AN IMPULSIVE NONLINEAR SINGULAR VERSION OF THE GRONWALL-BIHARI INEQUALITY

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We find bounds for a Gronwall-Bihari type inequality for piecewise continuous functions. Unlike works in the prior literature, here we consider inequalities involving singular kernels in addition to functions with delays.

1. Introduction

In this paper, we are concerned with the following impulsive integral inequality:

$$u(t) \leq a(t) + b(t) \int_0^t k_1(t,s) u^m(s) ds$$

$$+ c(t) \int_0^t k_2(t,s) u^n(s - \tau) ds + d(t) \sum_{0 < t_k < t} \eta_k u(t_k), \quad t \geq 0, \tag{1.1}$$

$$u(t) \leq \varphi(t), \quad t \in [-\tau, 0], \ \tau > 0,$$

where $a(t)$, $b(t)$, $c(t)$, and $d(t)$ are nonnegative continuous functions, $m, n > 1$, $\eta_k \geq 0$, the points t_k (called "instants of impulse effect") are in the increasing order, and $\lim_{k \to \infty} t_k = +\infty$. The kernels $k_i(t,s)$, $i = 1,2$, are of the form

$$k_i(t,s) = (t-s)^{\beta_i - 1} s^{\gamma_i} F_i(s), \quad i = 1,2, \tag{1.2}$$

where $\beta_i > 0$, $\gamma_i > -1$, $F_i(t)$, $i = 1,2$, and $\varphi(t)$ are nonnegative continuous functions. For this reason, we say that we are in the presence of an *impulsive nonlinear singular* version of the Gronwall inequality with *delay*.

We would like to find bounds for solutions to this inequality in the space of piecewise continuous functions $u : X \to Y$ ($X \subset \mathbb{R}, Y \subset \mathbb{R}^N$), with points of discontinuity of the first

kind at the points $t_k \in X$. Our functions will also be assumed to be left continuous at the points t_k. This space will be denoted by $PC(X, Y)$.

Integral inequalities are an important tool to investigate some qualitative and quantitative properties of solutions to differential equations such as existence, uniqueness, boundedness, and stability. Among these integral inequalities, we cite the famous Gronwall inequality and its different generalizations (see [3, 13]).

Impulsive integral equations, impulsive integro-differential equations, and impulsive differential equations arise naturally in various fields such as population dynamics and optimal control (see the monographs [2, 9, 15]). It seems that the first treatment of impulsive systems goes back to the monograph by Krylov and Bogolyubov [8].

The following impulsive integral inequality:

$$u(t) \leq a + \int_c^t b(s)u(s)ds + \sum_{c < t_k < t} \eta_k u(t_k), \quad t \geq 0, \tag{1.3}$$

has been first used by Samoilenko and Perestyuk [14] to investigate problems of the form

$$\begin{aligned} x' &= f(t, x), \quad t \neq t_k, \\ \Delta x &= I_k(x), \quad t = t_k. \end{aligned} \tag{1.4}$$

Then, a similar inequality with constant delay was considered by Bainov and Hristova in [1]. Recently, Hristova in [5] treated a more general inequality with nonlinear functions in u. However, in all previous works, the functions (kernels) involved in the integrals are regular, even in the case of integrals of convolution or nonconvolution types (see [3, 13]).

In this work, we consider the case of singular kernels of the form (1.2). The type of inequalities we are going to discuss arise for instance when we study impulsive evolution problems of the form

$$\begin{aligned} \frac{du}{dt} + Au &= f(t, u, u_t), \quad t > 0, \ t \neq t_k, \\ u(0) &= u_0 \in X, \\ \Delta u(t_k) &= u(t_k^+) - u(t_k^-), \quad k = 1, 2, \ldots, \end{aligned} \tag{1.5}$$

where A is a sectorial operator (see, for instance, [17] where the case without delay and with globally Lipschitzian right-hand side is treated).

We point out here that nonlinear singular versions of the Gronwall-Bihari inequality have been already considered by the present author in [6, 7, 10, 16] and Medved in [11, 12] to investigate problems of the form (1.5) and perturbed problems of (1.5) but without impulse effects.

The plan of the paper is as follows. In the next section we present some lemmas and notation which will be needed in the proof of our result. Section 3 contains the statement and proof of our theorem. It is ended with some important remarks.

2. Preliminaries

In this section, we prepare some lemmas and notation which we will use in the next section.

LEMMA 2.1. *For all $\beta > 0$ and $\gamma > -1$,*

$$\int_0^t (t-s)^{\beta-1}s^\gamma ds = Ct^{\beta+\gamma}, \quad t \geq 0, \tag{2.1}$$

where $C = C(\beta,\gamma) = \Gamma(\beta)\Gamma(\gamma+1)/\Gamma(\beta+\gamma+1)$.

LEMMA 2.2. *If $\beta,\gamma,\delta > 0$, then for any $t > 0$,*

$$t^{1-\beta}\int_0^t (t-s)^{\beta-1}s^{\gamma-1}e^{-\delta s}ds \leq C, \tag{2.2}$$

where $C = C(\beta,\gamma,\delta)$ is a positive constant independent of t. In fact,

$$C = \max\{1, 2^{1-\beta}\}\Gamma(\gamma)\left(1+\frac{\gamma}{\beta}\right)\delta^{-\gamma}. \tag{2.3}$$

See [6] for the proof.

LEMMA 2.3. *Let a, b, K, ψ be nonnegative continuous functions on the interval $I = (0,T)$ $(0 < T \leq \infty)$, let $\omega : (0,\infty) \to \mathbb{R}$ be a continuous, nonnegative, and nondecreasing function with $\omega(0) = 0$ and $\omega(u) > 0$ for $u > 0$, and let $A(t) := \max_{0 \leq s \leq t} a(s)$ and $B(t) := \max_{0 \leq s \leq t} b(s)$. Assume that*

$$\psi(t) \leq a(t) + b(t)\int_0^t K(s)\omega(\psi(s))ds, \quad t \in I. \tag{2.4}$$

Then

$$\psi(t) \leq H^{-1}\left[H(A(t)) + B(t)\int_0^t K(s)ds\right], \quad t \in (0,T_1), \tag{2.5}$$

where $H(v) := \int_{v_0}^v d\tau/\omega(\tau)$ $(v \geq v_0 > 0)$, H^{-1} is the inverse of H, and $T_1 > 0$ is such that $H(A(t)) + B(t)\int_0^t K(s)ds \in D(H^{-1})$ for all $t \in (0,T_1)$.

See [4] or [3, 13].

In order to lighten the statement of our result, we adopt the following notation. Let $V(\tau) := 1 + \int_0^\tau F_2^2(s)\varphi^{2n}(s-\tau)ds$, $r := \max\{m,n\} > 1$, $t_0 := 0$.

For p and q such that $1/p + 1/q = 1$, we define

$$\begin{aligned}
f_p(t) := \sup\{&a^q(t), C^{q/p}(p\beta_1 - p + 1, p\gamma_1)b^q(t)t^{q(\beta_1+\gamma_1)-1}, \\
&C^{q/p}(p\beta_2 - p + 1, p\gamma_2)c^q(t)t^{q(\beta_2+\gamma_2)-1}, d^q(t)\},
\end{aligned} \tag{2.6}$$

with $C(p\beta_1 - p + 1, p\gamma_1)$ and $C(p\beta_2 - p + 1, p\gamma_2)$ the constants from Lemma 2.1, and T_p

to be the sup of all values of t for which

$$\sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (i+2)^{(q-1)r} \prod_{j=1}^{i-1} (1+(j+2)^{q-1} \eta_j^q f(t_j))^r$$

$$\times \{F_1^q(s) f^m(s) + F_2^q(s) f^n(s-\tau)\} ds + (k+3)^{(q-1)r}$$

$$\times \prod_{j=1}^{k} (1+(j+2)^{q-1} \eta_j^q f(t_j))^r \int_{t_k}^{t} \{F_1^q(s) f^m(s) + F_2^q(s) f^n(s-\tau)\} ds < \frac{V(\tau)^{1-r}}{(r-1)}.$$

$$(2.7)$$

If $p = q = 2$, put $f(t) := f_2(t)$ and $T := T_2$.

3. The bounds

Without loss of generality, we will suppose that the t_k are such that $\tau < t_{k+1} - t_k \le 2\tau$, $k = 0, 1, 2, \dots$. For the general case, see Remark 3.2 below.

THEOREM 3.1. *Let the above assumptions on the different parameters and functions hold. Suppose that u is in $PC([-\tau, +\infty], [0, +\infty])$ and satisfies (1.1), then*
   (a) *if $\beta_i > 1/2$ and $\gamma_i > -1/2$, $i = 1, 2$, it holds that for $t \in (t_k, t_{k+1}]$,*

$$u(t) \le \left[(k+3) f(t) \prod_{l=1}^{k} (1+(k+2) \eta_l^2 f(t_l)) \right]^{1/q}$$

$$\times \left[V(\tau)^{1-r} - (r-1) \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (i+2)^r \prod_{j=1}^{i-1} (1+(j+2) \eta_j^2 f(t_j))^r \right.$$

$$\times \{F_1^2(s) f^m(s) + F_2^2(s) f^n(s-\tau)\} ds - (r-1)(k+3)^r$$

$$\left. \times \prod_{j=1}^{k} (1+(j+2) \eta_j^2 f(t_j))^r \int_{t_k}^{t} \{F_1^2(s) f^m(s) + F_2^2(s) f^n(s-\tau)\} ds \right]^{1/2(1-r)}$$

$$(3.1)$$

as long as the expression between the second brackets is positive, that is, on $(0, T)$;
   (b) *if $0 < \beta_i \le 1/2$ and $-1 < \gamma_i \le -1/2$, then it holds that for $t \in (t_k, t_{k+1}]$,*

$$u(t) \le \left[(k+3)^{q-1} f_p(t) \prod_{l=1}^{k} (1+(k+2)^{q-1} \eta_l^q f(t_l)) \right]^{1/q}$$

$$\times \left[V(\tau)^{1-r} - (r-1) \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (i+2)^{(q-1)r} \prod_{j=1}^{i-1} (1+(j+2)^{q-1} \eta_j^q f(t_j))^r \right.$$

$$\times \{F_1^q(s) f_p^m(s) + F_2^q(s) f_p^n(s-\tau)\} ds - (r-1)(k+3)^{(q-1)r}$$

$$\left. \times \prod_{j=1}^{k} (1+(j+2)^{q-1} \eta_j^q f(t_j))^r \int_{t_k}^{t} \{F_1^q(s) f_p^m(s) + F_2^q(s) f_p^n(s-\tau)\} ds \right]^{1/q(1-r)}$$

$$(3.2)$$

as long as the expression between the second brackets is positive, that is, on $(0, T_p)$.

Proof. We will use a mathematical induction.

(a) *Step 1.* We start by proving the validity of (3.1) in the interval $[0, t_1]$ (in fact, the argument we present is valid within the interval $(0, T)$, this fact will be mentioned in every occasion by indicating the right interval over which the estimate is valid). For $t \in [0, \tau] \subset [0, t_1]$ (see assumptions on t_k), we have

$$
\begin{aligned}
u(t) \leq a(t) + b(t) \int_0^t (t-s)^{\beta_1 - 1} s^{\gamma_1} F_1(s) u^m(s) ds \\
+ c(t) \int_0^t (t-s)^{\beta_2 - 1} s^{\gamma_2} F_2(s) u^n(s-\tau) ds.
\end{aligned}
\tag{3.3}
$$

If $\beta_i > 1/2$ and $\gamma_i > -1/2$, $i = 1, 2$, then by the Cauchy-Schwarz inequality and Lemma 2.1, we obtain

$$
\begin{aligned}
u(t) \leq a(t) + C^{1/2}(2\beta_1 - 1, 2\gamma_1) b(t) t^{\beta_1 + \gamma_1 - 1/2} \left(\int_0^t F_1^2(s) u^{2m}(s) ds \right)^{1/2} \\
+ C^{1/2}(2\beta_2 - 1, 2\gamma_2) c(t) t^{\beta_2 + \gamma_2 - 1/2} \left(\int_0^t F_2^2(s) u^{2n}(s-\tau) ds \right)^{1/2},
\end{aligned}
\tag{3.4}
$$

where $C(2\beta_1 - 1, 2\gamma_1)$ and $C(2\beta_2 - 1, 2\gamma_2)$ are the constants from Lemma 2.1. Squaring both sides of (3.4), we find

$$
\begin{aligned}
u^2(t) \leq 3a^2(t) + 3C(2\beta_1 - 1, 2\gamma_1) b^2(t) t^{2(\beta_1 + \gamma_1) - 1} \int_0^t F_1^2(s) u^{2m}(s) ds \\
+ 3C(2\beta_2 - 1, 2\gamma_2) c^2(t) t^{2(\beta_2 + \gamma_2) - 1} \int_0^t F_2^2(s) u^{2n}(s-\tau) ds.
\end{aligned}
\tag{3.5}
$$

Therefore

$$
\begin{aligned}
u^2(t) \leq 3f(t) \left(1 + \int_0^t F_1^2(s) u^{2m}(s) ds + \int_0^t F_2^2(s) u^{2n}(s-\tau) ds \right) \\
\leq 3f(t) \left(1 + \int_0^t F_1^2(s) u^{2m}(s) ds + \int_0^\tau F_2^2(s) \varphi^{2n}(s-\tau) ds \right).
\end{aligned}
\tag{3.6}
$$

Putting

$$
v_1(t) := 1 + \int_0^\tau F_2^2(s) \varphi^{2n}(s-\tau) ds + \int_0^t F_1^2(s) u^{2m}(s) ds,
\tag{3.7}
$$

we see that $v_1(t)$ is a nondecreasing positive differentiable function on $[0, \tau]$, $v_1(0) = 1 + \int_0^\tau F_2^2(s) \varphi^{2n}(s-\tau) ds =: V(\tau)$,

$$
u^2(t) \leq 3f(t) v_1(t),
\tag{3.8}
$$

$$
v_1'(t) = F_1^2(t) u^{2m}(t) \leq 3^m F_1^2(t) f^m(t) v_1^m(t) \leq 3^r F_1^2(t) f^m(t) v_1^r(t).
\tag{3.9}
$$

An integration of (3.9) (or using Lemma 2.3 directly) leads to

$$
v_1(t) \leq \left[V(\tau)^{1-r} - 3^r(r-1) \int_0^t F_1^2(s) f^m(s) ds \right]^{1/(1-r)}
\tag{3.10}
$$

as long as $\int_0^t F_1^2(s)f^m(s)ds < V(\tau)^{1-r}/3^r(r-1)$. Therefore, for $t \in [0,\tau]$,

$$u(t) \le \sqrt{3f(t)}\left[V(\tau)^{1-r} - 3^r(r-1)\int_0^t F_1^2(s)f^m(s)ds\right]^{1/2(1-r)} \tag{3.11}$$

as long as $\int_0^t F_1^2(s)f^m(s)ds < V(\tau)^{1-r}/3^r(r-1)$.

Let $t \in (\tau, t_1]$. Then, from (3.6) and (3.7), we have

$$u^2(t) \le 3f(t)\left(v_1(\tau) + \int_\tau^t F_1^2(s)u^{2m}(s)ds + \int_\tau^t F_2^2(s)u^{2n}(s-\tau)ds\right). \tag{3.12}$$

Let us designate

$$w_1(t) := v_1(\tau) + \int_\tau^t F_1^2(s)u^{2m}(s)ds + \int_\tau^t F_2^2(s)u^{2n}(s-\tau)ds. \tag{3.13}$$

Then $w_1(t)$ is a nondecreasing positive differentiable function on $(\tau, t_1]$,

$$w_1(\tau) = v_1(\tau) \le w_1(t), \qquad u^2(t) \le 3f(t)w_1(t), \tag{3.14}$$

$$w_1'(t) = F_1^2(t)u^{2m}(t) + F_2^2(t)u^{2n}(t-\tau). \tag{3.15}$$

Since $0 < t - \tau \le \tau$ (see Remark 3.2) and from (3.7), (3.8), (3.14), and (3.15),

$$u^2(t-\tau) \le 3f(t-\tau)v_1(t-\tau) \le 3f(t-\tau)v_1(\tau) \le 3f(t-\tau)w_1(t), \tag{3.16}$$

and we can write that

$$\begin{aligned} w_1'(t) &\le F_1^2(t)\big(3f(t)w_1(t)\big)^m + F_2^2(t)\big(3f(t-\tau)w_1(t)\big)^n \\ &\le 3^r\big[F_1^2(t)f^m(t) + F_2^2(t)f^n(t-\tau)\big]w_1^r(t). \end{aligned} \tag{3.17}$$

Integrating (3.17) from τ to t and using (3.10), we obtain

$$\begin{aligned} w_1(t) &\le \left[w_1(\tau)^{1-r} - 3^r(r-1)\int_\tau^t \big[F_1^2(s)f^m(s) + F_2^2(s)f^n(s-\tau)\big]ds\right]^{1/(1-r)} \\ &\le \Big[V(\tau)^{1-r} - 3^r(r-1)\int_0^\tau F_1^2(s)f^m(s)ds \\ &\quad - 3^r(r-1)\int_\tau^t \big[F_1^2(s)f^m(s) + F_2^2(s)f^n(s-\tau)\big]ds\Big]^{1/(1-r)} \\ &\le \left[V(\tau)^{1-r} - 3^r(r-1)\int_0^t \big[F_1^2(s)f^m(s) + F_2^2(s)f^n(s-\tau)\big]ds\right]^{1/(1-r)} \end{aligned} \tag{3.18}$$

and hence, for $t \in (\tau, t_1]$,

$$u(t) \le \sqrt{3f(t)}\left[V^{1-r} - 3^r(r-1)\int_0^t \big[F_1^2(s)f^m(s) + F_2^2(s)f^n(s-\tau)\big]ds\right]^{1/2(1-r)} \tag{3.19}$$

as long as

$$\int_0^t [F_1^2(s) f^m(s) + F_2^2(s) f^n(s - \tau)] ds < \frac{V^{1-r}}{3^r (r-1)}. \tag{3.20}$$

We define the function $\psi_1 : [0, t_1] \to \mathbb{R}$ by

$$\psi_1(t) := \begin{cases} v_1(t), & t \in [0, \tau], \\ w_1(t), & t \in (\tau, t_1]. \end{cases} \tag{3.21}$$

It can be easily seen that (3.1) in the statement of the theorem is satisfied over $[0, t_1]$ (recall that $t_0 := 0$).

Step 2. Let $t \in (t_1, t_2]$. If $t \in (t_1, t_1 + \tau]$, then

$$u(t) \le a(t) + b(t) \int_0^t (t - s)^{\beta_1 - 1} s^{\gamma_1} F_1(s) u^m(s) ds$$
$$+ c(t) \int_0^t (t - s)^{\beta_2 - 1} s^{\gamma_2} F_2(s) u^n(s - \tau) ds + \eta_1 d(t) u(t_1). \tag{3.22}$$

Squaring both sides of (3.22) after applying the Cauchy-Schwarz inequality and Lemma 2.1, as in the previous steps from (3.4) to (3.6), we find

$$u^2(t) \le 4f(t) \left(1 + \int_0^t F_1^2(s) u^{2m}(s) ds + \int_0^t F_2^2(s) u^{2n}(s - \tau) ds + \eta_1^2 u^2(t_1) \right)$$
$$\le 4f(t) \left(v_1(\tau) + \int_\tau^{t_1} F_1^2(s) u^{2m}(s) ds + \int_\tau^{t_1} F_2^2(s) u^{2n}(s - \tau) ds \right.$$
$$\left. + \int_{t_1}^t F_1^2(s) u^{2m}(s) ds + \int_{t_1}^t F_2^2(s) u^{2n}(s - \tau) ds + \eta_1^2 u^2(t_1) \right). \tag{3.23}$$

Note here that we have used definition (3.7) of $v_1(t)$. Thanks to (3.13) and (3.14), we entail that

$$u^2(t) \le 4f(t) \left(w_1(t_1) + \int_{t_1}^t F_1^2(s) u^{2m}(s) ds + \int_{t_1}^t F_2^2(s) u^{2n}(s - \tau) ds + 3\eta_1^2 f(t_1) w_1(t_1) \right)$$
$$\le 4f(t) [1 + 3\eta_1^2 f(t_1)] \left(w_1(t_1) + \int_{t_1}^t F_1^2(s) u^{2m}(s) ds + \int_{t_1}^t F_2^2(s) u^{2n}(s - \tau) ds \right). \tag{3.24}$$

We define

$$v_2(t) := w_1(t_1) + \int_{t_1}^t F_1^2(s) u^{2m}(s) ds + \int_{t_1}^t F_2^2(s) u^{2n}(s - \tau) ds. \tag{3.25}$$

It is clear that $v_2(t)$ is a nondecreasing positive differentiable function on $(t_1, t_1 + \tau]$,

$$v_2(t_1) = w_1(t_1) \le v_2(t), \qquad u^2(t) \le 4f(t)[1 + 3\eta_1^2 f(t_1)]v_2(t). \tag{3.26}$$

Since $t - \tau \le t_1$, by (3.6), (3.12), (3.13), and (3.25), we see that

$$u^2(t - \tau) \le 3f(t - \tau)\psi_1(t - \tau) \le 3f(t - \tau)w_1(t_1) \le 3f(t - \tau)v_2(t), \tag{3.27}$$

and thus from this estimation, (3.25) and (3.26), we get

$$
\begin{aligned}
v_2'(t) &= F_1^2(t)u^{2m}(t) + F_2^2(t)u^{2n}(t - \tau) \\
&\le \{4^m[1 + 3\eta_1^2 f(t_1)]^m f^m(t)F_1^2(t) + 3^n F_2^2(t) f^n(t - \tau)\}v_2^r(t).
\end{aligned} \tag{3.28}
$$

An integration of (3.28) from t_1 to t together with (3.18) leads to

$$
\begin{aligned}
v_2(t) \le \Bigg[& v_2(t_1)^{1-r} - (r - 1) \\
&\times \int_{t_1}^t \{4^m[1 + 3\eta_1^2 f(t_1)]^m f^m(s)F_1^2(s) + 3^n F_2^2(s) f^n(s - \tau)\}ds \Bigg]^{1/(1-r)} \\
\le \Bigg[& V(\tau)^{1-r} - 3^r(r - 1) \int_0^{t_1} [F_1^2(s) f^m(s) + F_2^2(s) f^n(s - \tau)]ds - (r - 1) \\
&\times \int_{t_1}^t \{4^m[1 + 3\eta_1^2 f(t_1)]^m f^m(s)F_1^2(s) + 3^n F_2^2(s) f^n(s - \tau)\}ds \Bigg]^{1/(1-r)}
\end{aligned} \tag{3.29}
$$

and hence, for $t \in (t_1, t_1 + \tau]$, we have

$$
\begin{aligned}
u(t) \le 2\sqrt{[1 + 3\eta_1^2 f(t_1)]f(t)} \\
\times \Bigg[& V(\tau)^{1-r} - 3^r(r - 1) \int_0^{t_1} [F_1^2(s) f^m(s) + F_2^2(s) f^n(s - \tau)]ds \\
& - (r - 1) \int_{t_1}^t \{4^m[1 + 3\eta_1^2 f(t_1)]^m f^m(s)F_1^2(s) + 3^n F_2^2(s) f^n(s - \tau)\}ds \Bigg]^{1/2(1-r)}
\end{aligned} \tag{3.30}
$$

as long as

$$
\begin{aligned}
3^r \int_0^{t_1} & [F_1^2(s) f^m(s) + F_2^2(s) f^n(s - \tau)]ds \\
& + \int_{t_1}^t \{4^m[1 + 3\eta_1^2 f(t_1)]^m f^m(s)F_1^2(s) + 3^n F_2^2(s) f^n(s - \tau)\}ds \le \frac{V^{1-r}}{r - 1}.
\end{aligned} \tag{3.31}
$$

Now let $t \in (t_1 + \tau, t_2]$, then from (3.7), (3.13), (3.14), (3.25), and

$$u^2(t) \le 4f(t)\left(1 + \int_0^t F_1^2(s)u^{2m}(s)ds + \int_0^t F_2^2(s)u^{2n}(s - \tau)ds + \eta_1^2 u^2(t_1)\right), \tag{3.32}$$

we deduce that

$$
\begin{aligned}
u^2(t) &\le 4f(t)\left(v_2(t_1+\tau) + \int_{t_1+\tau}^t F_1^2(s)u^{2m}(s)ds + \int_{t_1+\tau}^t F_2^2(s)u^{2n}(s-\tau)ds + 3\eta_1^2 f(t_1)v_2(t_1+\tau)\right) \\
&\le 4f(t)[1+3\eta_1^2 f(t_1)]\left(v_2(t_1+\tau) + \int_{t_1+\tau}^t F_1^2(s)u^{2m}(s)ds + \int_{t_1+\tau}^t F_2^2(s)u^{2n}(s-\tau)ds\right)
\end{aligned}
\tag{3.33}
$$

because $w_1(t_1) \le v_2(t_1) \le v_2(t_1+\tau)$. At this stage, we denote

$$
w_2(t) := v_2(t_1+\tau) + \int_{t_1+\tau}^t F_1^2(s)u^{2m}(s)ds + \int_{t_1+\tau}^t F_2^2(s)u^{2n}(s-\tau)ds.
\tag{3.34}
$$

Then, clearly $w_2(t)$ is a nondecreasing positive differentiable function on $(t_1+\tau, t_2]$, $w_2(t_1+\tau) = v_2(t_1+\tau) \le w_2(t)$, and

$$
w_2'(t) = F_1^2(t)u^{2m}(t) + F_2^2(t)u^{2n}(t-\tau).
\tag{3.35}
$$

Observe that by (3.33) and (3.34), we have the estimates

$$
u^2(t) \le 4f(t)[1+3\eta_1^2 f(t_1)]w_2(t),
\tag{3.36}
$$

and since $t_1 < t-\tau < t_1+\tau$, it follows from (3.24) that

$$
\begin{aligned}
u^2(t-\tau) &\le 4f(t-\tau)[1+3\eta_1^2 f(t_1)]v_2(t-\tau) \\
&\le 4f(t-\tau)[1+3\eta_1^2 f(t_1)]v_2(t_1+\tau) \\
&\le 4f(t-\tau)[1+3\eta_1^2 f(t_1)]w_2(t).
\end{aligned}
\tag{3.37}
$$

Consequently,

$$
w_2'(t) \le 4^r[1+3\eta_1^2 f(t_1)]^r\{f^m(t)F_1^2(t) + f^n(t-\tau)F_2^2(t)\}w_2^r(t).
\tag{3.38}
$$

Again by an integration of (3.38), we end up with

$$
\begin{aligned}
w_2(t) &\le \left[w_2^{1-r}(t_1+\tau) - 4^r(r-1)[1+3\eta_1^2 f(t_1)]^r \int_{t_1+\tau}^t \{f^m(s)F_1^2(s) + f^n(s-\tau)F_2^2(s)\}ds\right]^{1/(1-r)} \\
&\le \left[V(\tau)^{1-r} - 3^r(r-1)\int_0^{t_1} [F_1^2(s)f^m(s) + F_2^2(s)f^n(s-\tau)]ds \right. \\
&\quad - (r-1)\int_{t_1}^{t_1+\tau} \{4^m[1+3\eta_1^2 f(t_1)]^m f^m(s)F_1^2(s) + 3^n F_2^2(s)f^n(s-\tau)\}ds \\
&\quad \left. - 4^r(r-1)[1+3\eta_1^2 f(t_1)]^r \int_{t_1+\tau}^t \{f^m(s)F_1^2(s) + f^n(s-\tau)F_2^2(s)\}ds\right]^{1/(1-r)}
\end{aligned}
\tag{3.39}
$$

or simply

$$w_2(t) \leq \left[V(\tau)^{1-r} - 3^r (r-1) \int_0^{t_1} [F_1^2(s) f^m(s) + F_2^2(s) f^n(s-\tau)] ds \right.$$

$$\left. - 4^r (r-1) [1 + 3\eta_1^2 f(t_1)]^r \int_{t_1}^t \{f^m(s) F_1^2(s) + f^n(s-\tau) F_2^2(s)\} ds \right]^{1/(1-r)}.$$

$$(3.40)$$

Hence,

$$u(t) \leq 2\sqrt{f(t)[1 + 3\eta_1^2 f(t_1)]}$$

$$\times \left[V(\tau)^{1-r} - 3^r (r-1) \int_0^{t_1} [F_1^2(s) f^m(s) + F_2^2(s) f^n(s-\tau)] ds \right.$$

$$\left. - 4^r (r-1) [1 + 3\eta_1^2 f(t_1)]^r \int_{t_1}^t \{f^m(s) F_1^2(s) + f^n(s-\tau) F_2^2(s)\} ds \right]^{1/2(1-r)}.$$

$$(3.41)$$

provided that the expression between brackets is positive. We define $\psi_2 : (t_1, t_2] \to \mathbb{R}$ by

$$\psi_2(t) := \begin{cases} v_2(t), & t \in (t_1, t_1 + \tau], \\ w_2(t), & t \in (t_1 + \tau, t_2]. \end{cases} \tag{3.42}$$

It is clear that (3.1) holds on $(t_1, t_2]$.

Step 3. Finally, suppose that (3.1) is valid over $(t_k, t_{k+1}]$, then if $t \in (t_{k+1}, t_{k+2}]$, we define

$$\psi_{k+2}(t) := \begin{cases} v_{k+2}(t), & t \in [t_{k+1}, t_{k+1} + \tau], \\ w_{k+2}(t), & t \in (t_{k+1} + \tau, t_{k+2}], \end{cases} \tag{3.43}$$

with

$$v_{k+2}(t) := w_{k+1}(t_{k+1}) + \int_{t_{k+1}}^t F_1^2(s) u^{2m}(s) ds + \int_{t_{k+1}}^t F_2^2(s) u^{2n}(s-\tau) ds,$$

$$w_{k+2}(t) := v_{k+2}(t_{k+1} + \tau) + \int_{t_{k+1}+\tau}^t F_1^2(s) u^{2m}(s) ds + \int_{t_{k+1}+\tau}^t F_2^2(s) u^{2n}(s-\tau) ds.$$

$$(3.44)$$

In a similar manner as in Steps 1 and 2, we can see that (3.1) is valid over $(t_{k+1}, t_{k+2}]$.

(b) If $0 < \beta_i \leq 1/2$ and $-1 < \gamma_i \leq -1/2$, then instead of the Cauchy-Schwarz inequality we use the Hölder inequality with

$$1 < p < \min\left\{ \frac{1}{1-\beta_i}, -\frac{1}{\gamma_i}, i = 1, 2 \right\}, \tag{3.45}$$

and $q > 1$ such that $1/p + 1/q = 1$. We find

$$
\begin{aligned}
u(t) \leq a(t) &+ C^{1/p}\left(p\beta_1 - p + 1, p\gamma_1\right)b(t)t^{\beta_1+\gamma_1-1/q}\left(\int_0^t F_1^q(s)u^{qm}(s)ds\right)^{1/q} \\
&+ C^{1/p}\left(p\beta_2 - p + 1, p\gamma_2\right)c(t)t^{\beta_2+\gamma_2-1/q}\left(\int_0^t F_2^q(s)u^{qn}(s-\tau)ds\right)^{1/q}.
\end{aligned}
\tag{3.46}
$$

Then, we raise both sides to the power q and we use the inequality

$$
\left(\sum_{i=1}^n a_i\right)^r \leq n^{r-1}\left(\sum_{i=1}^n a_i^r\right), \quad n \in \mathbb{N}^*, \ r, a_i \in \mathbb{R}^+, \ i = 1,\dots,n.
\tag{3.47}
$$

The rest of the proof remains the same. The proof is now complete. $\square$

Remark 3.2. Apart from the case treated in the proof, that is, when $\tau < t_{k+1} - t_k \leq 2\tau$, $k = 0,1,2,\dots$, there are several other cases, but each and every one of them can fit in the one considered above or one of the following cases.

Case 1. There exists an $n_k > 1$ such that $t_{k+1} - t_k \geq n_k\tau$, that is, $t_k < t_k + n_k\tau \leq t_{k+1}$. In this case, we argue in a similar fashion over $(t_k, t_k + \tau]$, $(t_k + \tau, t_k + 2\tau], \dots, (t_k + (n_k - 1)\tau, t_k + n_k\tau]$ and then over $(t_k + n_k\tau, t]$ with $t > t_k + n_k\tau$. Therefore the function $\psi_{k+1}(t)$ will have $n_k + 1$ components.

Case 2. There exists $k_0 \geq 1$ such that $(t_{k_0}, t_{k_0+1}]$ does not contain any $t_{k_0} + n\tau$, $n = 1,2,\dots$, that is, $t_{k_0+1} - t_{k_0} < \tau$. Here we deal with this interval in a single step using only a function of the form $v_{k_0+1}(t)$, that is, $\psi_{k_0+1}(t) := v_{k_0+1}(t)$.

Case 3. $\tau \in (t_{k_1}, t_{k_1+1}]$ with $k_1 > 0$, that is, $\tau \notin (0, t_1]$ as in Case 1. Again, in this situation, we consider only functions of the form $v_k(t)$ until we reach the interval $(t_{k_1}, t_{k_1+1}]$ where we consider both $v_{k_1+1}(t)$ and $w_{k_1+1}(t)$.

Remark 3.3. Obviously, if $k_i(t,s) = (t-s)^{\beta_i-1}s^{\gamma_i}e^{-\delta_i s}F_i(s)$, $\delta_i > 0$, $i = 1,2\dots$, the proof still works. However, using Lemma 2.2 instead of Lemma 2.1 throughout the proof, we can have much larger intervals over which the estimations are valid.

Remark 3.4. It is clear that our result can be easily extended to other nonlinearities than the polynomial ones, iterated integrals and the case of several variables. One may use the Gronwall-Bihari lemma (Lemma 2.3) in case of a nondecreasing nonlinearity. See also [3, 13] for other classes of nonlinearities.

Acknowledgments

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