [0, x_i]$ , where  $x_i$  are positive real numbers. Let  $\phi_i, \alpha_i, \alpha'_i, \alpha$ , and  $\alpha'$  be as in Theorem 1.3. Then

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n \phi_i(f_i(s_i))}{\left(\sum_{i=1}^n \alpha'_i s_i\right)^{\alpha'}} ds_1 \cdots ds_n \leq L(x_1, \dots, x_n) \prod_{i=1}^n \left( \int_0^{x_i} (x_i - s_i) \left( p_i(s_i) \phi_i \left( \frac{f'_i(s_i)}{P_i(s_i)} \right) \right)^{1/\alpha_i} ds_i \right)^{\alpha_i}, \tag{1.7}$$

where

$$L(x_1, \dots, x_n) = \frac{1}{(\alpha')^{\alpha'}} \prod_{i=1}^n \left( \int_0^{x_i} \left( \frac{\phi_i(P_i(s_i))}{P_i(s_i)} \right)^{1/\alpha'_i} ds_i \right)^{\alpha'_i}. \tag{1.8}$$

The main purpose of the present paper is to establish their reverse versions, which are more extensive results for this type of inequalities. Our main results are given in the following theorems.

**THEOREM 1.5.** Let  $\{a_{s_i, t_i, m_{s_i}, m_{t_i}}\}$  ( $i = 1, 2, \dots, n$ ) be  $n$  sequences of nonnegative numbers defined for  $m_{s_i} = 1, 2, \dots, k_{s_i}$ ,  $m_{t_i} = 1, 2, \dots, k_{t_i}$ , with  $a_{s_i, t_i, 0, m_{t_i}} = a_{s_i, t_i, m_{s_i}, 0} = 0$  ( $i = 1, 2, \dots, n$ ), where  $k_{s_i}$  and  $k_{t_i}$  ( $i = 1, 2, \dots, n$ ) are natural numbers. Let  $\{p_{s_i, t_i, m_{s_i}, m_{t_i}}\}$  be  $n$  sequences of positive real numbers defined for  $m_{s_i}, m_{t_i}$ . Set

$$P_{s_i, t_i, m_{s_i}, m_{t_i}} = \sum_{m_{\xi_i}=1}^{m_{s_i}} \sum_{m_{\eta_i}=1}^{m_{t_i}} p_{s_i, t_i, m_{\xi_i}, m_{\eta_i}} \quad (i = 1, 2, \dots, n). \tag{1.9}$$

Let  $\phi_i$  ( $i = 1, 2, \dots, n$ ) be  $n$  real-valued nonnegative concave and supermultiplicative functions defined on  $\mathbb{R}_+$ . Let  $\alpha_i \in (1, \infty)$ . Set  $\alpha'_i = 1 - \alpha_i$  ( $i = 1, 2, \dots, n$ ),  $\alpha = \sum_{i=1}^n \alpha_i$ , and  $\alpha' = \sum_{i=1}^n \alpha'_i = n - \alpha$ . Define operators

$$\begin{aligned} \nabla_1 a_{s_i, t_i, m_{s_i}, m_{t_i}} &= a_{s_i, t_i, m_{s_i}, m_{t_i}} - a_{s_i, t_i, m_{s_i}-1, m_{t_i}}, \\ \nabla_2 a_{s_i, t_i, m_{s_i}, m_{t_i}} &= a_{s_i, t_i, m_{s_i}, m_{t_i}} - a_{s_i, t_i, m_{s_i}, m_{t_i}-1}, \\ \nabla_2 \nabla_1 a_{s_i, t_i, m_{s_i}, m_{t_i}} &= \nabla_2 ( \nabla_1 a_{s_i, t_i, m_{s_i}, m_{t_i}} ). \end{aligned} \tag{1.10}$$

Then

$$\begin{aligned} &\sum_{m_{s_1}=1}^{k_{s_1}} \sum_{m_{t_1}=1}^{k_{t_1}} \cdots \sum_{m_{s_n}=1}^{k_{s_n}} \sum_{m_{t_n}=1}^{k_{t_n}} \frac{\prod_{i=1}^n \phi_i(a_{s_i, t_i, m_{s_i}, m_{t_i}})}{\left( (1/\alpha') \sum_{i=1}^n \alpha'_i m_{s_i} m_{t_i} \right)^{\alpha'}} \\ &\geq C(k_{s_1} k_{t_1}, \dots, k_{s_n} k_{t_n}) \\ &\times \prod_{i=1}^n \left( \sum_{m_{s_i}=1}^{k_{s_i}} \sum_{m_{t_i}=1}^{k_{t_i}} (k_{s_i} - m_{s_i} + 1) (k_{t_i} - m_{t_i} + 1) \left( p_{s_i, t_i, m_{s_i}, m_{t_i}} \phi_i \left( \frac{\nabla_2 \nabla_1 a_{s_i, t_i, m_{s_i}, m_{t_i}}}{P_{s_i, t_i, m_{s_i}, m_{t_i}}} \right) \right)^{1/\alpha_i} \right)^{\alpha_i}, \end{aligned} \tag{1.11}$$

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where

$$C(k_{s_1}k_{t_1}, \dots, k_{s_n}k_{t_n}) = \prod_{i=1}^n \left( \sum_{m_{s_i}=1}^{k_{s_i}} \sum_{m_{t_i}=1}^{k_{t_i}} \left( \frac{\phi_i(P_{s_i, t_i, m_{s_i}, m_{t_i}})}{P_{s_i, t_i, m_{s_i}, m_{t_i}}} \right)^{1/\alpha'_i} \right)^{\alpha'_i}. \quad (1.12)$$

**THEOREM 1.6.** Let  $f_i(s_i, t_i)$  ( $i = 1, 2, \dots, n$ ) be real-valued continuous functions defined on  $[0, x_i] \times [0, y_i]$ , where  $x_i \in (0, \infty)$ ,  $y_i \in (0, \infty)$ , and with  $f_i(0, t_i) = f_i(s_i, 0) = 0$  ( $i = 1, 2, \dots, n$ ). Let  $p_i(\sigma_i)$  and  $q_i(\tau_i)$  be positive continuous functions defined for  $\sigma_i \in (0, s_i)$ ,  $\tau_i \in (0, t_i)$ . Set

$$P_i(s_i, t_i) = \int_0^{t_i} \int_0^{s_i} p_i(\sigma_i) q_i(\tau_i) d\sigma_i d\tau_i. \quad (1.13)$$

For the function  $f_i(s_i, t_i)$ , denote the partial derivatives  $(\partial/\partial s_i) f_i(s_i, t_i)$ ,  $(\partial/\partial t_i) f_i(s_i, t_i)$ , and  $(\partial^2/\partial s_i \partial t_i) f_i(s_i, t_i)$  by  $D_1 f_i(s_i, t_i)$ ,  $D_2 f_i(s_i, t_i)$ , and  $D_2 D_1 f_i(s_i, t_i) = D_1 D_2 f_i(s_i, t_i)$ , respectively. Let  $\phi_i$ ,  $\alpha_i$ ,  $\alpha'_i$ ,  $\alpha$ , and  $\alpha'$  be as in Theorem 1.5. Then

$$\begin{aligned} & \int_0^{x_1} \int_0^{y_1} \cdots \int_0^{x_n} \int_0^{y_n} \prod_{i=1}^n \frac{\phi_i(f_i(s_i, t_i))}{((1/\alpha') \sum_{i=1}^n \alpha'_i s_i t_i)^{\alpha'}} ds_1 dt_1 \cdots ds_n dt_n \\ &= G(x_1 y_1, \dots, x_n y_n) \\ & \times \prod_{i=1}^n \left( \int_0^{x_i} \int_0^{y_i} (x_i - s_i)(y_i - t_i) \left( p_i(s_i) q_i(t_i) \cdot \phi_i \left( \frac{D_2 D_1 f_i(s_i, t_i)}{p_i(s_i) q_i(t_i)} \right) \right)^{1/\alpha_i} ds_i dt_i \right)^{\alpha_i}, \end{aligned} \quad (1.14)$$

where

$$G(x_1 y_1, \dots, x_n y_n) = \prod_{i=1}^n \left( \int_0^{x_i} \int_0^{y_i} \left( \frac{\phi_i(P_i(s_i, t_i))}{P_i(s_i, t_i)} \right)^{1/\alpha'_i} ds_i dt_i \right)^{\alpha'_i}. \quad (1.15)$$

These results provide new estimates on these types of inequalities.

## 2. Proofs and remarks

*Proof of Theorem 1.5.* From the operators defined in Theorem 1.5, we obtain

$$\begin{aligned} a_{s_i, t_i, m_{s_i}, m_{t_i}} &= \sum_{m_{\xi_i}=1}^{m_{s_i}} \{ a_{s_i, t_i, m_{\xi_i}, m_{t_i}} - a_{s_i, t_i, m_{\xi_i}-1, m_{t_i}} \} \\ &= \sum_{m_{\xi_i}=1}^{m_{s_i}} \sum_{m_{\eta_i}=1}^{m_{t_i}} \{ a_{s_i, t_i, m_{\xi_i}, m_{\eta_i}} - a_{s_i, t_i, m_{\xi_i}-1, m_{\eta_i}} - a_{s_i, t_i, m_{\xi_i}, m_{\eta_i}-1} + a_{s_i, t_i, m_{\xi_i}-1, m_{\eta_i}-1} \} \\ &= \sum_{m_{\xi_i}=1}^{m_{s_i}} \sum_{m_{\eta_i}=1}^{m_{t_i}} \nabla_2 \nabla_1 a_{s_i, t_i, m_{\xi_i}, m_{\eta_i}}. \end{aligned} \quad (2.1)$$

From (2.1) and the hypotheses of Theorem 1.5, and in view of Jensen's inequality and inverse Hölder's inequality [6], we obtain

$$\begin{aligned}
& \phi_i(a_{s_i, t_i, m_{s_i}, m_{t_i}}) \\
&= \phi_i\left(\frac{P_{s_i, t_i, m_{s_i}, m_{t_i}} \sum_{m_{\xi_i}=1}^{m_{s_i}} \sum_{m_{\eta_i}=1}^{m_{t_i}} P_{s_i, t_i, m_{\xi_i}, m_{\eta_i}} (\nabla_2 \nabla_1 a_{s_i, t_i, m_{\xi_i}, m_{\eta_i}} / p_{s_i, t_i, m_{\xi_i}, m_{\eta_i}})}{\sum_{m_{\xi_i}=1}^{m_{s_i}} \sum_{m_{\eta_i}=1}^{m_{t_i}} P_{s_i, t_i, m_{\xi_i}, m_{\eta_i}}}\right) \\
&\geq \phi_i(P_{s_i, t_i, m_{s_i}, m_{t_i}}) \cdot \phi_i\left(\frac{\sum_{m_{\xi_i}=1}^{m_{s_i}} \sum_{m_{\eta_i}=1}^{m_{t_i}} P_{s_i, t_i, m_{\xi_i}, m_{\eta_i}} (\nabla_2 \nabla_1 a_{s_i, t_i, m_{\xi_i}, m_{\eta_i}} / p_{s_i, t_i, m_{\xi_i}, m_{\eta_i}})}{\sum_{m_{\xi_i}=1}^{m_{s_i}} \sum_{m_{\eta_i}=1}^{m_{t_i}} P_{s_i, t_i, m_{\xi_i}, m_{\eta_i}}}\right) \\
&\geq \frac{\phi_i(P_{s_i, t_i, m_{s_i}, m_{t_i}})}{P_{s_i, t_i, m_{s_i}, m_{t_i}}} \left(\sum_{m_{\xi_i}=1}^{m_{s_i}} \sum_{m_{\eta_i}=1}^{m_{t_i}} P_{s_i, t_i, m_{\xi_i}, m_{\eta_i}} \phi_i\left(\frac{\nabla_2 \nabla_1 a_{s_i, t_i, m_{\xi_i}, m_{\eta_i}}}{P_{s_i, t_i, m_{\xi_i}, m_{\eta_i}}}\right)\right) \\
&\geq \frac{\phi_i(P_{s_i, t_i, m_{s_i}, m_{t_i}})}{P_{s_i, t_i, m_{s_i}, m_{t_i}}} (m_{s_i} m_{t_i})^{\alpha'_i} \times \left(\sum_{m_{\xi_i}=1}^{m_{s_i}} \sum_{m_{\eta_i}=1}^{m_{t_i}} \left(P_{s_i, t_i, m_{\xi_i}, m_{\eta_i}} \phi_i\left(\frac{\nabla_2 \nabla_1 a_{s_i, t_i, m_{\xi_i}, m_{\eta_i}}}{P_{s_i, t_i, m_{\xi_i}, m_{\eta_i}}}\right)\right)^{1/\alpha_i}\right)^{\alpha_i}. \tag{2.2}
\end{aligned}$$

On the other hand, noticing  $\alpha'_i < 0$  ( $i = 1, 2, \dots, n$ ),  $\lambda_i > 0$  ( $i = 1, 2, \dots, n$ ), and applying the well-known means inequality, we have

$$\prod_{i=1}^n \lambda_i^{\alpha'_i} \geq \left(\frac{1}{\alpha'} \sum_{i=1}^n \alpha'_i \lambda_i\right)^{\alpha'}. \tag{2.3}$$

Hence

$$\begin{aligned}
& \frac{\prod_{i=1}^n \phi_i(a_{s_i, t_i, m_{s_i}, m_{t_i}})}{((1/\alpha') \sum_{i=1}^n \alpha'_i m_{s_i} m_{t_i})^{\alpha'}} \\
&\geq \prod_{i=1}^n \frac{\phi_i(P_{s_i, t_i, m_{s_i}, m_{t_i}})}{P_{s_i, t_i, m_{s_i}, m_{t_i}}} \left(\sum_{m_{\xi_i}=1}^{m_{s_i}} \sum_{m_{\eta_i}=1}^{m_{t_i}} \left(P_{s_i, t_i, m_{\xi_i}, m_{\eta_i}} \phi_i\left(\frac{\nabla_2 \nabla_1 a_{s_i, t_i, m_{\xi_i}, m_{\eta_i}}}{P_{s_i, t_i, m_{\xi_i}, m_{\eta_i}}}\right)\right)^{1/\alpha_i}\right)^{\alpha_i}. \tag{2.4}
\end{aligned}$$

Taking the sum of both sides of (2.4) over  $m_{s_i}, m_{t_i}$  ( $i = 1, 2, \dots, n$ ) from 1 to  $k_{s_i}, k_{t_i}$ , respectively, and in view of inverse Hölder's inequality, we get

$$\begin{aligned}
& \sum_{m_{s_1}=1}^{k_{s_1}} \sum_{m_{t_1}=1}^{k_{t_1}} \cdots \sum_{m_{s_n}=1}^{k_{s_n}} \sum_{m_{t_n}=1}^{k_{t_n}} \frac{\prod_{i=1}^n \phi_i(a_{s_i, t_i, m_{s_i}, m_{t_i}})}{((1/\alpha') \sum_{i=1}^n \alpha'_i m_{s_i} m_{t_i})^{\alpha'}} \\
&\geq \prod_{i=1}^n \sum_{m_{s_i}=1}^{k_{s_i}} \sum_{m_{t_i}=1}^{k_{t_i}} \left(\frac{\phi_i(P_{s_i, t_i, m_{s_i}, m_{t_i}})}{P_{s_i, t_i, m_{s_i}, m_{t_i}}}\right) \left(\sum_{m_{\xi_i}=1}^{m_{s_i}} \sum_{m_{\eta_i}=1}^{m_{t_i}} \left(P_{s_i, t_i, m_{\xi_i}, m_{\eta_i}} \phi_i\left(\frac{\nabla_2 \nabla_1 a_{s_i, t_i, m_{\xi_i}, m_{\eta_i}}}{P_{s_i, t_i, m_{\xi_i}, m_{\eta_i}}}\right)\right)^{1/\alpha_i}\right)^{\alpha_i}
\end{aligned}$$

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$$\begin{aligned}
&\geq \prod_{i=1}^n \left( \sum_{m_{s_i}=1}^{k_{s_i}} \sum_{m_{t_i}=1}^{k_{t_i}} \left( \frac{\phi_i(P_{s_i,t_i,m_{s_i},m_{t_i}})}{P_{s_i,t_i,m_{s_i},m_{t_i}}} \right)^{1/\alpha'_i} \right)^{\alpha'_i} \\
&\quad \times \left( \sum_{m_{s_i}=1}^{k_{s_i}} \sum_{m_{t_i}=1}^{k_{t_i}} \left( \sum_{m_{\xi_i}=1}^{m_{s_i}} \sum_{m_{\eta_i}=1}^{m_{t_i}} \left( p_{s_i,t_i,m_{\xi_i},m_{\eta_i}} \phi_i \left( \frac{\nabla_2 \nabla_1 a_{s_i,t_i,m_{\xi_i},m_{\eta_i}}}{p_{s_i,t_i,m_{\xi_i},m_{\eta_i}}} \right) \right)^{1/\alpha_i} \right)^{\alpha_i} \\
&= C(k_{s_1} k_{t_1}, \dots, k_{s_n} k_{t_n}) \\
&\quad \times \prod_{i=1}^n \left( \sum_{m_{\xi_i}=1}^{k_{s_i}} \sum_{m_{\eta_i}=1}^{k_{t_i}} (k_{s_i} - m_{\xi_i} + 1)(k_{t_i} - m_{\eta_i} + 1) \left( p_{s_i,t_i,m_{\xi_i},m_{\eta_i}} \phi_i \left( \frac{\nabla_2 \nabla_1 a_{s_i,t_i,m_{\xi_i},m_{\eta_i}}}{p_{s_i,t_i,m_{\xi_i},m_{\eta_i}}} \right) \right)^{1/\alpha_i} \right)^{\alpha_i} \\
&= C(k_{s_1} k_{t_1}, \dots, k_{s_n} k_{t_n}) \\
&\quad \times \prod_{i=1}^n \left( \sum_{m_{s_i}=1}^{k_{s_i}} \sum_{m_{t_i}=1}^{k_{t_i}} (k_{s_i} - m_{s_i} + 1)(k_{t_i} - m_{t_i} + 1) \left( p_{s_i,t_i,m_{s_i},m_{t_i}} \phi_i \left( \frac{\nabla_2 \nabla_1 a_{s_i,t_i,m_{s_i},m_{t_i}}}{p_{s_i,t_i,m_{s_i},m_{t_i}}} \right) \right)^{1/\alpha_i} \right)^{\alpha_i}. \tag{2.5}
\end{aligned}$$

This completes the proof.  $\square$

*Remark 2.1.* Let  $a_{s_i,t_i,m_{s_i},m_{t_i}}$ ,  $m_{s_i}$ ,  $m_{t_i}$ ,  $k_{s_i}$ ,  $k_{t_i}$ ,  $a_{s_i,t_i,0,m_{t_i}}$ ,  $a_{s_i,t_i,m_{s_i},0}$ ,  $P_{s_i,t_i,m_{s_i},m_{t_i}}$ ,  $\nabla_1$ , and  $\nabla_2$  be as in Theorem 1.5. Let  $\phi_i$ ,  $\alpha_i$ ,  $\alpha'_i$ ,  $\alpha$ , and  $\alpha'$  ( $i = 1, 2, \dots, n$ ) be as in Theorem 1.3. Similar to the proof of Theorem 1.5, we obtain

$$\begin{aligned}
&\sum_{m_{s_1}=1}^{k_{s_1}} \sum_{m_{t_1}=1}^{k_{t_1}} \cdots \sum_{m_{s_n}=1}^{k_{s_n}} \sum_{m_{t_n}=1}^{k_{t_n}} \frac{\prod_{i=1}^n \phi_i(a_{s_i,t_i,m_{s_i},m_{t_i}})}{(\sum_{i=1}^n \alpha'_i m_{s_i} m_{t_i})^{\alpha'}} \\
&\leq \bar{C}(k_{s_1} k_{t_1}, \dots, k_{s_n} k_{t_n}) \\
&\quad \times \prod_{i=1}^n \left( \sum_{m_{s_i}=1}^{k_{s_i}} \sum_{m_{t_i}=1}^{k_{t_i}} (k_{s_i} - m_{s_i} + 1)(k_{t_i} - m_{t_i} + 1) \left( p_{s_i,t_i,m_{s_i},m_{t_i}} \phi_i \left( \frac{\nabla_2 \nabla_1 a_{s_i,t_i,m_{s_i},m_{t_i}}}{p_{s_i,t_i,m_{s_i},m_{t_i}}} \right) \right)^{1/\alpha_i} \right)^{\alpha_i}, \tag{2.6}
\end{aligned}$$

where

$$\bar{C}(k_{s_1} k_{t_1}, \dots, k_{s_n} k_{t_n}) = \frac{1}{(\alpha')^{\alpha'}} \prod_{i=1}^n \left( \sum_{m_{s_i}=1}^{k_{s_i}} \sum_{m_{t_i}=1}^{k_{t_i}} \left( \frac{\phi_i(P_{s_i,t_i,m_{s_i},m_{t_i}})}{P_{s_i,t_i,m_{s_i},m_{t_i}}} \right)^{1/\alpha'_i} \right)^{\alpha'_i}. \tag{2.7}$$

This is just a *general form* of inequality (1.5) and the *inverse form* of inequality (1.11).

Moreover, let  $\{a_{i,m_i}\}$ ,  $m_i$ ,  $k_i$ ,  $a_{i,0}$ ,  $p_{i,m_i}$ ,  $\nabla$ , and  $P_{i,m_i}$  ( $i = 1, 2, \dots, n$ ) be as in Theorem 1.3. Let  $\phi_i$ ,  $\alpha_i$ ,  $\alpha'_i$ ,  $\alpha$ , and  $\alpha'$  ( $i = 1, 2, \dots, n$ ) be as in Theorem 1.5. Similar to the proof of Theorem 1.5, we obtain

$$\begin{aligned} & \sum_{m_1=1}^{k_1} \dots \sum_{m_n=1}^{k_n} \frac{\prod_{i=1}^n \phi_i(a_{i,m_i})}{((1/\alpha') \sum_{i=1}^n \alpha'_i m_i)^{\alpha'}} \\ & \geq \overline{M}(k_1, \dots, k_n) \prod_{i=1}^n \left( \sum_{m_i=1}^{k_i} (k_i - m_i + 1) \left( p_{i,m_i} \phi_i \left( \frac{\nabla a_{i,m_i}}{p_{i,m_i}} \right) \right)^{1/\alpha_i} \right)^{\alpha_i}, \end{aligned} \tag{2.8}$$

where

$$\overline{M}(k_1, \dots, k_n) = \prod_{i=1}^n \left( \sum_{m_i=1}^{k_i} \left( \frac{\phi_i(P_{i,m_i})}{P_{i,m_i}} \right)^{1/\alpha'_i} \right)^{\alpha'_i}. \tag{2.9}$$

This is just an *inverse form* of inequality (1.5) which was given by Handley et al. [4]. Let  $n = 2$ ,  $\alpha_1 = \alpha_2 = 2$ , then  $\alpha'_1 = \alpha'_2 = -1$ , putting them in inequality (2.8), we have

$$\begin{aligned} & \sum_{m_1=1}^{k_1} \sum_{m_2=1}^{k_2} \frac{\phi_1(a_{1,m_1}) \phi_2(a_{2,m_2})}{(m_1 + m_2)^{-2}} \\ & \geq \overline{M}(k_1, k_2) \left( \sum_{m_1=1}^{k_1} (k_1 - m_1 + 1) \left( p_{1,m_1} \phi_1 \left( \frac{\nabla a_{1,m_1}}{p_{1,m_1}} \right) \right)^{1/2} \right)^2 \\ & \quad \times \left( \sum_{m_2=1}^{k_2} (k_2 - m_2 + 1) \left( p_{2,m_2} \phi_2 \left( \frac{\nabla a_{2,m_2}}{p_{2,m_2}} \right) \right)^{1/2} \right)^2, \end{aligned} \tag{2.10}$$

where

$$\overline{M}(k_1, k_2) = 4 \left( \sum_{m_1=1}^{k_1} \left( \frac{\phi_1(P_{1,m_1})}{P_{1,m_1}} \right)^{-1} \right)^{-1} \left( \sum_{m_2=1}^{k_2} \left( \frac{\phi_2(P_{2,m_2})}{P_{2,m_2}} \right)^{-1} \right)^{-1}. \tag{2.11}$$

This is just an *inverse form* of inequality (1.1) which was given by Pachpatte [9].

*Proof of Theorem 1.6.* From the hypotheses of Theorem 1.6, we obtain

$$f_i(s_i, t_i) = \int_0^{s_i} \int_0^{t_i} D_2 D_1 f_i(\sigma_i, \tau_i) d\sigma_i d\tau_i. \tag{2.12}$$

## 8 Inverses of new Hilbert-Pachpatte-type inequalities

From (2.12) and using Jensen's integral inequality and Hölder's integral inequality [6], we obtain

$$\begin{aligned}
 \phi_i(f_i(s_i, t_i)) &= \phi_i\left(\frac{P_i(s_i, t_i) \int_0^{t_i} \int_0^{s_i} p_i(\sigma_i) q_i(\tau_i) (D_2 D_1 f_i(\sigma_i, \tau_i) / p_i(\sigma_i) q_i(\tau_i)) d\sigma_i d\tau_i}{\int_0^{t_i} \int_0^{s_i} p_i(\sigma_i) q_i(\tau_i) d\sigma_i d\tau_i}\right) \\
 &\geq \phi_i(P_i(s_i, t_i)) \cdot \phi_i\left(\frac{\int_0^{t_i} \int_0^{s_i} p_i(\sigma_i) q_i(\tau_i) (D_2 D_1 f_i(\sigma_i, \tau_i) / p_i(\sigma_i) q_i(\tau_i)) d\sigma_i d\tau_i}{\int_0^{t_i} \int_0^{s_i} p_i(\sigma_i) q_i(\tau_i) d\sigma_i d\tau_i}\right) \\
 &\geq \frac{\phi_i(P_i(s_i, t_i))}{P_i(s_i, t_i)} \int_0^{t_i} \int_0^{s_i} p_i(\sigma_i) q_i(\tau_i) \cdot \phi_i\left(\frac{D_2 D_1 f_i(\sigma_i, \tau_i)}{p_i(\sigma_i) q_i(\tau_i)}\right) d\sigma_i d\tau_i \\
 &\geq \frac{\phi_i(P_i(s_i, t_i))}{P_i(s_i, t_i)} (s_i t_i)^{\alpha'_i} \left(\int_0^{t_i} \int_0^{s_i} \left(p_i(\sigma_i) q_i(\tau_i) \cdot \phi_i\left(\frac{D_2 D_1 f_i(\sigma_i, \tau_i)}{p_i(\sigma_i) q_i(\tau_i)}\right)\right)^{1/\alpha_i} d\sigma_i d\tau_i\right)^{\alpha_i}.
 \end{aligned} \tag{2.13}$$

From the well-known inequality for means, we have

$$\begin{aligned}
 \prod_{i=1}^n \frac{\phi_i(f_i(s_i, t_i))}{((1/\alpha') \sum_{i=1}^n \alpha'_i s_i t_i)^{\alpha'}} &\geq \prod_{i=1}^n \frac{\phi_i(P_i(s_i, t_i))}{P_i(s_i, t_i)} \left(\int_0^{t_i} \int_0^{s_i} \left(p_i(\sigma_i) q_i(\tau_i) \cdot \phi_i\left(\frac{D_2 D_1 f_i(\sigma_i, \tau_i)}{p_i(\sigma_i) q_i(\tau_i)}\right)\right)^{1/\alpha_i} d\sigma_i d\tau_i\right)^{\alpha_i}.
 \end{aligned} \tag{2.14}$$

Integrating both sides of (2.14) over  $s_i, t_i$  from 1 to  $x_i, y_i$  ( $i = 1, 2, \dots, n$ ) and in view of Hölder's integral inequality and Fubini's theorem, we observe that

$$\begin{aligned}
 &\int_0^{x_1} \int_0^{y_1} \cdots \int_0^{x_n} \int_0^{y_n} \prod_{i=1}^n \frac{\phi_i(f_i(s_i, t_i))}{((1/\alpha') \sum_{i=1}^n \alpha'_i s_i t_i)^{\alpha'}} ds_1 dt_1 \cdots ds_n dt_n \\
 &\geq \prod_{i=1}^n \int_0^{x_i} \int_0^{y_i} \frac{\phi_i(P_i(s_i, t_i))}{P_i(s_i, t_i)} \\
 &\quad \times \left(\int_0^{t_i} \int_0^{s_i} \left(p_i(\sigma_i) q_i(\tau_i) \cdot \phi_i\left(\frac{D_2 D_1 f_i(\sigma_i, \tau_i)}{p_i(\sigma_i) q_i(\tau_i)}\right)\right)^{1/\alpha_i} d\sigma_i d\tau_i\right)^{\alpha_i} ds_i dt_i \\
 &\geq \prod_{i=1}^n \left(\int_0^{x_i} \int_0^{y_i} \left(\frac{\phi_i(P_i(s_i, t_i))}{P_i(s_i, t_i)}\right)^{1/\alpha'_i} ds_i dt_i\right)^{\alpha'_i} \\
 &\quad \times \left(\int_0^{x_i} \int_0^{y_i} \left(\int_0^{t_i} \int_0^{s_i} \left(p_i(\sigma_i) q_i(\tau_i) \cdot \phi_i\left(\frac{D_2 D_1 f_i(\sigma_i, \tau_i)}{p_i(\sigma_i) q_i(\tau_i)}\right)\right)^{1/\alpha_i} d\sigma_i d\tau_i\right)^{\alpha_i} ds_i dt_i\right)^{\alpha_i}
 \end{aligned}$$

$$\begin{aligned}
 &= G(x_1 y_1, \dots, x_n y_n) \\
 &\quad \times \prod_{i=1}^n \left( \int_0^{x_i} \int_0^{y_i} (x_i - s_i)(y_i - t_i) \left( p_i(s_i) q_i(t_i) \cdot \phi_i \left( \frac{D_2 D_1 f_i(s_i, t_i)}{p_i(s_i) q_i(t_i)} \right) \right)^{1/\alpha_i} ds_i dt_i \right)^{\alpha_i}.
 \end{aligned} \tag{2.15}$$

The proof is complete. □

*Remark 2.2.* Let  $f_i(s_i, t_i)$ ,  $x_i$ ,  $y_i$ ,  $f_i(0, t_i)$ ,  $f_i(s_i, 0)$ ,  $D_1 f_i(s_i, t_i)$ ,  $D_2 f_i(s_i, t_i)$ ,  $D_2 D_1 f_i(s_i, t_i)$ ,  $p_i(\sigma_i)$ ,  $q_i(\tau_i)$ , and  $P_i(s_i, t_i)$  be as in Theorem 1.6. Let  $\phi_i$ ,  $\alpha_i$ ,  $\alpha'_i$ ,  $\alpha$ , and  $\alpha'$  be as in Theorem 1.3. Similar to the proof of Theorem 1.6, we have

$$\begin{aligned}
 &\int_0^{x_1} \int_0^{y_1} \cdots \int_0^{x_n} \int_0^{y_n} \prod_{i=1}^n \frac{\phi_i(f_i(s_i, t_i))}{(\sum_{i=1}^n \alpha'_i s_i t_i)^{\alpha'}} ds_1 dt_1 \cdots ds_n dt_n \\
 &\leq \bar{G}(x_1 y_1, \dots, x_n y_n) \\
 &\quad \times \prod_{i=1}^n \left( \int_0^{x_i} \int_0^{y_i} (x_i - s_i)(y_i - t_i) \left( p_i(s_i) q_i(t_i) \cdot \phi_i \left( \frac{D_2 D_1 f_i(s_i, t_i)}{p_i(s_i) q_i(t_i)} \right) \right)^{1/\alpha_i} ds_i dt_i \right)^{\alpha_i},
 \end{aligned} \tag{2.16}$$

where

$$\bar{G}(x_1 y_1, \dots, x_n y_n) = \frac{1}{(\alpha')^{\alpha'}} \prod_{i=1}^n \left( \int_0^{x_i} \int_0^{y_i} \left( \frac{\phi_i(P_i(s_i, t_i))}{P_i(s_i, t_i)} \right)^{1/\alpha'_i} ds_i dt_i \right)^{\alpha'_i}. \tag{2.17}$$

This is just a *general form* of inequality (1.7) and an *inverse form* of the inequality (1.14).

On the other hand, let  $f_i$ ,  $f_i(0)$ ,  $f'_i(s_i)$ ,  $p_i(\sigma_i)$ ,  $\sigma_i$ ,  $k_i$ ,  $s_i$ ,  $x_i$ ,  $P_i(s_i)$  be as in Theorem 1.4, and let  $\phi_i$ ,  $\alpha_i$ ,  $\alpha'_i$ ,  $\alpha$ ,  $\alpha'$  be as in Theorem 1.5. Then

$$\begin{aligned}
 &\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n \phi_i(f_i(s_i))}{(1/\alpha')(\sum_{i=1}^n \alpha'_i s_i)^{\alpha'}} ds_1 \cdots ds_n \\
 &\geq \bar{L}(x_1, \dots, x_n) \prod_{i=1}^n \left( \int_0^{x_i} (x_i - s_i) \left( p_i(s_i) \phi_i \left( \frac{f'_i(s_i)}{P_i(s_i)} \right) \right)^{1/\alpha_i} ds_i \right)^{\alpha_i},
 \end{aligned} \tag{2.18}$$

where

$$\bar{L}(x_1, \dots, x_n) = \prod_{i=1}^n \left( \int_0^{x_i} \left( \frac{\phi_i(P_i(s_i))}{P_i(s_i)} \right)^{1/\alpha'_i} ds_i \right)^{\alpha'_i}. \tag{2.19}$$

This is just an *inverse form* of inequality (1.7) which was given by Handley et al. [4].

## 10 Inverses of new Hilbert-Pachpatte-type inequalities

Let  $n = 2$ ,  $\alpha_1 = \alpha_2 = 2$ , then  $\alpha'_1 = \alpha'_2 = -1$ , taking them in inequality (2.18), we have

$$\begin{aligned} & \int_0^{x_1} \int_0^{x_2} \frac{\phi_1(f_1(s_1))\phi_2(f_2(s_2))}{(s_1 + s_2)^{-2}} ds_1 ds_2 \\ & \geq \bar{L}(x_1, x_2) \left( \int_0^{x_1} (x_1 - s_1) \left( p_1(s_1) \phi_1 \left( \frac{f'_2(s_1)}{p_1(s_1)} \right) \right)^{1/2} ds_1 \right)^2 \\ & \quad \times \left( \int_0^{x_2} (x_2 - s_2) \left( p_2(s_2) \phi_2 \left( \frac{f'_2(s_2)}{p_2(s_2)} \right) \right)^{1/2} ds_2 \right)^2, \end{aligned} \quad (2.20)$$

where

$$\bar{L}(x_1, x_2) = 4 \left( \int_0^{x_1} \left( \frac{\phi_1(P_1(s_1))}{P_1(s_1)} \right)^{-1} ds_1 \right)^{-1} \left( \int_0^{x_2} \left( \frac{\phi_2(P_2(s_2))}{P_2(s_2)} \right)^{-1} ds_2 \right)^{-1}. \quad (2.21)$$

This is just an *inverse form* of inequality (1.3) which was given by Pachpatte [9].

For interrelated research on similar inequalities, one is referred to [2, 8, 11, 14, 15], and the references cited therein.

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